The emergence of the triangular lattice near a flat wall

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Abstract

The bifurcation of periodic solutions near a flat wall for applied magnetic fields which are slightly weaker than H_{C_2} is considered for a reduced Ginzburg-Landau model obtained in the large κ limit. We formally demonstrate that following the bifurcation of the first mode, when the applied magnetic field is further decreased, there is a second bifurcation, after which the solution develops continuously into the well-known triangular lattice.

1 Introduction

Consider a planar superconducting body at a sufficiently low temperature (below the critical one) under the action of an applied magnetic field . Its energy is given by the Ginzburg-Landau energy functional which can be represented in the dimensionless form [10]

$$E = \int_\Omega \left(-|\Psi|^2 + \frac{|\Psi|^4}{2} + |h - h_{ex}|^2 + \left| \frac{i}{\kappa}\nabla\Psi + A\Psi \right|^2 \right) dx_1 dx_2 \,, \tag{1.1}$$

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in which Ψ is the (complex) superconducting order parameter, such that $|\Psi|$ varies from $|\Psi| = 0$ (when the material is at a normal state) to $|\Psi| = 1$ (for the purely superconducting state). The magnetic vector potential is denoted by A (the magnetic field is thus $h = \nabla \times A$), h_{ex} is the constant applied magnetic field, and κ denotes the Ginzburg-Landau parameter which is a material property. The superconductor lies in Ω, which is a connected domain. Its Gibbs free energy is given by E, which is invariant under the gauge transformation

$$\Psi \to e^{i\kappa\eta}\Psi \qquad ; \qquad A \to A + \nabla\eta \, . \tag{1.2}$$

wherein η is any smooth function.

For sufficiently large magnetic fields it is well known, from both experimental observations [20] and theoretical predictions [14], that superconductivity is destroyed and the material must be at the normal state. If the applied magnetic field is then decreased there exists a critical field where the material enters the superconducting phase. This field is called "the onset field" and is denoted by H_{C_3}.

It is well-known that at the bifurcation from the normal state, superconductivity is concentrated near the boundary. Alternatively we can say that Ψ decays exponentially fast away from the boundaries as either κ or the size of Ω tend to infinity, which is why the phenomenon has been termed surface superconductivity. This result has first been established for a half-plane [24], then for disks [6], and for general smooth bounded domains in $\mathbb{R}^2$ [7, 19, 12, 16]. It has later been extended to weakly non-linear cases in the large κ limit [17].

In the absence of boundaries the critical field at which superconductivity nucleates is denoted by H_{C_2} and is smaller than H_{C_3} ($H_{C_3} \approx 1.7\kappa$ whereas $H_{C_2} = \kappa$). Furthermore, the bifurcating modes are periodic Abrikosov lattices [1, 9, 2] which have been observed experimentally [13]. Rubinstein [23] has therefore conjectured that superconductivity remains concentrated near the boundary for $H_{C_2} < h_{ex} < H_{C_3}$; when $h_{ex} \approx H_{C_2}$ (either for κ large or for large domains) Abrikosov's lattices bifurcate away from the wall.

Recently, it has been proved both in the large κ limit [22, 5], and in the large domain limit [3] that as long as $H_{C_2} < h_{ex} < H_{C_3}$ superconductivity remains concentrated near the boundaries. Furthermore, Pan [22] proved that when $\kappa \gg 1$, the solution near the boundary is close to the solution of the problem

$$\left(i\nabla + x_1\hat{i}_2\right)^2 \psi = \lambda\psi\left(1 - |\psi|^2\right) \quad \text{in } \mathbb{R}^2_+ \tag{1.3a}$$

$$\left.\frac{\partial\psi}{\partial x_1}\right|_{x_1=0} = 0, \tag{1.3b}$$

where $\lambda = \kappa/h_{ex}$. In addition, it is conjectured in [22], that the unique solution when $\lambda < 1$ is essentially one-dimensional, i.e.,

$$\psi = f(x_1, \lambda)e^{i\omega_0 x_2}, \tag{1.4}$$

for some $\omega_0 \in \mathbb{R}$ and $f(x_1, \lambda)$ which satisfies

$$-f'' + (x - \omega_0)^2 f = \lambda f(1 - f^2) \quad ; \quad f'(0) = 0.$$

Non-trivial positive solutions exist for all $\lambda > \beta(\omega_0)$, where

$$\beta(z) = \inf_{\phi\in H^1(z,\infty)} \frac{\int_z^\infty |\phi'|^2 + x^2|\phi|^2}{\int_z^\infty |\phi|^2}. \tag{1.5}$$

Furthermore, $f < 1$, and [22]

$$f(x) \sim x^{-\frac{1-\lambda}{2}} e^{-\frac{1}{2}x^2} \quad \text{as } x \to \infty \,, \tag{1.6}$$

(cf. [21] for the definition of $\sim$). In a previous contribution [4], we studied (1.3), after applying the transformation

$$x_1 \to x_1 - \omega_0 \,,$$

in the space

$$\mathcal{P}_L^{\omega_0} = \left\{ \phi \in H^1_{mag}([-\omega_0, \infty) \times \mathbb{R}) \mid \phi(x_1, x_2 + L) = \phi(x_1, x_2) \right\} \,. \tag{1.7}$$

In this setting the solutions of (1.3a) may be treated as critical points of the functional

$$\mathcal{E}(\psi) = \int_0^\infty \int_0^{2\pi/\omega} \left| (i\nabla + x_1 \hat{i}_2)\psi \right|^2 + \lambda \left(\frac{1}{2}|\psi|^4 - |\psi|^2 \right) dx_2 dx_1 \,. \tag{1.8}$$

We proved in [4] that (1.4) must undergo a bifurcation for λ slightly greater than unity, i.e., we proved the existence of a sequence of critical values $\{\lambda_n\}_{n=1}^\infty$ such that $\lambda_n \to 1$ as $n \to \infty$ where a bifurcation from (1.4) can take place. Furthermore, we proved that near the bifurcation, the bifurcating branch is given in the form

$$\lambda \cong \lambda_n + \epsilon^2 \left(\frac{\omega}{2\pi} \right)^{3/2} + O(\epsilon^3) \tag{1.9a}$$

$$\psi \cong e^{i\omega_0 x_2} \left\{ f(x_1, \lambda) + \epsilon \left[\phi_n(x_1) e^{in\omega x_2} + \phi_{-n}(x_1) e^{-in\omega x_2} \right] \right\} + O(\epsilon^2) \,, \tag{1.9b}$$

where $\epsilon \ll 1$, $\omega = 2\pi/L$, and $\phi_{\pm n}$ satisfy a system of ordinary differential equations which is described in the next section. Finally, we proved that near the bifurcation, the energy of the bifurcating branch is lower than the energy of (1.4). Hence, for $\lambda > 1$ (1.4) must lose its stability.

While the results in [4] prove the bifurcation of a single Fourier mode, they do not address the behaviour of the bifurcating branch with increasing λ. It was expected in [4] that with growing λ the solution tends to become periodic in the x_1 direction as well. In other words, the solution should approach an Abrikosov lattice [9, 2]. The manner by which (1.9) develops into a periodic solution is not clear: it may either result from a sequence of bifurcations, or it may evolve continuously, or else undergo some combination of the above.

The present contribution focuses on the evolution of (1.9) with increasing λ. We assume that the first bifurcation takes place at $\lambda = \lambda_N$, where $N \gg 1$ (we discuss this choice in § 5). Then, by using a combination of formal and rigorous arguments, we find

1. The value of λ at which the next bifurcation (hereafter referred to as the "second" bifurcation) should take place.

2. The bifurcating mode.

3. The behaviour of the solution with increasing λ following the second bifurcation. We find that if no other bifurcation occurs after the second bifurcation, then, as $N \to \infty$, for $\lambda > \lambda_{N-l}$ where $1 \ll l \ll N$, the solution tends pointwise to the triangular lattice [18] for
$$\left(N - \frac{5}{8}l\right)\omega < x_1 < \left(N - \frac{3}{8}l\right)\omega.$$

We note that there are many indications, both theoretical [2, 9, 18] and experimental [13], that the minimizer of (1.1) is indeed the triangular lattice. It is thus plausible that the transition from (1.9) to a periodic solution in the x_1 direction is indeed described by the results in this work, despite a number of gaps that must be addressed in order to prove these results rigorously.

The rest of this contribution is arranged as follows: In the next section we prove the exact asymptotic behaviour of λ_n and $\phi_{\pm n}$ as $n \to \infty$ and conjecture that $\lambda_n > 1$ for all n. In § 3 we make use of the asymptotic formulas and the above conjecture to analyze the second bifurcation. In § 4 we formally analyze the behaviour of the solution with increasing λ after the second bifurcation. Finally, in § 5, we briefly summarize the main results of this work, emphasize some additional key points, and list the gaps that must be bridged in order to rigorously prove the main results.

2 Preliminaries

Consider the problem

$$\left(i\nabla + x_1\hat{i}_2\right)^2 \psi = \lambda\psi\left(1 - |\psi|^2\right) \quad \text{in } \mathbb{R}^2_+ \tag{2.1a}$$

$$\left.\frac{\partial\psi}{\partial x_1}\right|_{x_1=0} = 0 \quad \psi(x_1, x_2 + L) = e^{i\theta}\psi(x_1, x_2)\,, \tag{2.1b}$$

where θ is constant. Pan [22] conjectured that for $\lambda < 1$ the unique solution of (2.1) is given by (1.4). In [4] we prove that there exists a sequence of critical values $\{\lambda_n\}_{n=n_0}^{\infty}$ where solutions of (2.1) can bifurcate from (1.4) such that $\lambda_n \downarrow 1$. Furthermore, near the bifurcation,

$$\psi \approx f(x_1)e^{i\omega_0 x_2} + C(\lambda)\left[\phi_n(x_1)e^{in\omega x_2} + \phi_{-n}(x_1)e^{-in\omega x_2}\right]\,,$$

where upon applying the transformation $x \to x - \omega_0$, ϕ_n, ϕ_{-n} satisfy

$$-\phi_n'' + \left[(x_1 - n\omega)^2 - \lambda\right] \phi_n + \lambda f^2 \left(2\phi_n + \overline{\phi_{-n}}\right) = 0 \quad x_1 > -\omega_0\,, \tag{2.2a}$$

$$-\phi_{-n}'' + \left[(x_1 + n\omega)^2 - \lambda\right] \phi_{-n} + \lambda f^2 \left(2\phi_{-n} + \overline{\phi_n}\right) = 0 \quad x_1 > -\omega_0\,, \tag{2.2b}$$

$$\phi_n'(-\omega_0) = \phi_{-n}'(-\omega_0) = 0\,, \tag{2.2c}$$

and $C(\lambda) \sim O\left(|\lambda - \lambda_n|^{1/2}\right)$. The asymptotic behaviour of λ_n has been studied as well. In particular, it is proved in [4], that

$$1 + C_1 e^{-\frac{1}{2}(n\omega+\omega_0)^2} < \lambda_n < 1 + C_2 e^{-\frac{1}{2}(n\omega+\omega_0)^2}. \tag{2.3}$$

We now prove the exact asymptotic behaviour of λ_n.

Lemma 2.1 *Let λ_n be the lowest critical value of* (2.2)*. Then*

$$\lambda_n \sim 1 + 2a^2 e^{-(n\omega+\omega_0)^2/2} + O\left(e^{-2(n\omega+\omega_0)^2/3}\right) \quad \textit{as } n \to \infty, \tag{2.4a}$$

where

$$a = \lim_{x\to\infty} e^{x^2/2} f(x,1). \tag{2.4b}$$

Furthermore, let ϕ_n, ϕ_{-n} be the corresponding eigenmodes. Then, for sufficiently large n we have

$$\left\|\phi_n - e^{-(x-n\omega)^2/2}\right\|_2^2 + \|\phi_{-n}\|_2^2 \le e^{-2(n\omega+\omega_0)^2/3}\,, \tag{2.5}$$

where $\|\cdot\|_2$ denotes the L^2 norm on $(-\omega_0, \infty)$.

Proof: Let

$$\mathcal{H} = \left\{\phi \in H^2(-\omega_0,\infty) \mid x\phi \in L^2(-\omega_0,\infty)\ ;\ \phi'(-\omega_0) = 0\right\},$$

and $\mathcal{P}_n : \mathcal{H} \times \mathcal{H} \to L^2 \times L^2$ denote the operator

$$\mathcal{P}_n = \begin{bmatrix} -\frac{d^2}{dx_1^2} + (x_1 - n\omega)^2 & 0 \\ 0 & -\frac{d^2}{dx_1^2} + (x_1 + n\omega)^2 \end{bmatrix} + \lambda_n f^2(x_1, \lambda_n) \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

Clearly,

$$(\mathcal{P}_n - \lambda_n) \begin{bmatrix} \phi_n \\ \bar{\phi}_{-n} \end{bmatrix} = 0\,.$$

We now choose the quasi-mode

$$v_n = C_n \begin{bmatrix} \chi(x + \omega_0) e^{(-x-n\omega)^2/2} \\ 0 \end{bmatrix},$$

where C_n is chosen such that $\int |v_n|^2 = 1$ and χ is a smooth cutoff function satisfying

$$\chi(x) = \begin{cases} 0 & x \leq \frac{1}{2} \\ 1 & x \geq 1 \end{cases}.$$

Let

$$\alpha_n = 1 + 2\lambda_n \int f^2(x, \lambda_n)\, |v_n|^2\,.$$

Then,

$$|\langle v_n, (\mathcal{P}_n - \alpha_n)\, v_n \rangle| \leq \int |-\chi'' + 2(x - n\omega)\chi'|\, e^{-(x-n\omega)^2} \leq C e^{-(n\omega+\omega_0)^2}\,. \tag{2.6}$$

We now represent the quasi-mode v_n

$$v_n = a_n \Phi_n + \tilde{v}_n\,,$$

where

$$\Phi_n = \begin{bmatrix} \phi_n \\ \phi_{-n} \end{bmatrix},$$

and

$$a_n = \langle v_n, \Phi_n \rangle \ .$$

From (2.6) we have

$$a_n^2 \left| \lambda_n - \alpha_n \right| \leq \left| \langle v_n, (\mathcal{P}_n - \alpha_n) \, v_n \rangle \right| + \left| \langle \tilde{v}_n, (\mathcal{P}_n - \alpha_n) \, \tilde{v}_n \rangle \right| \leq$$

$$\leq \|\tilde{v}_n\|_2 \, \|(\mathcal{P}_n - \alpha_n) \, v_n\|_2 + Ce^{-(n\omega+\omega_0)^2}. \quad (2.7)$$

Let

$$\mu_n = \inf_{\substack{u \in \tilde{\mathcal{H}}_n \\ \|u\|_2 = 1}} \langle \mathcal{P}_n u, u \rangle \ ,$$

where $\tilde{\mathcal{H}}_n$ is the orthogonal complement of span $\{\Phi_n\}$ in $\mathcal{H} \times \mathcal{H}$.

Then,

$$\|\tilde{v}_n\|_2^2 \leq \frac{\|(\mathcal{P}_n - \alpha_n) \, \tilde{v}_n\|_2^2}{\left|\mu_n - \alpha_n\right|^2} \leq \frac{\|(\mathcal{P}_n - \alpha_n) \, v_n\|_2^2}{\left|\mu_n - \alpha_n\right|^2} \, . \quad (2.8)$$

We claim that

$$\liminf_{n \to \infty} \mu_n - \alpha_n = 2 \, . \quad (2.9)$$

To prove (2.9) we define the operator

$$\mathcal{Q}_n = \begin{bmatrix} -\frac{d^2}{dx_1^2} + (x_1 - n\omega)^2 & 0 \\ 0 & -\frac{d^2}{dx_1^2} + (x_1 + n\omega)^2 \end{bmatrix} . \quad (2.10)$$

Clearly,

$$\nu_n = \inf_{\substack{u \in \tilde{\mathcal{H}}_n \\ \|u\|_2 = 1}} \langle \mathcal{Q}_n u, u \rangle \leq \mu_n. \quad (2.11)$$

We now prove that

$$\liminf_{n\to\infty} \nu_n \geq 3\,. \tag{2.12}$$

Denote by U_n the minimizer of $< \mathcal{Q}_n u, u >$ in $\mathcal{H}\times\mathcal{H}$. For sufficiently large n we have

$$U_n = \begin{bmatrix} u_n \\ \\ 0 \end{bmatrix}\,,$$

where u_n satisfies

$$-u_n'' + (x-n\omega)^2 u_n = \beta_n u_n \quad u_n'(-\omega_0) = 0\,,$$

in which $\beta_n = \beta(-n\omega-\omega_0)$ is given by (1.5) and $\|u_n\|_2 = 1$. Let w denote a unit vector in $\tilde{\mathcal{H}}_n$. Then,

$$\langle w, U_n\rangle = \langle w, -\Phi_n + \langle \Phi_n, U_n\rangle\, U_n\rangle + (1-\langle \Phi_n, U_n\rangle)\,\langle w, U_n\rangle\ .$$

We seek to estimate the norm of $\Phi_n - \langle \Phi_n, U_n\rangle\, U_n$. To this end we apply to it first $\mathcal{Q}_n - \beta_n$ and then estimate the norm of the outcome. We have

$$\int |(\mathcal{Q}_n-\beta_n)\,(\Phi_n - \langle \Phi_n, U_n\rangle\, U_n)|^2 \leq (\lambda_n-\beta_n)^2 + 4\lambda_n^2\int f^2\left(|\phi_n|^2 + |\phi_{-n}|^2\right).$$

To estimate the second term on the right-hand-side, we recall from [4] that

$$\begin{aligned}
\lambda_n\int f^2\left(|\phi_n|^2 + |\phi_{-n}|^2 + |\phi_n+\bar{\phi}_{-n}|^2\right) = \\
= -\int \left[|\phi_n'|^2 + |\phi_{-n}'|^2 + (x-n\omega)^2|\phi_n|^2 + (x+n\omega)^2|\phi_{-n}|^2\right] + \lambda_n \leq (\lambda_n-\beta_n)\,.
\end{aligned}$$

From [8, 19] we know that

$$\beta_n \sim 1 - O\left(e^{-(n\omega+\omega_0)^2}\right).$$

Hence, in view of (2.3) we obtain

$$\int \left|(\mathcal{Q}_n - \beta_n)\left(\Phi_n - \langle \Phi_n, U_n\rangle U_n\right)\right|^2 \leq C e^{-(n\omega+\omega_0)^2/2}\,. \tag{2.13}$$

Since $\Phi_n - \langle \Phi_n, U_n\rangle U_n$ is orthogonal to U_n we have

$$\int \left|(\mathcal{Q}_n - \beta_n)\left(\Phi_n - \langle \Phi_n, U_n\rangle U_n\right)\right|^2 \geq (\sigma_n - \beta_n)^2 \left\|\Phi_n - \langle \Phi_n, U_n\rangle U_n\right\|_2^2 \,,$$

where

$$\sigma_n = \inf_{\substack{u\in span\{U_n\}^\perp \\ \|u\|_2=1}} \langle \mathcal{Q}_n u, u\rangle\,.$$

Semi-classical analysis [15, 8] shows that

$$\sigma_n \xrightarrow[n\to\infty]{} 3. \tag{2.14}$$

Combining the above with (2.13) yields

$$\left\|\Phi_n - \langle \Phi_n, U_n\rangle U_n\right\|_2^2 \leq C e^{-(n\omega+\omega_0)^2/2},$$

and consequently

$$\langle w, U_n\rangle \leq C e^{-(n\omega+\omega_0)^2/4}\,, \tag{2.15}$$

where C is independent of w and n.

We now present w in the form

$$w = \langle w, U_n\rangle U_n + w_n.$$

Clearly, $w_n \perp U_n$, and hence

$$\langle \mathcal{Q}_n w, w\rangle = \beta_n \langle w, U_n\rangle^2 + \langle \mathcal{Q}_n w_n, w_n\rangle \geq \sigma_n \|w_n\|_2^2 \geq \sigma_n(1 - e^{-(n\omega+\omega_0)^2/2})\,,$$

which proves (2.12). In view of (2.11), we have proved (2.9) too.

We now substitute (2.9) into (2.8) and then into (2.7) to obtain

$$a_n^2 \left|\lambda_n - \alpha_n\right| \leq \left\|\left(\mathcal{P}_n - \alpha_n\right) v_n\right\|_2^2 + Ce^{-(n\omega+\omega_0)^2},$$

or, equivalently

$$a_n^2 \left|\lambda_n - \alpha_n\right| \leq 2\left(\alpha_n - 1\right)^2 + 5\int f^4 \left|v_n\right|^2 + Ce^{-(n\omega+\omega_0)^2} \leq Ce^{-2(n\omega+\omega_0)^2/3}.$$

Furthermore,

$$a_n^2 = 1 - \|v_n\|_2^2 \geq 1 - \left\|\left(\mathcal{P}_n - \alpha_n\right) v_n\right\|_2^2 \geq 1 - Ce^{-2(n\omega+\omega_0)^2/3}.$$

Hence,

$$\lambda_n \sim 1 + 2\lambda_n \int f^2(x, \lambda_n) \left|v_n\right|^2 + O\left(e^{-2(n\omega+\omega_0)^2/3}\right) \quad \text{as } n \to \infty. \tag{2.16}$$

In [4] we proved that $|\partial f/\partial\lambda| \leq C$ for all x in some neighborhood of $\lambda = 1$. Utilizing this result together with (2.3) we obtain

$$\lambda_n \sim 1 + 2\int f^2(x, 1)e^{-(x-n\omega)^2} + O\left(e^{-2(n\omega+\omega_0)^2/3}\right) \quad \text{as } n \to \infty,$$

from which (2.4) readily follows. Similarly, from (2.8) we obtain (2.5).

□

We conclude this section by stating the following conjecture, and by making a simple observation.

Conjecture 1 *Let*

$$\gamma_n(\lambda) = \inf_{\substack{(\chi_n, \chi_{-n}) \in \mathcal{H}\times\mathcal{H} \\ \|\chi_n\|_2^2 + \|\chi_{-n}\|_2^2 = 1}} \mathcal{J}_n(\chi_n, \chi_{-n}), \tag{2.17a}$$

where

$$\mathcal{J}_n(\chi_n,\chi_{-n}) =: \int_{-\omega_0}^{\infty} |\chi_n'|^2 + (x-n\omega)^2|\chi_n|^2 + \left|\chi_{-n}'\right|^2 + (x+n\omega)^2|\chi_{-n}|^2 -$$

$$-\lambda\left[|\chi_n|^2+|\chi_{-n}|^2 - f^2\left(|\chi_n|^2 + |\chi_n+\bar{\chi}_{-n}|^2 + |\chi_{-n}|^2\right)\right]\,. \quad (2.17b)$$

Then,

$$\forall n\in\mathbb{N}, \exists\delta_n>0 : \lambda<1+\delta_n \Rightarrow \gamma_n(\lambda)>0\,. \quad (2.18)$$

Note that by (2.4) the above conjecture is correct for sufficiently large n, since $\gamma_n(\lambda)>0$ for all $\lambda<\lambda_n$. For n which is not necessarily large, it is still expected that (2.18) remains valid since otherwise the surface superconductivity solution (1.4) would loose its stability for $\lambda\leq 1$ for some $n\in\mathbb{N}$. This would contradict the physical intuition suggesting that (1.4) must be stable for $\lambda\leq 1$.

Finally, we prove the leading asymptotic behavior for $\gamma_n(\lambda)$ as $n\to\infty$.

Lemma 2.2 *Let $\lambda<2$. Then,*

$$\gamma_n(\lambda)\sim(\lambda_n-\lambda)\left(1+Ce^{-\frac{1}{3}(n\omega+\omega_0)^2}\right)\,. \quad (2.19)$$

Proof: It is easy to show that

$$-\phi_n'' + \left[(x_1-n\omega)^2-\lambda-\gamma_n\right]\phi_n + \lambda f^2\left(2\phi_n+\overline{\phi_{-n}}\right)=0 \quad x_1>-\omega_0$$

$$-\phi_{-n}'' + \left[(x_1+n\omega)^2-\lambda-\gamma_n\right]\phi_{-n} + \lambda f^2\left(2\phi_{-n}+\overline{\phi_n}\right)=0 \quad x_1>-\omega_0$$

$$\phi_n'(-\omega_0)=\phi_{-n}'(-\omega_0)=0\,.$$

Consequently, in the same manner used to derive (2.16) we can obtain that

$$\gamma_n(\lambda)\sim 1+2\lambda\int f^2(x,\lambda)\,|v_n|^2 + O\left(e^{-2(n\omega+\omega_0)^2/3}\right)-\lambda \quad \text{as } n\to\infty.$$

Utilizing (2.16) we have

$$\gamma_n(\lambda) \sim (\lambda_n - \lambda)\left(1 + 2\int f^2(x,\lambda_n)\,|v_n|^2\right) + 2\lambda \int \left[f^2(x,\lambda) - f^2(x,\lambda_n)\right]|v_n|^2$$

and since $|\partial f/\partial\lambda| \leq C$ (cf. [4]) we readily obtain (2.19).

3 The second bifurcation

In the preceding section we have considered the bifurcation from the one-dimensional solution (1.4). In this section we study the linear bifurcation from the bifurcating branch, i.e. the second bifurcation. Let then

$$\psi = \psi_0 + u \tag{3.1}$$

where $\psi_0(\lambda)$ denotes the solution which bifurcates from (1.4) at $\lambda = \lambda_N$ for $N \gg 1$. It is convenient to present ψ_0 using the parametric form

$$\psi_0 \cong f_0 + \epsilon V_0 + O\left(\epsilon^2, e^{-(N\omega+\omega_0)^2/2}\right) \tag{3.2a}$$

$$\lambda \cong \lambda_N + \epsilon^2\lambda_N^{(2)} + O\left(\epsilon^2\right)\,, \tag{3.2b}$$

in which

$$V_n = \phi_{n+N}(x_1)e^{i(n+N)\omega x_2} + \phi_{-n-N}(x_1)e^{-i(n+N)\omega x_2}\,, \tag{3.2c}$$

$$f_0(x_1) = f(x_1, 1)\,, \tag{3.2d}$$

$$\lambda_N^{(2)} \cong \left(\frac{\omega}{2\pi}\right)^{3/2} + O\left(e^{-(\omega N^{+}\omega_0)^2/2}\right)\,. \tag{3.2e}$$

We shall consider in the sequel positive ϵ values which are of $O\left(e^{-[(N-N_0)\omega+\omega_0]^2/4}\right)$ for $N_0 \ll N$. In view of (3.2c) we have

$$\left(i\nabla + x_1\hat{i}_2\right)^2 v_n - \lambda_{N+n}\left[v_n - f^2\big|_{\lambda=\lambda_{N+n}}(2v_n + \bar{v}_n)\right] = 0. \tag{3.3}$$

Consider now the bifurcation from the branch which bifurcated at $\lambda = \lambda_{\tilde{N}}$ from the surface superconductivity solution (1.4) for $\tilde{N} \neq N$ which is still much greater than 1. In this case we obtain from (2.5) that apart from an exponentially small error

$$\tilde{\psi}_0 - f_0\Big|_{(x_1,x_2)} \cong \tilde{\psi}_0 - f_0\Big|_{(x_1+\omega(\tilde{N}-N),x_2)},$$

where $\tilde{\psi}_0$ is the mode bifurcating from f at $\lambda = \lambda_{\tilde{N}}$. Thus, the choice of N has little impact on the results obtained in this section. We shall return to this problem in the last section.

Substituting (3.1) into (2.1) while keeping in mind that ψ_0 is a solution of (2.1), we obtain

$$\left(i\nabla + x_1\hat{i}_2\right)^2 u - \lambda\left[u - \left(2|\psi_0|^2 u + \psi_0^2\bar{u} + 2|u|^2\psi_0 + u^2\bar{\psi}_0 + |u|^2 u\right)\right] = 0 \quad x \in \mathbb{R}^2_{\omega_0}, \tag{3.4a}$$

$$u_{x_1}(-\omega_0, x_2) = 0 \quad ; \quad u(x_1, x_2 + L) = u(x_1, x_2), \tag{3.4b,c}$$

where $L = 2\pi/\omega$, and $\mathbb{R}^2_{\omega_0} = \{(x_1, x_2) \mid -\omega_0 < x_1,\ x_2 \in \mathbb{R}\}$. We look for solutions in $\mathcal{P}_L^{\omega_0}$ which bifurcate from $u \equiv 0$. We thus assume that such a bifurcation takes place at $\lambda = \mu$ and linearize (3.4) by introducing the expansion

$$\lambda \cong \mu + \delta\mu^{(1)} + \delta^2\mu^{(2)} + O\left(\delta^3\right), \tag{3.5a}$$

$$u \cong \delta u^{(0)} + \delta^2 u^{(1)} + \delta^3 u^{(2)} + O\left(\delta^4\right), \tag{3.5b}$$

to obtain

$$\left(i\nabla + x_1\hat{i}_2\right)^2 u^{(0)} - \mu\left[u^{(0)} - \left(2|\psi_0|^2 u^{(0)} + \psi_0^2\bar{u}^{(0)}\right)\right] = 0\,, \tag{3.6}$$

together with (3.4b,c).

We shall now obtain a necessary condition for the existence of non-trivial solutions of (3.6) involving V_n with $N+n \gg 1$.

Lemma 3.1 *Let*

$$\mu = \lambda_N + \epsilon^2\lambda_N^{(2)}\,,$$

where $\lambda_N^{(2)}$ *is given by (3.2e). Let further* ψ_0 *be given by* (3.2) *and* $u^{(0)}$ *satisfy* (3.6)*. Denote by* $\hat{u}_n$ *the Fourier coefficient*

$$\hat{u}_n = \int_0^{2\pi/\omega} e^{-i\omega n x_2}dx_2\,.$$

Then, if there exist $\epsilon \leq \exp\left\{-\frac{1}{4}[(N-l)\omega+\omega_0]^2\right\}$ *(where* $l \ll N$*) such that a non-trivial* $u^{(0)}$ *satisfying*

$$\int_{-\omega_0}^{\infty} |\hat{u}_{N+n}|^2 + |\hat{u}_{N-n}|^2\,dx_1 \geq \frac{1}{2} \tag{3.7}$$

exists for some $0 \leq n \leq l$*, we must have*

$$\epsilon^2 \cong \eta_{-n}^2 \frac{1-2q^{n^2}}{\left(1-2q^{n^2}\right)^2 - q^{4n^2}} \tag{3.8a}$$

and

$$\frac{2}{\omega}\pi^{3/2}u_{(0)} \cong \frac{1-2q^{n^2}}{\sqrt{\left(1-2q^{n^2}\right)^2+q^{4n^2}}}\exp\left\{-\frac{1}{2}\left[x_1-(N-n)\omega\right]^2 + i\omega(N-n)x_2\right\} +$$

$$+\frac{q^{2n^2}}{\sqrt{\left(1-2q^{n^2}\right)^2+q^{4n^2}}}\exp\left\{-\frac{1}{2}\left[x_1-(N+n)\omega\right]^2 + i\omega(N+n)x_2\right\}\,, \tag{3.8b}$$

where

$$\eta_n^2 = \left(\frac{2\pi}{\omega}\right)^{3/2} (\lambda_N - \lambda_{N+n}) \quad ; \quad \eta_{-n}^2 = \left(\frac{2\pi}{\omega}\right)^{3/2} (\lambda_N - \lambda_{N+n}) \,, \tag{3.8c}$$

and

$$q = e^{-\frac{1}{2}\omega^2}. \tag{3.8d}$$

Proof: Multiplying (3.6) by $\bar{V}_n$ and integrating by parts over $D = (-\omega_0, \infty) \times (0, L)$ we obtain

$$\begin{aligned}(\mu - \lambda_{N+n}) \int u^{(0)} \bar{V}_n &= (\mu - \lambda_{N+n}) \int \bar{V}_n \left[2|\psi_0|^2 u^{(0)} + \psi_0^2 \bar{u}^{(0)}\right] + \\ + \lambda_{N+n} \int f^2 \left(u^{(0)} \bar{V}_n - \bar{u}^{(0)} V_n\right) &+ \lambda_{N+n} \int \bar{V}_n \left[2\left(|\psi_0|^2 - f^2\right) u^{(0)} + \left(\psi_0^2 - f^2\right) \bar{u}^{(0)}\right] \,. \end{aligned} \tag{3.9}$$

We need first an estimate for the first two integrals on the right-hand-side of (3.9). Without loss of generality we assume that $\|u^{(0)}\|_{L^2(D)} = 1$ (which may always be achieved through appropriately adjusting δ). By (3.6) we also have

$$\int \left|(i\nabla + x_1 \hat{i}_2) u^{(0)}\right|^2 \leq C \int \left|u^{(0)}\right|^2 \leq C \,. \tag{3.10}$$

For the first integral on the right-hand-side of (3.9) we obtain by using (3.2)

$$\int \bar{V}_n |\psi_0|^2 u^{(0)} \cong \int \bar{V}_n f_0^2 u^{(0)} + 2\epsilon \int \bar{V}_n f_0 \Re(V_0) u^{(0)} + O(\epsilon^2).$$

Then , utilizing (2.5) we have

$$\left|\int \bar{V}_n f_0^2 u^{(0)}\right|^2 \leq \int f_0^4 |V_n|^2 \leq C e^{-2[(N+n)\omega + \omega_0]^2/3} \,,$$

$$\left|\int \bar{V}_n f_0 V_0 u^{(0)}\right|^2 \leq \int f_0^2 |V_n|^2 |V_0|^2 \leq C e^{-2[(N+n/2)\omega + \omega_0]^2/3}.$$

To estimate the second integral on the right-hand-side of (3.9) we multiply (3.6) once by $e^{-i(N+n)x_2}$ and once by $e^{i(N+n)x_2}$ and integrate by parts to obtain the system

$$-\hat{u}''_{N+n}+\left[(x-n\omega)^2-\mu\right]\hat{u}_{N+n}+\mu f^2\left(2\hat{u}_{N+n}+\overline{\hat{u}_{-N-n}}\right)=$$
$$=-\int_0^{2\pi/\omega}\left[\left(|\psi_0|^2-f^2\right)2u^{(0)}+\left(\psi_0^2-f^2\right)\bar{u}^{(0)}\right]e^{-i\omega(N+n)x_2}dx_2 \tag{3.11a}$$
$$-\hat{u}''_{-N-n}+\left[(x+n\omega)^2-\mu\right]\hat{u}_{-N-n}+\mu f^2\left(2\hat{u}_{-N-n}+\overline{\hat{u}_{N+n}}\right)=$$
$$=-\int_0^{2\pi/\omega}\left[\left(|\psi_0|^2-f^2\right)2u^{(0)}+\left(\psi_0^2-f^2\right)\bar{u}^{(0)}\right]e^{i\omega(N+n)x_2}dx_2 \tag{3.11b}$$
$$\hat{u}'_{N+n}(-\omega_0)=\hat{u}'_{-N-n}(-\omega_0)=0\,. \tag{3.11c}$$

Multiplying (3.11a) by $\bar{\hat{u}}_{N+n}$ and (3.11b) by $\bar{\hat{u}}_{-N-n}$, summing the resulting equations, and integrating with respect to x_1 we obtain

$$\mathcal{J}_{N+n}(\hat{u}_{N+n},\hat{u}_{-N-n},\mu)=\int\left[\overline{e^{-i\omega(N+n)x_2}\hat{u}_{N+n}+e^{i\omega(N+n)x_2}\hat{u}_{-N-n}}\right]\cdot$$
$$\left[\left(|\psi_0|^2-f^2\right)2u^{(0)}+\left(\psi_0^2-f^2\right)\bar{u}^{(0)}\right]\,,$$

where $\mathcal{J}$ is defined in (2.17). Since $\hat{u}_n e^{i\omega n x_2}$ is the projection of $u^{(0)}$ on the subspace of the n'th Fourier harmonic, it is easy to show that its H^1_{mag} norm is uniformly bounded for all N and n. Consequently,

$$\|u^{(0)}\|_{L^4}\leq C\quad;\quad\|\hat{u}_n e^{i\omega n x_2}\|_{L^4}\leq C.$$

Therefore, it is not difficult to show that for sufficiently large $N+n$

$$\int\left|\hat{u}_{-N-n}\right|^2+f^2\left|\hat{u}_{N+n}\right|^2-\left(\mu-\beta\big|_{z=\omega(N+n)+\omega_0}\right)\leq$$
$$\leq\mathcal{J}_{N+n}(\hat{u}_{N+n},\hat{u}_{-N-n},\mu)\leq C\left(\epsilon+e^{-(N\omega+\omega_0)^2/4}\right)^2. \tag{3.12}$$

where $\beta(z)$ is defined in (1.5). It is thus easy to show from (3.12) that

$$\int |\hat{u}_{-N-n}|^2 + f^2 |\hat{u}_{N+n}|^2 \leq C \left(\epsilon + e^{-(N\omega+\omega_0)^2/4}\right)^2 .$$

Let U_n be the minimizer of $< \mathcal{Q}_n u, u >$ in $\mathcal{H} \times \mathcal{H}$ where $\mathcal{Q}_n$ is defined in (2.10). Let further

$$\hat{u}_{N+n} = \langle \hat{u}_{N+n}, U_{N+n} \rangle U_{N+n} + \tilde{u}_{N+n}. \tag{3.13}$$

Then, by (3.12) and (2.14) we also have

$$\int |\tilde{u}_{N+n}|^2 \leq C \left(\epsilon + e^{-(N\omega+\omega_0)^2/4}\right)^2 . \tag{3.14}$$

We now return to the second integral on the right-hand-side of (3.9). Obviously,

$$\int f^2 \bar{u}^{(0)} \bar{V}_n = \int_{-\omega_0}^{\infty} f^2 \left[\bar{\phi}_{N+n} \bar{\hat{u}}_{-N-n} + \bar{\phi}_{-N-n} \bar{\hat{u}}_{N+n}\right] dx_1. \tag{3.15}$$

From (3.12), (2.5) and (2.4) we obtain

$$\begin{aligned} \left| \int_{-\omega_0}^{\infty} f^2 \bar{\phi}_{N+n} \bar{\hat{u}}_{-N-n} \right|^2 &\leq \int_{-\omega_0}^{\infty} f^4 |\phi_{N+n}|^2 \int_{-\omega_0}^{\infty} |\hat{u}_{-N-n}|^2 \leq \\ &\leq C \left(\epsilon + e^{-[\omega(N+n)+\omega_0]^2/4}\right)^2 e^{-2[\omega(N+n)+\omega_0]^2/3} , \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} \left| \int_{-\omega_0}^{\infty} f^2 \bar{\phi}_{-N-n} \bar{\hat{u}}_{N+n} \right|^2 &\leq \int_{-\omega_0}^{\infty} f^4 |\hat{u}_{N+n}|^2 \int_{-\omega_0}^{\infty} |\phi_{-N-n}|^2 \leq \\ &\leq C \left(\epsilon + e^{-[\omega(N+n)+\omega_0]^2/4}\right)^2 e^{-2[\omega(N+n)+\omega_0]^2/3} . \end{aligned} \tag{3.17}$$

Consequently,

$$(\mu - \lambda_{N+n}) \int u^{(0)} \bar{V}_n = \lambda_{N+n} \int \bar{V}_n \left[2 \left(|\psi_0|^2 - f^2\right) u^{(0)} + \left(\psi_0^2 - f^2\right) \bar{u}^{(0)}\right] + \tilde{\epsilon} , \tag{3.18a}$$

where

$$|\tilde{\epsilon}| \leq C \left\{ \epsilon^2 \left(\epsilon + e^{-[\omega(N+n)+\omega_0]^2/3} \right)^2 + \left(\epsilon + e^{-[\omega(N+n)+\omega_0]^2/4} \right)^2 e^{-[\omega(N+n)+\omega_0]^2/3} \right\} . \qquad (3.18b)$$

We now estimate the remaining terms on the right-hand side of (3.9). Evidently,

$$\int \bar{V}_n \left(|\psi_0|^2 - f^2 \right) u^{(0)} \cong \int \bar{V}_n \left(f_0^2 - f^2 \right) u^{(0)} + 2\epsilon \int f_0 \Re \left\{ V_0 u^{(0)} \bar{V}_n \right\} + \epsilon^2 \int |V_0|^2 u^{(0)} \bar{V}_n + O\left(\epsilon^3\right) . \qquad (3.19)$$

For the first term on the right-hand-side we have

$$\left| \int \bar{V}_n \left(f_0^2 - f^2 \right) u^{(0)} \right|^2 \leq \left\| \frac{\partial f}{\partial \lambda} \right\|^2_{L^\infty(-\omega_0,\infty)} |\lambda_n - 1|^2 \int (f_0 + f)^2 |V_n|^2 .$$

In view of (2.4) and since $\partial f / \partial \lambda$ is uniformly bounded [4], we have

$$\left| \int \bar{V}_n \left(f_0^2 - f^2 \right) u^{(0)} \right|^2 \leq C e^{-3[\omega(N+n)+\omega_0]^2/2}. \qquad (3.20)$$

For the second term on the right-hand-side of (3.19) we have

$$\left| \int f_0 \Re \left\{ V_0 u^{(0)} \bar{V}_n \right\} \right|^2 \leq \int f_0^2 |V_0 V_n|^2 \leq C e^{-2[\omega(N+n)+\omega_0]^2/3}. \qquad (3.21)$$

Combining (3.19), (3.20), and (3.21) we obtain

$$\int \bar{V}_n \left(|\psi_0|^2 - f^2 \right) u^{(0)} \cong \epsilon^2 \int |V_0|^2 u^{(0)} \bar{V}_n + O\left(\epsilon^2 e^{-[\omega(N+n)+\omega_0]^2/12} \right).$$

In a similar manner we can obtain a similar estimate for the second term on the right-hand side of (3.18a). Thus,

$$(\mu - \lambda_{N+n}) \int u^{(0)} \bar{V}_n = \epsilon^2 \int \bar{V}_n \left[2 |V_0|^2 u^{(0)} + V_0^2 \bar{u}^{(0)} \right] + \epsilon^2 \tilde{\epsilon}_n , \qquad (3.22a)$$

where

$$|\tilde{\epsilon}_n| \leq e^{-[\omega(N+n)+\omega_0]^2/12}. \tag{3.22b}$$

Let

$$\tilde{d}_n = \int u^{(0)} \bar{V}_n.$$

Then, we can write (3.22) in the form

$$\left(\epsilon^2 + \eta_n^2\right) \tilde{d}_n = \epsilon^2 q^{n^2} \left[2\tilde{d}_n + \bar{\tilde{d}}_{-n} q^{n^2}\right] + C\epsilon^2 \tilde{\epsilon}_n \tag{3.23a}$$

$$\left(\epsilon^2 - \eta_{-n}^2\right) \tilde{d}_{-n} = \epsilon^2 q^{n^2} \left[2\tilde{d}_{-n} + \bar{\tilde{d}}_n q^{n^2}\right] + C\epsilon^2 \tilde{\epsilon}_{-n}\,. \tag{3.23b}$$

Let $[d_n, d_{-n}]^T$ denote a non-trivial solution and $\epsilon_n^{(0)}$ be a critical value of

$$\left(\epsilon^2 - \eta_{-n}^2\right) d_{-n} = \epsilon^2 q^{n^2} \left[2d_{-n} + \bar{d}_n q^{n^2}\right] \tag{3.24a}$$

$$\left(\epsilon^2 - \eta_{-n}^2\right) d_{-n} = \epsilon^2 q^{n^2} \left[2d_{-n} + \bar{d}_n q^{n^2}\right]. \tag{3.24b}$$

Let further μ_{N-n} denote a critical value of μ for which a non-trivial solution of (3.6) satisfying (3.7) exists. Let ϵ_n be given by

$$\epsilon_n^2 = \frac{\mu_{N-n} - \lambda_N}{\lambda_N^{(2)}}\,.$$

By (3.7) we have to look for a solution of (3.23) which satisfies

$$\left|\tilde{d}_n\right|^2 + \left|\tilde{d}_{-n}\right|^2 \geq \frac{1}{2}\,.$$

It is not difficult to show that

$$\left|\epsilon_n - \epsilon_n^{(0)}\right| + \left|\tilde{d}_n - d_n\right| + \left|\tilde{d}_{-n} - d_{-n}\right| \leq C\tilde{\epsilon}_n\,, \tag{3.25}$$

where C is independent of n and N. Furthermore, one can easily show that whenever $1 \leq k \leq 2l$ and $k \neq n$, the solutions of (3.23) with $\epsilon = \epsilon_k$ must satisfy

$$\left|\tilde{d}_n\right| + \left|\tilde{d}_{-n}\right| \leq C\tilde{\epsilon}_n. \tag{3.26}$$

To solve (3.24) we multiply (3.24a) by $\bar{d}_{-n}$ and (3.24b) by $\bar{d}_n$ to obtain

$$\left[\epsilon^2\left(1-2q^{n^2}\right)-\eta_{-n}^2\right]|d_{-n}|^2 = \epsilon^2 q^{2n^2}\bar{d}_n\bar{d}_{-n} \tag{3.27a}$$

$$\left[\epsilon^2\left(1-2q^{n^2}\right)+\eta_n^2\right]|d_n|^2 = \epsilon^2 q^{2n^2}\bar{d}_n\bar{d}_{-n}. \tag{3.27b}$$

We confine the subsequent discussion to the case $q < 1/2$, (and later also to $q < \sqrt{2}-1$). This is done because most of the analysis in the next section will be devoted to the limit $q \to 0$. Furthermore, the periodic solution with the minimal energy in $\mathbb{R}^2$ is the triangular lattice [1, 9, 2] for which q is either $e^{-\pi\sqrt{3}}$ or $e^{-\pi/\sqrt{3}}$ which are both smaller than 1/2.

For $q < 1/2$ we must have by (3.27b)

$$\bar{d}_n\bar{d}_{-n} = |d_n|\ |d_{-n}|\,.$$

Hence,

$$\det\begin{bmatrix}\left(\epsilon_n^{(0)}\right)^2\left(1-2q^{n^2}\right)-\eta_{-n}^2 & -\left(\epsilon_n^{(0)}\right)^2 q^{2n^2}\\ -\left(\epsilon_n^{(0)}\right)^2 q^{2n^2} & \left(\epsilon_n^{(0)}\right)^2\left(1-2q^{n^2}\right)+\eta_n^2\end{bmatrix} = 0\,.$$

We now observe that by (2.4)

$$\frac{\eta_n^2}{\eta_{-n}^2} \leq Ce^{-\omega nN}.$$

Hence, we can approximate $\epsilon_n^{(0)}$ by

$$\left(\epsilon_n^{(0)}\right)^2 \cong \eta_{-n}^2\frac{1-2q^{n^2}}{\left(1-2q^{n^2}\right)^2-q^{4n^2}}\,.$$

Using (3.2b) the critical values of λ are accordingly

$$\mu_{N-n} = \lambda_N + \frac{1-2q^{n^2}}{\left(1-2q^{n^2}\right)^2 - q^{4n^2}} \left(\lambda_{N-n} - \lambda_N\right) .$$

As long as $q < \sqrt{2} - 1$ we have

$$\lambda_{N-n} < \mu_{N-n} < \lambda_{N-n-1},$$

and

$$\frac{\mu_{N-n} - \lambda_{N-n}}{\lambda_{N-n-1} - \lambda_{N-n}} \leq Ce^{-\omega N}.$$

Consequently, we can assert that the next bifurcation takes place at $\lambda = \mu_{N-1}$ where

$$\mu_{N-1} \cong \lambda_N + \frac{1-2q}{\left(1-2q\right)^2 - q^4} \left(\lambda_{N-1} - \lambda_N\right). \tag{3.28}$$

The corresponding eigenmode is

$$\frac{2}{\omega}\pi^{3/2} \begin{bmatrix} d_{-1} \\ \\ d_1 \end{bmatrix} = \frac{1}{\sqrt{(1-2q)^2 + q^4}} \begin{bmatrix} 1-2q \\ \\ q^2 \end{bmatrix}. \tag{3.29}$$

Combining (3.13),(3.14),(3.25), and (3.26) we obtain

$$\begin{aligned} \frac{2}{\omega}\pi^{3/2} u^{(0)} \cong & \frac{1-2q}{\sqrt{(1-2q)^2+q^4}} \exp\left\{ -\frac{1}{2}\left[x_1 - (N-1)\omega\right]^2 + i\omega(N-1)x_2 \right\} + \\ & + \frac{q^2}{\sqrt{(1-2q)^2+q^4}} \exp\left\{ -\frac{1}{2}\left[x_1 - (N+1)\omega\right]^2 + i\omega(N+1)x_2 \right\} + \tilde{v}_l \, , \end{aligned} \tag{3.30}$$

where $\tilde{v}_l$ satisfies

$$\int_0^{2\pi/\omega} e^{-i\omega n x_2} \tilde{v}_l dx_2 = 0 \; \forall x_1 \in (-\omega_0, \infty) \quad N - 2l \leq |n| \leq N + 2l \, .$$

We first consider the case $|n| \leq N - l - 1$. By (3.12) and (2.17) we have

$$\gamma_n(\mu) \left[\|\hat{u}_n\|_2^2 + \|\hat{u}_{-n}\|_2^2\right] \leq C \left(\epsilon + e^{-(N\omega+\omega_0)^2/4}\right)^2 .$$

By (2.19) and (2.18) there exists n_0 such that

$$n > n_0 \Rightarrow \gamma_n(\mu) \geq \frac{1}{2}\left(\lambda_n - \mu\right).$$

Let first $n_0 < n \leq N - l - 1$. Then

$$\|\hat{u}_n\|_2^2 + \|\hat{u}_{-n}\|_2^2 \leq C\frac{\epsilon^2}{\lambda_n - \mu} \leq C\frac{\mu - \lambda_N}{\lambda_n - \mu} \leq C\frac{\mu - \lambda_N}{\lambda_n - \mu} \leq C\frac{\lambda_{N-l} - \lambda_N}{\lambda_{N-2l} - \lambda_{N-l}} \leq Ce^{-\frac{1}{2}lN\omega}.$$

Consider now the case $n \leq n_0$. Let

$$\tilde{\gamma} = \min_{\substack{1 \leq n \leq n_0 \\ 1 \leq \mu \leq \lambda_{N-l}}} \gamma_n(\mu).$$

By (2.18) $\tilde{\gamma}$ must be positive for sufficiently large N. Consequently,

$$\|\hat{u}_n\|_2^2 + \|\hat{u}_{-n}\|_2^2 \leq C\frac{\epsilon^2}{\tilde{\gamma}}.$$

Finally, we consider the case $n \geq N + 2l + 1$. In this case we can write instead of (3.23a)

$$\left(\epsilon^2 + \eta_n^2\right)\tilde{d}_n = 2\epsilon^2 q^{n^2}\tilde{d}_n + C\epsilon^2\tilde{\epsilon}_n,$$

wherein

$$\tilde{\epsilon}_n \leq e^{-\frac{1}{4}lN\omega}.$$

Hence, $\tilde{d}_n \leq C\tilde{\epsilon}_n$. Since (3.14) is still valid we have

$$\|\tilde{v}_l\| \leq Ce^{-\frac{1}{4}lN\omega},$$

which completes the proof of (3.8).

□

Note that as $q \to 0$

$$\mu \to \lambda_{N-1} \quad ; \quad \begin{bmatrix} d_{-1} \\ \\ d_1 \end{bmatrix} \to \begin{bmatrix} 1 \\ \\ 0 \end{bmatrix} .$$

This is the limit of weak interaction between the modes: As $q \to 0$, ω tends to ∞ and hence, since $\phi_n \sim \exp\{-(x - n\omega)^2/2\}$, we obtain exponentially small interaction between the V_0 and its adjacent modes V_1 and V_{-1}. Consequently, as $q \to 0$, the next bifurcation is almost identical in nature with the first bifurcation: It takes place at $\mu = \lambda_{N-1}$ and the bifurcating mode is V_{N-1}.

The fact that by (3.30) $u^{(0)}$ depends only on a finite number of modes is surprising: One expects that the bifurcating branch would include infinitely many modes since the term $|\psi|^2\psi$ on the right-hand-side of (2.1) does not allow us to separate a finite number of modes from the others. It is thus expected that if we consider additional terms in the expansion (3.5), we shall obtain additional modes, so that u, the solution of (3.4) would consist of an infinite number of modes.

Upon multiplying (3.4) by $\bar{V}_n$ and integrating by parts we obtain

$$\begin{aligned}
(\lambda - \lambda_{N+n}) \int u\bar{V}_n = (\lambda - \lambda_{N+n}) \int \bar{V}_n \left[2|\psi_0|^2 u + \psi_0^2 \bar{u} \right] + \\
+ \lambda_{N+n} \int f^2 \left(u\bar{V}_n - \bar{u}V_n \right) + \lambda_{N+n} \int \bar{V}_n \left[2\left(|\psi_0|^2 - f^2 \right) u + \left(\psi_0^2 - f^2 \right) \bar{u} \right] + \\
+ \lambda \int \bar{V}_n \left[2|u|^2 \psi_0 + u^2 \bar{\psi}_0 \right] + \lambda \int \bar{V}_n |u|^2 u \, .
\end{aligned}$$

We now substitute (3.5) into the above and obtain for the $O(\delta^2)$ balance for $n = \pm 1$, making the same approximations as before

$$\left(\frac{\omega}{2\pi}\right)^{3/2}\left[\epsilon_1^2 \pm \eta_{\pm 1}^2\right)\int u^{(0)}\bar{V}_n + \mu_{N-1}^{(1)}\int u^{(0)}\bar{V}_n =$$
$$= \epsilon_1^2\int \bar{V}_n\left[2\left|V_0\right|^2 u^{(1)} + V_0^2\bar{u}^{(1)}\right] + \epsilon_1\int \bar{V}_n\left[2\left|u^{(0)}\right|^2 V_0 + V_0\left(u^{(0)}\right)^2\right] +$$
$$+ \mu_{N-1}^{(1)}\left(\frac{2\pi}{\omega}\right)^{3/2}\int \bar{V}_n\left[2\left|V_0\right|^2 u^{(0)} + V_0^2\bar{u}^{(0)}\right] + \epsilon_1^2\hat{\epsilon}_n. \quad (3.31)$$

It is possible to show that $\hat{\epsilon}_n$ is exponentially small as $N \to \infty$, since it results from interactions between "distant" modes, e.g. f_0 and V_n. We skip the details here and proceed by formally obtaining the next-order term.

To find $\mu_{N-1}^{(1)}$ we write the equations for $n = \pm 1$ neglecting the exponentially small terms

$$\left(\epsilon_1^2 - \eta_{-1}^2\right) d_{-1}^{(1)} = \epsilon_1^2 q\left[2d_{-1}^{(1)} + \bar{d^{(1)}}_1 q\right] + \mu_{N-1}^{(1)}\left[(1-2q)^2 - q^4\right], \quad (3.32)$$

$$\left(\epsilon_1^2 + \eta_1^2\right) d_1^{(1)} = \epsilon_1^2 q\left[2d_1^{(1)} + \bar{d^{(1)}}_{-1} q\right], \quad (3.33)$$

where

$$d_n^{(1)} = \int u^{(1)}\bar{V}_n. \quad (3.34)$$

It is easy to show that (3.32) can have solutions only if

$$\mu_{N-1}^{(1)} = 0. \quad (3.35)$$

In this case the solutions are proportional to (3.29) and are therefore of no interest. For $n = 0$ we have

$$d_0^{(1)} = 2d_0^{(1)} + \bar{d}_0^{(1)} + \frac{2q}{\epsilon_1}\left[(1-2q)^2 + q^3 - q^4\right].$$

Hence,

$$d_0^{(1)} = -\frac{q}{\epsilon_1}\left[(1-2q)^2 + q^3 - q^4\right] + iC, \tag{3.36}$$

where the last term can be eliminated using the gauge transformation (1.2). For $n = \pm 2$ we obtain

$$\left(\epsilon_1^2 - \eta_{-2}^2\right) d_{-2}^{(1)} = \epsilon_1^2 q^4 \left[2d_{-2}^{(1)} + \bar{d^{(1)}}_2 q^4\right] + \epsilon_1 q^2 (1-2q)\left[2q^5 + 1 - 2q\right] ,$$
$$\left(\epsilon_1^2 + \eta_2^2\right) d_2^{(1)} = \epsilon_1^2 q^4 \left[2d_2^{(1)} + \bar{d^{(1)}}_{-2} q^4\right] + \epsilon_1 q^6 [2q(1-2q)+1].$$

Since $\epsilon_1^2/\eta_{-2}^2 \sim O\left(e^{-\omega N}\right)$, we obtain

$$\left|d_{-2}^{(1)}\right| \sim O\left(e^{-\omega N}\right).$$

One can then obtain

$$d_2^{(1)} \cong \frac{q^6}{\epsilon_1}[2q(1-2q)+1] + \left(e^{-\omega N}\right).$$

For $n \geq 3$ it is easy to show that $d_{\pm n}^{(1)} = 0$.

We can proceed in the same manner to obtain the next order term in the expansion (3.5). However, this solution will not provide any significant information except for the fact that

$$\mu^{(2)} \cong \left(\frac{\omega}{2\pi}\right)^{3/2}\left[\epsilon_1\left(4d_{-1}^{(0)}d_0^{(1)}q + 2d_1^{(0)}d_0^{(1)}q^2 + 2d_1^{(0)}d_2^{(1)}q^5\right) + \right.$$
$$\left. + \left(\left|d_{-1}^{(0)}\right|^2 + 2q^4\left|d_1^{(0)}\right|^2\right) d_{-1}^{(0)}\right] \frac{1 + O\left(e^{-\omega N}\right)}{(1-2q)^2 - q^4} \cong 1 - 2q + O\left(q^2, e^{-\omega N}\right) \qquad \text{as } q \to 0. \tag{3.37}$$

We see that while $u^{(0)}$ contains the modes V_1 and V_{-1}, $u^{(1)}$ contains V_2 and V_{-2} as well. It can be shown that $u^{(n-1)}$ contains V_n and V_{-n} and hence by (3.5) u contains infinitely many modes. However, the coefficients of V_{-n} for $n \geq 2$ are exponentially small.

If we increase δ in (3.5) such that $\delta \gg \epsilon_1$, then the series (3.5) does not seem to converge: It is possible to show from (3.31) that $u^{(n)} \sim O\left(q^n/\epsilon_1^n\right)$. Therefore, convergence of (3.5) can be guaranteed only when $\delta \ll \epsilon_1/q$. We are interested, however, in the behaviour of the bifurcating branch when $\delta \gg \epsilon_1$ as well and therefore need to apply a different approach to obtain this behaviour, which is what we do in the next section.

To this end we discuss here the behaviour of u near the bifurcation in the limit $q \to 0$. Let then,

$$a_n = \lim_{q \to 0} \int u \bar{V}_n.$$

Formally, we have

$$a_n = \sum_{k=1}^{\infty} a_n^{(k)} \delta^k,$$

where

$$a_n^{(k)} = \lim_{q \to 0} \int u^{(k-1)} \bar{V}_n.$$

By (3.8b) we have

$$a_n^{(1)} = \begin{cases} 1 & n = -1 \\ & \\ 0 & n \neq -1 \end{cases}.$$

Furthermore, since

$$u^{(n)} \sim O\left(q^n\right) \quad \text{as } q \to 0,$$

we must have $a_n^{(k)} = 0$ for all n and $k \geq 2$. Consequently,

$$\psi \sim f_0 + \epsilon V_0 + \delta V_{-1} + O(q) \text{as } q \to 0. \tag{3.38}$$

4 Post-bifurcation behaviour

Consider again (2.1). We multiply the equation by $\bar{V}_n$ and integrate by parts to obtain

$$\lambda \int \psi \left(1 - |\psi|^2\right) \bar{V}_n = \lambda_{N+n} \int \psi \left[\bar{V}_n - f^2 \left(2\bar{V}_n + V_n\right)\right],$$

or

$$\begin{aligned}(\lambda - \lambda_{N+n}) \int \psi \bar{V}_n = (\lambda - \lambda_{N+n}) \int \psi \, |\psi|^2 \, \bar{V}_n + \\ + \lambda_{N+n} \int \psi \, |\psi - f|^2 \, \bar{V}_n - \lambda_{N+n} \int f^2 \left[(\bar{\psi} - f)\bar{V}_n - (\psi - f) V_n\right]. \quad (4.1)\end{aligned}$$

We look for solutions of (2.1) for

$$0 < \lambda - \mu_{N-1} \ll 1,$$

which are close to (1.4), i.e.

$$\lambda = \lambda_N + \left(\frac{\omega}{2\pi}\right)^{3/2} \epsilon^2 \tag{4.2a}$$

and

$$\psi \cong f_0 + \epsilon \psi^{(1)} + \epsilon^2 \psi^{(2)}, \tag{4.2b}$$

where

$$\psi^{(1)} = \sum_{n=-N+1} C_n V_n, \tag{4.2c}$$

$\epsilon_1 < \epsilon \ll 1$, and $\|\psi^{(2)}\|$ is bounded as $\epsilon \to \epsilon_1$ and $N \to \infty$. (Note that (4.2) is an extrapolation of the results in the preceding section.)

We now estimate the various terms on the right hand side of (4.1). For the first term we have

$$\int \psi \left|\psi\right|^2 \bar{V}_n = \epsilon \int f^2 \left[2\psi^{(1)} + \bar{\psi}^{(1)}\right] \bar{V}_n + \epsilon^2 \int f^2 \left[2\psi^{(2)} + \bar{\psi^{(2)}}\right] \bar{V}_n + \epsilon^2 \int f \left[2\left|\psi^{(1)}\right|^2 + \left(\psi^{(1)}\right)^2\right] \bar{V}_n \, ,$$

in which we have the estimates

$$\left|\int f^2 \left[2\psi^{(1)} + \bar{\psi}^{(1)}\right] \bar{V}_n\right|^2 \leq C \int f^4 \left|V_n\right|^2 \int \left|\psi^{(1)}\right|^2 \leq Ce^{-2[(N+n)\omega+\omega_0]^2/3},$$

$$\left|\int f \left[2\left|\psi^{(1)}\right|^2 + \left(\psi^{(1)}\right)^2\right] \bar{V}_n\right|^2 \leq e^{-[(N+n)\omega+\omega_0]^2/2}.$$

Hence,

$$\left|(\lambda - \lambda_{N+n}) \int \psi|\psi|^2\bar{V}_n\right| \leq C \left|\lambda - \lambda_{N+n}\right| \left[\epsilon e^{-[(N+n)\omega+\omega_0]^2/3} + \epsilon^2 e^{-[(N+n)\omega+\omega_0]^2/4} + \epsilon^3\right]. \quad (4.3)$$

For the last term on the right-hand-side of (4.1) we have

$$\int f^2 \left[(\bar{\psi} - f)\bar{V}_n - (\psi - f)V_n\right] = \epsilon \int f^2 \left[\bar{\psi}^{(1)}\bar{V}_n - \psi^{(1)}V_n\right] + \epsilon^2 \int f^2 \left[\bar{\psi^{(2)}}\bar{V}_n - \psi^{(2)}V_n\right],$$

in which we have the estimates

$$\left|\int f^2 \left[\bar{\psi}^{(1)}\bar{V}_n - \psi^{(1)}V_n\right]\right| \leq C \left|\int_{-\omega_0}^{\infty} f^2\phi_n\phi_{-n}dx_1\right| \leq Ce^{-2[(N+n)\omega+\omega_0]^2/3},$$

$$\left|\int f^2\psi^{(2)}V_n\right| \leq Ce^{-[(N+n)\omega+\omega_0]^2/3}.$$

For the second term on the right-hand-side of (4.1) we have

$$\int \psi \left|\psi - f\right|^2 \bar{V}_n = \epsilon^2 \int f \left|\psi^{(1)}\right|^2 \bar{V}_n + 2\epsilon^3 \int f \Re\left\{\bar{\psi}^{(1)}\psi^{(2)}\right\} \bar{V}_n + \epsilon^3 \int \psi^{(1)} \left|\psi^{(1)}\right|^2 \bar{V}_n + O\left(\epsilon^4\right) \, ,$$

in which

$$\left|\int f \left|\psi^{(1)}\right|^2 \bar{V}_n\right| \leq C \sup_{k<n} \left|\int f V_k V_{n-k-N} \bar{V}_n\right| \leq Ce^{-[(N+n)\omega+\omega_0]^2/2} \, ,$$

and

$$\left| \int f \Re \left\{ \bar{\psi}^{(1)} \psi^{(2)} \right\} \bar{V}_n \right| \leq C e^{-[(N+n)\omega+\omega_0]^2/4} .$$

Combining the above estimates and (4.1) we obtain

$$(\lambda - \lambda_{N+n}) \int \psi^{(1)} \bar{V}_n = \epsilon^2 \int \psi^{(1)} \left| \psi^{(1)} \right|^2 \bar{V}_n + \epsilon^2 \tilde{\epsilon}_n , \tag{4.4}$$

where $\tilde{\epsilon}_n$ satisfies (3.22b).

Substituting (4.2) into (4.4) and neglecting the exponentially small terms yields

$$\sum_{r,m=n-N+1}^{\infty} C_{n+r} \bar{C}_{n+r+m} C_{n+m} q^{m^2+r^2} - \nu_n C_n = 0, \tag{4.5a}$$

where

$$\nu_n^2 = \frac{\lambda - \lambda_{N+n}}{\lambda - \lambda_N} \tag{4.5b}$$

which is formally valid only when $(N+n) \gg 1$. Let $l \in \mathbb{N}$ and

$$\frac{1}{2} \left(\lambda_{N-l+1} + \lambda_{N-l} \right) < \lambda \leq \frac{1}{2} \left(\lambda_{N-l} + \lambda_{N-l-1} \right) .$$

Then, by (2.4), we have

$$\nu_n \sim \begin{cases} 1 + O(e^{-\omega N}) & n \geq -l+1 \\ \\ -O(e^{\omega N}) & n \leq -l-1 \end{cases} . \tag{4.6}$$

For $n = -l$ the above ratio varies from a negative $O(e^{\omega N})$ value for $\lambda = \left(\lambda_{N-l+1} + \lambda_{N-l} \right) /2$ to a value close to unity for $\lambda = \frac{1}{2} \left(\lambda_{N-l} + \lambda_{N-l-1} \right)$. An immediate consequence of (4.6) is that

$$C_n \sim O\left(e^{-\omega N} \right) \quad \forall n \leq -l-1. \tag{4.7}$$

Equation (4.5) possesses infinitely many solutions. We first consider solutions which are analytic functions of q, i.e.

$$C_n = \sum_{k=0}^{\infty} a_{nk} q^k. \tag{4.8}$$

We first look at $\lambda(q)$ such that ν_{-l} is independent of q. Note that by (4.6) ν_n remains too independent of q except for an $O(e^{-\omega N})$ term, which is negligible for $\omega \ll N$. For $q = 0$ (4.5) has the form

$$a_{n0} = |a_{n0}|^2 a_{n0} \qquad -l+1 \leq n \tag{4.9a}$$

$$\nu_{-l} a_{-l,0} = |a_{-l,0}|^2 a_{-l,0}\,. \tag{4.9b}$$

Note that the real roots of (4.9) are all simple provided that $\lambda \neq \lambda_{N-l}$. Consequently, all real solutions of (4.5) are holomorphic in q in some neighborhood of $q = 0$ for $\lambda \neq \lambda_{N-l}$. We later show that the converse statement is also true, i.e. every solution of the form (4.8) is essentially real.

Consider first the case $l = -1$. We shall assume that (3.5) and (4.2) should match $\delta \sim o(\epsilon_1)$. If we continue (3.5) into the region where $\delta \gg \epsilon_1$ [where (3.5) does not necessarily converge], we obtain via the superposition $\psi = \psi_0 + u$ a solution in the form (4.2). Near the bifurcation we have by (3.38)

$$a_{n0} = 0 \quad \forall n \neq 0, -1. \tag{4.10}$$

By (3.38) we have

$$a_{00} = 1 \quad ; \quad a_{-1,0} = \lim_{\substack{q \to 0 \\ \nu_{-1} = \text{const.}}} \frac{\delta}{\epsilon}, \tag{4.11}$$

or, equivalently, that near the bifurcation

$$|a_{-1,0}|^2 = \nu_{-1} \,,$$

which is in accordance with (4.9).

We now make the following claim

Lemma 4.1 *Let* $\{C_n\}_{n=-N+1}^{\infty}$ *satisfy* (4.5) *and* (4.8)*. Then,*

$$C_n = e^{i[\theta_0(q)+n\theta_1(q)]} C'_n \,, \tag{4.12}$$

where $C'_n \in \mathbb{R}$ *for all* $n \geq -N+1$*, and* $\theta_0(q)$ *and* $\theta_1(q)$ *are holomorphic in* q *in some neighborhood of* $q = 0$*. Furthermore, let* $\{a_{n0}\}_{n=-N+1}^{\infty}$ *be real solutions of* (4.9)*. Then, there exists a unique solution of* (4.5) *which satisfies* (4.8) *in some neighborhood of* $q = 0$*.*

We bring the proof in appendix A.

In view of (4.12) we can replace $\{C_n\}_{n=-N+1}^{\infty}$ by $\{C'_n\}_{n=-N+1}^{\infty}$ by applying the transformation

$$\psi \to e^{i\theta_0(q)} \psi(x_1, x_2 + \theta_1(q)/\omega) \,.$$

We may thus assume that the C_n's are all real (all other solutions will be gauge equivalent by (4.12)).

Consequently, using (4.12), we can set

$$a_{-1,0} = \nu_{-1} \,.$$

When $\lambda - \lambda_{N-1} \gg \lambda_{N-1} - 1$ we obtain $a_{-1,0} = 1$.

We now set $l = -2$ to examine the behaviour of C_{-2} with increasing λ. As stated earlier C_{-2} is exponentially small when $\lambda - 1 \ll \lambda_{N-2} - 1$. However, we expect C_{-2} to become $O(1)$ for $\lambda > \lambda_{N-2}$.

Substituting (4.8) into (4.5) we obtain for the $O(q)$ balance

$$\nu_n a_{n1} = 3a_{n0}^2 a_{n1} + 2\left[a_{n+1,0}^2 + a_{n-1,0}^2\right] a_{n0}\,. \tag{4.13}$$

For $n \neq 0, -1$ we have $a_{n1} = 0$. For $n = 0, -1$ we have

$$a_{01} = -\nu_{-1} \quad ; \quad a_{-1,1} = -\frac{1}{\nu_{-1}^{1/2}}\,.$$

The $O(q^2)$ balance is

$$\begin{aligned}\nu_n a_{n2} = {} & 3a_{n0}^2 a_{n2} + 3a_{n1}^2 a_{n0} + 4\left(a_{n+1,0}a_{n+1,1} + a_{n-1,0}a_{n-1,1}\right) a_{n0} + \\ & + 2\left[a_{n+1,0}^2 + a_{n-1,0}^2\right] a_{n1} + \left[a_{n+1,0}^2 a_{n+2,0} + 2a_{n+1,0}a_{n0}a_{n-1,0} + a_{n-1,0}^2 a_{n-2,0}\right]\,. \end{aligned} \tag{4.14}$$

Since we are interested in the behaviour of C_{-2} with increasing λ we solve (4.14) for $n = -2$

$$\nu_{-2} a_{-2,2} = a_{-1,0}^2 a_{00}.$$

Consequently,

$$C_{-2} \sim \frac{\nu_{-1}}{\nu_{-2}} q^2\,. \tag{4.15}$$

As was expected in (4.7) C_{-2} is exponentially small provided that $\lambda - 1 \ll \lambda_{N-2} - 1$. If, however, $\lambda \uparrow \lambda_{N+2}$, then $a_{-2,2} \to -\infty$ and (4.15) ceases to be valid. To obtain the leading behaviour of C_{-2} in the limit $q \to 0$ when $\nu_{-2} \ll 1$ we consider first the case where $\nu_{-2} \sim O(q)$. Let then

$$\nu_{-2} = \rho q\,.$$

Substituting into (4.9), (4.13), and (4.14) (recalling that $\nu_{-1} \cong 1$ up to an exponentially small error) we obtain

$$a_{-2,0} = 0$$

$$\rho a_{-2,1} = 2\,.$$

Consequently,

$$C_{-2} \cong \frac{1}{\rho - 2} q\,. \tag{4.16}$$

It is not difficult to show that in the overlap range where $\nu_{-2} \sim O(q^\alpha)$ for $0 < \alpha < 1$ (4.15) and (4.16) match. We thus formally conclude that (4.15) develops into (4.16), which is valid as long as $\rho < 2$.

When $\rho \uparrow 2$, $a_{-2,1} \to -\infty$ and hence we must consider separately the case $|\rho - 2| \ll 1$. We thus consider ν_{-2} values satisfying

$$\nu_{-2} = 2q + \tau q^{4/3}.$$

In this case (4.8) is no longer valid. We therefore use the more general ansatz

$$C_n = \sum_{k=0}^{\infty} b_{nk} q^{k/3}.$$

Substituting in (4.5) and applying an appropriate gauge transformation we obtain

$$b_{00} = 1 \quad ; \quad b_{-1,0} = 1 \quad ; \quad b_{-2,0} = 0,$$

$$b_{-2,1} = 0,$$

and

$$\left(|b_{-2,2}|^2 - \tau\right) b_{-2,2} = -b_{-1,0}^2 b_{00} = -1. \tag{4.17}$$

Since our goal is to follow C_{-2} with increasing λ we look for the solution of (4.17) which matches (4.16) as $\tau \to -\infty$. Consequently, we must have

$$b_{-2,2} \sim \frac{1}{\tau} \quad \text{as } \tau \to -\infty.$$

Using the theory of cubic equations, it is easy to show that (4.17) has only one solution for

$$\tau < \tau_c = \frac{3}{2\sqrt[3]{2}},$$

since (4.17) admits only real solutions. We now follow this solution with increasing value of τ in order to find its behaviour as $\tau \to \infty$.

When $\tau = \tau_c$, (4.17) possesses two distinct solutions

$$b_{-2,2} = \frac{1}{\sqrt[3]{2}} \quad \text{with double multiplicity} \tag{4.18a}$$

and

$$b_{-2,2} = \sqrt[3]{4} \quad \text{with single multiplicity}\,. \tag{4.18b}$$

The former solution does not exist for $\tau < \tau_c$, and therefore, the latter solution is the one we follow. Since $|b_{-2,2}(\tau_c)|^2 > \tau_c$, we must have $|b_{-2,2}(\tau)|^2 > \tau$ for all $\tau \geq \tau_c$. Consequently, as $\tau \to \infty$,

$$C_{-2} \sim -\tau^{1/2} q^{2/3} C_{-1}^2 C_0 \sim -\tau^{1/2} q^{2/3}. \tag{4.19}$$

Consider now the case $\nu_{-2} = \rho q$ when $\rho > 2$. In this case we use the ansatz

$$C_n = \sum_{k=0}^{\infty} B_{nk} q^{k/2}.$$

Substituting into (4.5) yields, upon applying an appropriate gauge transformation,

$$B_{00} = 1 \quad ; \quad B_{-1,0} = 1 \quad ; B_{-2,0} = 0,$$

$$|B_{-2,1}| = \sqrt{\rho - 2}. \tag{4.20a}$$

To find the phase of $B_{-2,1}$ we match (4.20) with (4.19) to obtain

$$C_{-2} \sim -\sqrt{\rho-2}\, q^{1/2} C_{-1}^2 C_0 \sim -\sqrt{\rho-2}\, q^{1/2}\,. \tag{4.21}$$

Finally, we consider positive $\nu_{-2} \sim O(1)$. Here we assume (4.8) once again. By (4.9) we have

$$|a_{-2,0}| = \sqrt{\nu_{-2}}.$$

The sign of $a_{-2,0}$ is determined from matching with (4.21). We obtain

$$C_{-2} \sim -\nu_{-2}^{1/2} C_{-1}^2 C_0 \sim -\nu_{-2}^{1/2}. \tag{4.22}$$

If λ further increases so that $\lambda_{N-2} - 1 \ll \lambda - 1 \ll \lambda_{N-3} - 1$, then $\nu_{-2} \sim 1$ and $C_{-2} \sim -1$.

To summarize: we have followed the $C_{-2}(\lambda)$ and found that it varies from a small negative value when $\lambda - 1 \ll \lambda_{N-2} - 1$ to approximately -1 for $\lambda - 1 \gg \lambda_{N-2} - 1$. This procedure can be applied recursively to obtain

$$C_{-l} \sim -C_{-l+1}^2 C_{l+2}\,, \tag{4.23}$$

from which we obtain

$$C_n \sim \begin{cases} (-1)^{[(n+1)/2]} & -l \le n \le 0 \\ 0 & \text{otherwise} \end{cases} + O(q)\,, \tag{4.24}$$

wherein $[\cdot]$ denotes the integer value.

We have thus formally obtained the behaviour of the C_n's in the limit $q \to 0$. We now claim that for $l \gg 1$ and $N - l \gg 1$, when substituting (4.24) into (4.2c), ψ becomes close to the well-known triangular lattice [1, 18, 9, 2]. The following lemma proves a stronger result: If a_{n0} satisfy (4.24), then any solution of the form (4.8) is close to the triangular lattice, for all q where (4.8) converges and not only when $q \to 0$. We should, however, emphasize that the foregoing discussion demonstrates (4.24) only formally and in the limit $q \to 0$. No such result has been proved for a_{n0}.

Lemma 4.2 *Let*

$$\phi = \epsilon D(q) \sum_{n=-\infty}^{\infty} (-1)^{[(n+1)/2]} \exp\left\{-\frac{1}{2}(x_1 - n\omega)^2 + in\omega x_2\right\}, \tag{4.25a}$$

where

$$\frac{1}{D^2} = \sum_{(m,r)\in\mathbb{Z}^2} (-1)^{mr} q^{m^2+r^2}. \tag{4.25b}$$

Let ψ *be given by* (4.2) *and* $\{C_n\}_{n=-N+1}^{\infty}$ *satisfy* (4.5). *Then,*

$$\|\phi - \psi\|_\infty \le C\epsilon\left[q^{l^2/64} + \epsilon^{1/2} + e^{Cl^2 - \omega N l/4}\right], \tag{4.26a}$$

where

$$\|\cdot\|_\infty = \|\cdot\|_{L^\infty\{[(N-5l/8)\omega,(N-3l/8)\omega]\times[0,2\pi/\omega]\}}. \tag{4.26b}$$

Proof: Let

$$\tilde{\psi} = \epsilon \sum_{n=-\infty}^{\infty} e_n \exp\left\{-\frac{1}{2}(x_1 - n\omega)^2 + in\omega x_2\right\}, \tag{4.27}$$

where $\{e_n\}_{n=-\infty}^{\infty}$ denotes the solution of

$$\sum_{(r,m)\in\mathbb{Z}^2} e_r \bar{e}_{m-n+r} e_m q^{(r-n)^2+(m-n)^2} - e_n = 0\,, \tag{4.28}$$

which is holomorphic in q, or

$$e_n = \sum_{k=0}^{\infty} e_{nk} q^k,$$

such that

$$e_{n0} = \begin{cases} (-1)^{[(n+1)/2]} & -l \le n \le 0 \\ \\ 0 & \text{otherwise} \end{cases}. \tag{4.29}$$

We first prove that there exists $A > 0$ which is independent of k, l, ω, q, and N such that

$$|e_{nk} - a_{nk}| \le A^k e^{-\omega N l/4}\,, \tag{4.30}$$

for all $0 \le k \le l^2/16$ and $-3l/4 \le n \le -l/4$. This can be done by using the recurrence relation obtained by substituting (4.8) into (4.5):

$$\nu_n a_{nk} = \sum_{r^2+s^2\le k} \sum_{\substack{m,j\ge 0\\ m+j\le M}} a_{(n+r)j} a_{(n+r+s)(M-m-j)} a_{(n+s)m}\,, \tag{4.31}$$

where $M(r,s) = k - r^2 - s^2$. Note that the C_n's were assumed all real in view of (4.12). By (4.28), e_{nk} satisfy the same recurrence relation with $\nu_n = 1$. We can thus proceed by induction: We assume (4.30) for $0 \le k \le K-1$. Substituting in (4.31) we obtain

$$\nu_n a_{nK} = \sum_{r^2+s^2\le K} \sum_{\substack{m,j\ge 0\\ m+j\le M(K)}} e_{(n+r)j} e_{(n+r+s)(M-m-j)} e_{(n+s)m} + \Delta_{nK}\,,$$

where

$$\Delta_{nK} \le \sum_{j=1}^{K} 3A^{K-j} j^2 e^{-\omega N l/4}.$$

Any $A > 0$ satisfying

$$\sum_{n=1}^{\infty} A^{-n} n^2 \leq \frac{1}{3}$$

would then satisfy (4.30).

Using (4.27) and (4.2) we now have

$$\|\psi - \tilde{\psi}\|_\infty \leq \|f_0\|_\infty + \epsilon \sum_{n=-\infty}^{\infty} |C_n - e_n| \left\| e^{-\frac{1}{2}(x_1 - n\omega)^2} \right\|_\infty .$$

For the first term we have, since $l \ll N$,

$$\|f_0\|_\infty \leq C \exp\left\{-\frac{1}{2}\left(N - \frac{5}{8}l\right)^2 \omega^2\right\} \leq C \exp\left\{-\frac{3}{8}N^2\omega^2\right\} \leq C\epsilon^{3/2}.$$

For the second term we have the bound

$$\epsilon \sum_{n=-\infty}^{\infty} |C_n - e_n| \left\| e^{-\frac{1}{2}(x_1 - n\omega)^2} \right\|_\infty \leq$$

$$\leq \epsilon \sum_{n=-3l/4}^{-l/4} |C_n - e_n| + Ce^{-l^2\omega^2/128} \leq$$

$$\leq \epsilon \frac{l}{2} \sum_{k=0}^{l^2/64} (Aq)^k e^{-\omega N l/4} + Cq^{l^2/64} ,$$

from which we obtain

$$\|\psi - \tilde{\psi}\|_\infty \leq C\epsilon \left(q^{l^2/64} + \epsilon^{1/2} + e^{Cl^2 - \omega N l/4}\right). \tag{4.32}$$

To complete the proof we need to obtain a similar estimate for $\|\phi - \tilde{\psi}\|_\infty$. By (4.25) we can write

$$\phi = \epsilon \sum_{n=-\infty}^{\infty} g_n \exp\left\{-\frac{1}{2}(x_1 - n\omega)^2 + in\omega x_2\right\},$$

where $\{g_n\}_{n=-\infty}^{\infty}$ satisfies (4.28) and

$$g_n = \sum_{k=0}^{\infty} g_{nk} q^k ,$$

but in contrast with $\{e_n\}_{n=-\infty}^{\infty}$

$$g_{n0} = (-1)^{[(n+1)/2]} .$$

Substituting the above into (4.28) yields

$$g_{nk} = e_{nk}$$

for all $0 \le k \le l^2/16$ and $-3l/4 \le n \le -l/4$. The proof of the lemma then easily follows.

$\square$

5 Concluding remarks

In § 2 we prove the exact asymptotic behaviour of the critical values $\{\lambda_n\}_{n=1}^{\infty}$ of (2.2) and their corresponding modes. We proved that

$$\lambda_n \sim 1 + 2a^2 e^{-(n\omega+\omega_0)^2/2} + O\left(e^{-2(n\omega+\omega_0)^2/3}\right) \quad \text{as } n \to \infty,$$

where

$$a = \lim_{x\to\infty} e^{x^2/2} f(x,1) ,$$

and that

$$\left\| \phi_n - e^{-(x_1-n\omega)^2/2} \right\|_2^2 + \|\phi_{-n}\|_2^2 \le e^{-2(n\omega+\omega_0)^2/3} .$$

Consequently, immediately after the first bifurcation we have

$$\lambda \cong \lambda_n + \epsilon^2 \left(\frac{\omega}{2\pi}\right)^{3/2},$$
$$\psi \cong e^{i\omega_0 x_2} \left\{ f(x_1, \lambda) + \epsilon e^{-(x_1 - n\omega)^2/2} e^{in\omega x_2} \right\} + O\left(e^{-(n\omega+\omega_0)^2/3}\right),$$

Where f is given in (1.4).

At the conclusion of § 2 we conjecture that $\lambda_n > 1$ for all $n \in \mathbb{N}$. This appears plausible since we expect that (1.4) would serve as the global minimizer of (1.8) in $\mathcal{P}_L^{\omega_0}$ when $\lambda \leq 1$. In [4] it was shown that any bifurcating branch has lower energy than (1.4) independently of n and λ. Thus, no bifurcation should take place for $\lambda \leq 1$ if (1.4) is indeed the global minimizer.

In § 3 we consider the second bifurcation, while assuming that the first bifurcation takes place at $\lambda = \lambda_N$. We can explain this choice by considering (1.3a) not in $\mathbb{R}_2^+$ but in $[0, d] \times \mathbb{R}$ in the limit $d \to \infty$. In this case we have to add to (2.2) the boundary conditions

$$\phi_n'(d - \omega_0) = \phi_{-n}'(d - \omega_0) = 0.$$

As a result of introduction of the additional boundary we have

$$\lambda_n(d) \sim n^2\omega^2 \quad \text{when } n\omega \gg d. \tag{5.1}$$

However, because of continuity [11], one expects that

$$\lambda_n(d) \xrightarrow[d\to\infty]{} \lambda_n(\infty).$$

While the above convergence is clearly not uniform in n, it still implies that for sufficiently large d there is a large number of critical values $\lambda_n(d)$ which can be approximated by (2.4).

Let then,

$$\lambda_N = \min_{n\in\mathbb{N}} \lambda_n(d)\,.$$

In view of (5.1) such a minimum must exist. Furthermore, if (2.18) is correct then λ_N must be very close to 1. Since with increasing λ the first bifurcation from (1.4) must take place at $\lambda = \lambda_N$ we see that the assumption that the bifurcation takes place at $\lambda = \lambda_N$ is in accordance in principle with the situation in finite domains.

We show in § 3 that, if the second bifurcation exists and if (2.18) is correct, then the second bifurcation must take place at

$$\mu_{N-1} \cong \lambda_N + \frac{1-2q}{(1-2q)^2 - q^4}\left(\lambda_{N-1} - \lambda_N\right),$$

where $q = e^{-\omega^2/2}$, and the bifurcating mode must have the form

$$\frac{2}{\omega}\pi^{3/2} u_{(0)} \cong \frac{1-2q}{\sqrt{(1-2q)^2 + q^4}} \exp\left\{-\frac{1}{2}\left[x_1 - (N-n)\omega\right]^2 + i\omega(N-n)x_2\right\} +$$
$$+ \frac{q^2}{\sqrt{(1-2q)^2 + q^4}} \exp\left\{-\frac{1}{2}\left[x_1 - (N+n)\omega\right]^2 + i\omega(N+n)x_2\right\}.$$

By formally evaluating the next order terms in (3.5) we obtain that each term provides two additional Fourier modes to ψ. Thus, $u^{(1)}$ adds the modes V_{-2} and V_2 etc. However, the coefficients of V_{-k} have been shown to be exponentially small for all $k \geq 2$.

In § 4 we extrapolate the behaviour of ψ near the bifurcation into the region where

$$\lambda - \mu_{N-1} \sim O\left(\lambda - \lambda_N\right).$$

To this end we assume that

$$\psi \cong f + \epsilon \sum_{n=-N+1}^{\infty} C_n V_n, \tag{5.2}$$

and

$$\lambda \cong \lambda_N + \epsilon^2 \left(\frac{\omega}{2\pi}\right)^{3/2}.$$

Based upon this assumption we find that as $N \to \infty$ the C_n's must satisfy the system

$$\sum_{r,m=n-N+1}^{\infty} C_{n+r}\bar{C}_{n+r+m}C_{n+m}q^{m^2+r^2} - \nu_n^2 C_n = 0,$$

where

$$\nu_n^2 = \frac{\lambda - \lambda_{N+n}}{\lambda - \lambda_N}.$$

This system of polynomial equations is very similar to the one obtained by Abrikosov [1] in the absence of boundaries where

$$\sum_{r,m=-\infty}^{\infty} C_{n+r}\bar{C}_{n+r+m}C_{n+m}q^{m^2+r^2} - \nu C_n = 0,$$

in which ν is proportional to $\lambda - 1$.

To investigate the solution of the above system with increasing λ we first match (5.2) with the solution obtained in § 3. We obtain that near $\lambda = \mu_{N-1}$ as $q \to 0$ with fixed ν_{-1} we have

$$C_0 \sim 1 \quad C_{-1} \sim \sqrt{\nu_{-1}}\,.$$

By following the leading order of C_{-2} as $q \to 0$ (with fixed ν_{-2}) with increasing λ, we obtain that when $\nu_{-2} \sim 1$

$$C_{-2} \sim -C_0^2 C_{-1} = -1.$$

Since the same procedure can be applied again to derive the behaviour of C_{-n} when $\nu_{-n} \sim 1$, we obtain that

$$C_n \sim \begin{cases} (-1)^{[(n+1)/2]} & -l \leq n \leq 0 \\ & \\ 0 & \text{otherwise} \end{cases} + O(q).$$

Finally, we show that if C_n is holomorphic in q for all n, then the above asymptotic relation implies that (5.2) is closed to the triangular lattice, which is given by

$$\phi = \epsilon D(q) \sum_{n=-\infty}^{\infty} (-1)^{[(n+1)/2]} \exp\left\{ -\frac{1}{2}(x_1 - n\omega)^2 + in\omega x_2 \right\},$$

where

$$\frac{1}{D^2} = \sum_{(m,r)\in\mathbb{Z}^2} (-1)^{mr} q^{m^2+r^2}.$$

We prove that

$$\|\phi - \psi\|_\infty \leq C\epsilon \left[q^{l^2/64} + \epsilon^{1/2} + e^{Cl^2 - \omega N l/4} \right],$$

where $1 \ll l \ll N$ and

$$\| \cdot \|_\infty = \| \cdot \|_{L^\infty\{[(N-5l/8)\omega,(N-3l/8)\omega]\times[0,2\pi/\omega]\}}.$$

We conclude this section by listing the main gaps which need to be addressed in order to establish a rigorous proof of the main result of this work

1. Proof of (2.18).

2. Existence proof of the second bifurcation.

3. Proving that (5.2) is indeed a continuation of (3.1).

4. Proving that C_n must be holomorphic in q when $\nu_n \sim 1$ for $n \geq -l$ and $\nu_n \sim -O(e^{\omega N})$ for $n \leq -l-1$ (here l is any integer smaller than $N/2$).

5. Proof that either no other bifurcation exists after the second one or, if another bifurcation does exist, then (5.2) has lower energy than the supposed bifurcating branch.

A Proof of lemma 4.1

We prove here an equivalent statement to (4.12), i.e. that

$$C_{n-1}\bar{C}_n^2 C_{n+1} \in \mathbb{R}. \tag{A.1}$$

We prove (A.1) by invoking inductive arguments. We first prove that

$$\Im\left(C_{n-1}\bar{C}_n^2 C_{n+1}\right) \sim O(q)\ \forall n, \tag{A.2}$$

and then that

$$\Im\left(C_{n-1}\bar{C}_n^2 C_{n+1}\right) \sim O(q^k) \Rightarrow \Im\left(C_{n-1}\bar{C}_n^2 C_{n+1}\right) \sim O(q^{k+1})\ \forall n. \tag{A.3}$$

Substituting (4.8) into (4.5) we obtain from the $O(q^2)$ balance that [2]

$$\Im\left\{\bar{a}_{(n+2)0}a_{(n+1)0}^2\bar{a}_{n0} + 2a_{(n+1)0}(\bar{a}_{n0})^2 a_{(n-1)0} + \bar{a}_{(n-2)0}a_{(n-1)0}^2\bar{a}_{n0}\right\} = 0\ \forall n\,.$$

From this we easily conclude that

$$\Im\left\{a_{(n+1)0}(\bar{a}_{n0})^2 a_{(n-1)0}\right\} = 0\ \forall n,$$

which is exactly (A.2).

To prove (A.3) we assume by induction that

$$\Im\left(C_{n-1}\bar{C}_n^2 C_{n+1}\right) \sim O(q^k).$$

Equivalently we may assume the existence of θ_0, θ_1, holomorphic in q, such that

$$\Im\left\{C_n e^{i[\theta_0(q)+n\theta_1(q)]}\right\} \sim O\left(q^k\right) .$$

Let

$$C_n' = e^{i[\theta_0(q)+n\theta_1(q)]}C_n.$$

Then,

$$c_n' = \sum_{j=0}^{\infty} a_{nj}' q^j ,$$

where

$$a_{nj}' \in \mathbb{R} \ \forall 0 \leq j \leq k-1.$$

Let further

$$a_{nk}' = \left[a_n' + b_n^k\right] a_{n0}' \qquad -l \leq n \leq 0.$$

It is easy to show that when $a_{n0}' = 0$, then $a_{nk}' \in \mathbb{R}$. Thus, it remains necessary to show that $b_n^k = 0$ for all $-l \leq n \leq 0$. The recurrence relation (4.31) for complex a_{nj} becomes

$$\nu_n a_{nj}' = \sum_{r^2+s^2\leq j} \sum_{\substack{m,j\geq 0 \\ m+j\leq M}} a_{(n+r)j}' \bar{a}'_{(n+r+s)(M-m-j)} a_{(n+s)m}' \quad , \tag{A.4}$$

where $M(r,s) = j - r^2 - s^2$. It is easy to show that (A.4) is satisfied for $j = k$ and $j = k+1$ independently of the values of the b_n^k's. For $j = k+2$ we obtain after a tedious calculation that (A.4) is solvable if and only if

$$Pb = 0 \tag{A.5a}$$

where b is the vector

$$b = \begin{bmatrix} b_{0k} \\ \vdots \\ b_{-l,k} \end{bmatrix}, \tag{A.5b}$$

and P is the matrix

$$P = \begin{bmatrix} -p_{-l+1} & 2p_{-l+1} & -p_{-l+1} & 0 & \dots & 0 \\ 2p_{-l+1} & -4p_{-l+1} - p_{-l+2} & 2(p_{-l+1} + p_{-l+2}) & -p_{-l+2} & 0 & \dots \\ -p_{-l+1} & 2(p_{-l+1} + p_{-l+2}) & -p_{-l+1} - 4p_{-l+2} - p_{-l+3} & 2(p_{-l+2} + p_{-l+3}) & -p_{-l+3} & \dots \\ \vdots & \vdots & \dots & \dots & \dots & \vdots \\ 0 & \dots & 0 & -p_{-1} & 2p_{-1} & -p_{-1} \end{bmatrix}, \tag{A.5c}$$

in which,

$$p_n = a'_{n-1,0}\,(a'_{n0})^2\,a'_{n+1,0}. \tag{A.5d}$$

The matrix P can conveniently be decomposed into the product

$$P = P^{(1)}P^{(2)}\,, \tag{A.6a}$$

where

$$P^{(1)} = \begin{bmatrix} 0 & -p_{-l+1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2p_{-l+1} & -p_{-l+2} & 0 & \cdots & 0 & 0 \\ 0 & -p_{-l+1} & 2p_{-l+2} & -p_{-l+3} & \cdots & 0 & 0 \\ \vdots & & & & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -p_{-1} & 0 \end{bmatrix}, \tag{A.6b}$$

and

$$P^{(2)} = \begin{bmatrix} -2 & 1 & 0 & & \cdots & 0 & 1 \\ 1 & -2 & 1 & 0 & \cdots & & 0 \\ \vdots & & & & & & \vdots \\ 0 & \cdots & & 0 & 1 & -2 & 1 \end{bmatrix}. \tag{A.6c}$$

The matrix $P^{(2)}$ is circulant. Its kernel is spanned by $[1, \ldots, 1]^T$. For $P^{(1)}$ we have

$$\ker P^{(1)} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}; \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \right\}$$

Obviously $[1, \ldots, 1]^T \in \ker P$. Furthermore, any vector $b \in \mathbb{R}^{l+1}$ for which $P^{(2)}b \in \ker P^{(1)}$ belongs to $\ker P$ as well. Consequently, $r(P) = 2$ and

$$\ker P = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}; \begin{bmatrix} 0 \\ 1 \\ \vdots \\ l \end{bmatrix} \right\}$$

and hence

$$b_n^k = b^k + n\tilde{b}^k,$$

which proves (A.3). □

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