

# The stability of the normal state of superconductors in the presence of electric currents

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## Abstract

The stability of the normal state of superconductors in the presence of electric currents is studied in the large domain limit. The model being used is the time-dependent Ginzburg-Landau model, in the absence of an applied magnetic field, and with the effect of the induced magnetic field being neglected. We find that if the current is nowhere perpendicular to the boundary, or if the minimal current on the boundary, at points where it is perpendicular to it, is greater than the critical current in the one-dimensional case, then the normal state is stable. We also prove some short-time instability when the current is both perpendicular to the boundary and smaller than the one-dimensional critical current.

## 1 Introduction

It is well known that when a superconductor is placed in a temperature lower than the critical one, it loses its electrical resistivity. This means that current can flow through a superconducting sample with a vanishingly small voltage drop. If one raises the current above a certain critical level, superconductivity will be destroyed and the material would revert to the normal state, even if the temperature is kept fixed below the critical one.

The reverse experiment can also be considered. One can flow a strong current through the sample which would set it in the normal state. Then, if we lower the current, there is a critical current where the sample would abruptly become purely superconducting. Though the two experiments substantially differ from each other from a theoretical point of view, hysteresis

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was not experimentally observed in the current-voltage characteristics of the sample [8, 12, 13].

We consider here the second experiment. To this end we must analyze the stability of the normal state. The model we use in this work is the time-dependent Ginzburg-Landau model [7, 4], presented here in a dimensionless form

$$\frac{\partial \psi}{\partial t} + i\phi\psi = (\nabla - iA)^2 \psi + \psi(1 - |\psi|^2) \quad \text{in } \Omega \quad (1.1a)$$

$$-\kappa^2 \nabla \times (\nabla \times A) - \sigma \left( \frac{\partial A}{\partial t} + \nabla \phi \right) = \bar{\psi} \nabla \psi - \psi \nabla \bar{\psi} + |\psi|^2 A \quad \text{in } \Omega \quad (1.1b)$$

$$\psi = 0 \quad \text{on } \partial\Omega_c \quad (1.1c)$$

$$(i\nabla + A)\psi \cdot \nu = 0 \quad \text{on } \partial\Omega_i. \quad (1.1d)$$

In the above  $\psi$  is the superconducting order parameter, so that  $|\psi|$  represents the number density of superconducting electrons. Superconductors for which  $|\psi| = 1$  are said to be wholly superconducting, and those for which  $\psi = 0$  are said to be at the normal state.  $A$  is the magnetic vector potential and  $\phi$  is the electric scalar potential. The constant  $\sigma$  is a measure of the normal conductivity of the superconducting material so that  $-\sigma \nabla \phi$  is the normal current, and  $\kappa$  is the Ginzburg-Landau parameter. Length has been scaled with the coherence length  $\xi$ , which is the natural length-scale for variations in  $\psi$ . The domain  $\Omega \subset \subset \mathbb{R}^n$  ( $n = 1, 2, 3$ ), where the superconducting sample resides, is smooth, has interface, denoted by  $\partial\Omega_c$ , with a conducting metal which is at the normal state. The rest of the boundary, denoted by  $\partial\Omega_i$  is adjacent to an insulator. We allow non-smoothness of  $\partial\Omega$  in the sense that  $\partial\Omega_c$  and  $\partial\Omega_i$  are required to be perpendicular to each other in order to include cylindrical-like domains. Figure 1 presents a typical two dimensional sample, where the current flows into the sample from one part of  $\partial\Omega_c$ , and exits from another part, disconnected from the first one. Most wires would fall into the above class of domains.

Equations (1.1) are gauge invariant in the sense that they are invariant under transformations of the form

$$A \rightarrow A + \nabla \omega, \quad \phi \rightarrow \phi - \frac{\partial \omega}{\partial t}, \quad \psi \rightarrow \psi e^{i\kappa \omega}.$$

Note that none of the important physical properties:  $|\psi|$ , the magnetic field  $H = \nabla \times A$ , and the electric field  $E = -\partial A / \partial t - \nabla \phi$ , is altered by the above transformation.

To obtain a well-posed problem one must add to (1.1) initial conditions, and the equations satisfied by  $A$  and  $\phi$  outside  $\Omega$ , that is the Maxwell equations. Continuity of the tangential components of  $A$  and  $\nabla \times A$  through  $\partial\Omega$

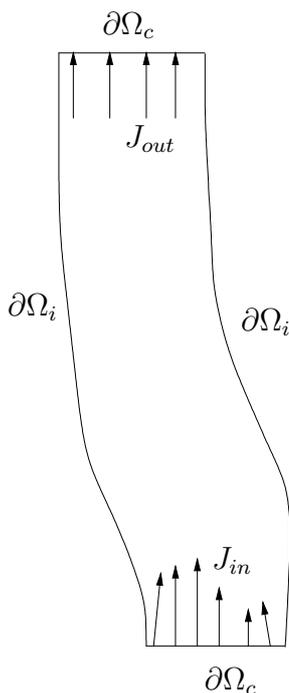


Figure 1: Typical superconducting sample. The arrows denote the direction of the current flow ( $J_{in}$  for the inlet, and  $J_{out}$  for the outlet).

and some conditions on  $\nabla \times A$  at infinity should be required as well. Since these details are irrelevant in the context of the present contribution we omit them. Interested readers may be able to find them in [4, 6].

We consider here the stability of the normal state. If  $\psi \equiv 0$ , we obtain that the steady state solution must satisfy

$$\nabla \times H = -\frac{\sigma}{\kappa^2} \nabla \phi,$$

and hence  $\phi$  is harmonic in  $\Omega$ . To obtain  $\phi$  we thus need to solve the following problem

$$\begin{cases} \Delta \phi = 0 & \text{in } \Omega \\ \phi = \phi_0(x) & \text{on } \partial\Omega_c . \\ \frac{\partial \phi}{\partial \nu} = 0 & \text{on } \partial\Omega_i \end{cases} \quad (1.2)$$

We note that instead of prescribing the potential on  $\partial\Omega_c$  we can prescribe the current in the normal direction to the boundary  $J_n(x) = -\sigma \partial \phi / \partial \nu$ . For simplicity we assume that  $\nabla \phi \neq 0$  everywhere in  $\Omega$ . This can easily be achieved: for instance, for the samples described in fig. 1, if  $\inf \phi$  on one connected component of  $\partial\Omega_c$  is greater than the  $\sup \phi$  on the other connected component  $\nabla \phi$  never vanishes.

Once (1.2) is solved, one can solve for the magnetic field. Here we need to solve a problem in  $\mathbb{R}^n$

$$\begin{cases} \nabla \times H = -\frac{\sigma}{\kappa^2} \nabla \phi = \frac{1}{\kappa^2} J & \text{in } \Omega \\ \nabla \times H = 0 & \text{in } \mathbb{R}^n \setminus \Omega \\ H \rightarrow h_{ex} & \text{as } |x| \rightarrow \infty \end{cases}, \quad (1.3)$$

with  $H$  continuous across  $\partial\Omega$ . We assume zero applied magnetic field ( $h_{ex} = 0$ ). To simplify our problem further we shall assume  $A = H = 0$ . This assumption can be justified in the case where  $\kappa \gg 1$  in view of (1.3). However, since we intend to consider large domains, one must assume that  $\kappa \gg \text{diam } \Omega$ . In real world coordinates this means that our domain size must be much larger than the coherence length  $\xi$  but also much smaller than the penetration depth  $\lambda$  which is the length scale characterizing variations in  $H$  ( $\kappa = \lambda/\xi$ ). While this assumption significantly limits the validity of our results, it has been made very often by physicists [13, 8] and it is reasonable to adopt it as a start point.

Once the above assumption is adopted one obtains

$$\begin{cases} \frac{\partial \psi}{\partial t} - \Delta \psi + i\phi\psi - \psi(1 - |\psi|^2) = 0 & \text{in } \Omega \\ \psi = 0 & \text{on } \partial\Omega_c \\ \frac{\partial \psi}{\partial \nu} = 0 & \text{on } \partial\Omega_i \\ \psi(x, 0) = \psi_0(x) & \text{in } \Omega \end{cases}.$$

It should be noted that in [6] Du and Gray proves, within the framework of a more general case, convergence in the limit  $\kappa \rightarrow \infty$  of (1.1) to a different limit problem where the magnetic field is not negligible. The domain size considered there is, however much larger than in our case, as it is comparable with the penetration depth.

Linearizing the above near the normal state we obtain

$$\begin{cases} \frac{\partial \psi}{\partial t} - \Delta \psi + i\phi\psi - \psi = 0 & \text{in } \Omega \\ \psi = 0 & \text{on } \partial\Omega_c \\ \frac{\partial \psi}{\partial \nu} = 0 & \text{on } \partial\Omega_i \\ \psi(x, 0) = \psi_0(x) & \text{in } \Omega \end{cases}. \quad (1.4)$$

The above problem has been analyzed by physicists in one-dimensional settings. Ivlev and Kopnin review these results in [8]. In these settings we have  $\phi = Jx + \mu$  where the current  $J$  and  $\mu$  are constants. Previous results include the closed form solution of (1.4) in  $\mathbb{R}$  for any initial condition. In  $\mathbb{R}_+$ , the first critical current  $J_c$  for which a steady state solution ( $\partial|\psi|/\partial t = 0$ )

exists is found. Then weakly non-linear analysis is performed, where it is shown that [9] the bifurcation taking place near  $J = J_c$  is sub-critical (i.e., unstable).

We consider (1.4) in three dimensional settings. While all the results are stated for three dimensional objects they are equally valid for two dimensional objects as well. We deal with (1.4) in the large domain limit, i.e., we consider a domain  $\Omega_R$  which is obtained from  $\Omega$  via the transformation

$$x \rightarrow Rx. \quad (1.5)$$

The portions of the boundaries  $\partial\Omega_R^c$  and  $\partial\Omega_R^i$  are similarly obtained from  $\partial\Omega_c$  and  $\partial\Omega_i$  respectively. To keep  $\nabla\phi$  unaltered we consider also potentials of the form

$$\phi_R = R\phi(x/R).$$

Thus, we consider the following problem

$$\begin{cases} \frac{\partial\psi_R}{\partial t} - \Delta\psi_R + i\phi_R\psi_R - \psi_R = 0 & \text{in } \Omega_R \\ \psi_R = 0 & \text{on } \partial\Omega_R^c \\ \frac{\partial\psi_R}{\partial\nu} = 0 & \text{on } \partial\Omega_R^i \\ \psi_R(x, 0) = \psi_0(x) & \text{in } \Omega_R \end{cases}. \quad (1.6)$$

Our main result is the following

**Theorem 1.1.** *Let  $\phi$  satisfy (1.2). Let  $\partial\Omega_c^n$  denote the portion of  $\partial\Omega_c$  on which  $\nabla\phi$  is perpendicular to the boundary. Let further*

$$J_m = \min_{x \in \partial\Omega_c^n} \left| \frac{\partial\phi}{\partial\nu} \right|.$$

*and let  $J_c$  denote the critical current for the problem in  $\mathbb{R}_+$  (which is precisely defined in (2.19)). Suppose further that  $|J| > 0$  everywhere on  $\partial\Omega_c$ . Then, if  $J_m > J_c$ , or if  $\partial\Omega_c^n$  is empty, there exists  $R_0 > 0$  such that  $\psi_R \equiv 0$  is a stable solution of (1.6) in  $L^2(\Omega_R)$  sense for all  $R > R_0$ .*

*Furthermore, if  $J_m < J_c$  there exists  $\psi_0 \in L^2(\Omega_R)$  and  $T_R > 0$  such that*

$$\liminf_{R \rightarrow \infty} \frac{T_R}{\ln R} > 0,$$

*and such that the solution  $\psi_R$  of (1.6) has the following property:*

$$t < T_R \Rightarrow \|\psi_R\|_{L^2(\Omega_R)} > \frac{1}{2} \|\psi_0\|_{L^2(\Omega_R)} e^{\beta t}, \quad (1.7)$$

*where*

$$\beta = 1 - (J/J_c)^{2/3}.$$

In the next section we review and enhance the results in [8] in the one-dimensional case. In §3 we extend some of the results in §2 to unbounded three-dimensional domains. In §4 we provide the proof of the theorem. Finally, in the last section we highlight some possible directions of future research.

## 2 One-dimensional problems

In this section we consider (1.4) in two different one-dimensional settings: on  $\mathbb{R}$  and on  $\mathbb{R}_+$ . The solution of these simple problems would provide us with some important intuition on the solution of (1.6) in three-dimensional bounded domains in the large domain limit. In both cases we shall assume  $\phi = Jx$ , i.e., that the current is uniform and equals  $J$  throughout the sample.

### 2.1 Infinite 1D domain

Here we consider the problem

$$\frac{\partial \psi}{\partial t} - \psi'' - \psi + iJx\psi = 0. \quad (2.1)$$

We consider here the only case  $J > 0$ . Otherwise, if  $J < 0$  we can consider the complex conjugate of (2.1). Applying the coordinate transformation

$$x \rightarrow J^{1/3}x \quad ; \quad t \rightarrow J^{2/3}t \quad (2.2)$$

we obtain the problem

$$\frac{\partial \psi}{\partial t} + \mathcal{L}\psi = \lambda_J \psi,$$

where

$$\mathcal{L}\psi = -\psi'' + ix\psi, \quad (2.3)$$

and  $\lambda_J = J^{-2/3}$ .

We first focus interest on the spectrum of the operator  $\mathcal{L} : \mathcal{D}_{\mathbb{R}}(\mathcal{L}) \rightarrow L^2(\mathbb{R}, \mathbb{C})$ , where  $\mathcal{D}_{\mathbb{R}}(\mathcal{L})$  is the dense subset of  $L^2(\mathbb{R}, \mathbb{C})$  defined as

$$\mathcal{D}_{\mathbb{R}}(\mathcal{L}) = \{u \in L^2(\mathbb{R}, \mathbb{C}) \mid -u'' + iXu \in L^2(\mathbb{R}, \mathbb{C})\}.$$

**Lemma 2.1.** *The operator  $\mathcal{L} - \lambda I$  is invertible for all  $\lambda \in \mathbb{C}$ . (Thus  $\sigma(\mathcal{L}) = \emptyset$ .)*

*Proof:* It is sufficient to consider here  $\lambda \in \mathbb{R}$ . Otherwise, if  $\lambda = \lambda_r + i\lambda_i$  we apply the transformation  $x \rightarrow x - \lambda_r$ .

Though it is not necessary, we first prove injectivity of  $\mathcal{L} - \lambda I$ . We shall later make use of a similar argument in three dimensions. To the problem

$$\mathcal{L}u = \lambda u$$

we apply the Fourier transform

$$\hat{u}(\omega) = \mathcal{F}(u) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\omega x} u(x) dx, \quad (2.4)$$

to obtain

$$\omega^2 \hat{u} - \frac{\partial \hat{u}}{\partial \omega} = \lambda \hat{u}.$$

Since the above equation doesn't possess any non-trivial solutions with bounded  $L^2(\mathbb{R})$  norm  $\mathcal{L} - \lambda I$  must be injective.

To find the spectrum of  $\mathcal{L}$  we construct the Green's function of  $\mathcal{L} - \lambda I$ . Let

$$\begin{cases} w_1(x, \lambda) = A_i(e^{i\pi/6}x + \lambda e^{i2\pi/3}) \\ w_2(x, \lambda) = A_i(-e^{-i\pi/6}x + \lambda e^{-i2\pi/3}) \end{cases} \quad (2.5)$$

where  $A_i$  denotes Airy's function [1] (cf. also equation (A.2) in the appendix for the asymptotic behaviour of  $A_i$ ). It is easy to show that  $w_1, w_2$  constitute a fundamental set of solutions to  $(\mathcal{L} - \lambda I)u = 0$  [11]. We can now write the Green's function in the form

$$G(x, \xi, \lambda) = \begin{cases} \frac{w_2(\xi, \lambda)}{W(w_1, w_2)} w_1(x, \lambda) & x > \xi \\ \frac{w_1(\xi, \lambda)}{W(w_1, w_2)} w_2(x, \lambda) & x < \xi \end{cases}.$$

where  $W(w_1, w_2)$  denote the Wronskian (which is a constant that clearly doesn't vanish since  $w_1$  and  $w_2$  are linearly independent).

Using the asymptotic behaviour of Airy's functions [1], we can show, using the same procedure applied in Appendix A to the semi-infinite case, that  $G(\cdot, \lambda) \in L^2(\mathbb{R}^2, \mathbb{C})$  for all  $\lambda \in \mathbb{R}$ . The lemma is proved.

□

The spectrum of the operator  $\mathcal{L} : \mathcal{D}_{\mathbb{R}}(\mathcal{L}) \rightarrow L^2(\mathbb{R}, \mathbb{C})$  can teach us therefore very little on the stability of the normal state in this case. The following lemma proves its stability in a direct manner.

**Lemma 2.2.** *The trivial solution of (2.1),  $\psi \equiv 0$  is globally stable in  $L^2$ , i.e., if  $f(x) \in L^2(\mathbb{R}, \mathbb{C})$ , then the solution of (2.1)  $\psi(x, t)$  satisfying the initial condition  $\psi(x, 0) = f(x)$  satisfies  $\psi(x, t) \xrightarrow[t \rightarrow \infty]{} 0$  in  $L^2(\mathbb{R}, \mathbb{C})$ .*

*Proof:* As long as  $\psi(x, t) \in L^2(\mathbb{R}, \mathbb{C})$  we can apply to (2.1) the Fourier transform (2.4). We obtain

$$\frac{\partial \hat{\psi}}{\partial t} + \omega^2 \hat{\psi} - \frac{\partial \hat{\psi}}{\partial \omega} - \lambda_J \hat{\psi} = 0.$$

The unique solution to the above problem is given by

$$\hat{\psi}(\omega, t) = \hat{f}(\omega) \exp \left\{ -\omega^2 t - \omega t^2 - \frac{1}{3} t^3 + \lambda_J t \right\}, \quad (2.6)$$

in which  $\hat{f}(\omega)$  denotes the Fourier transform of  $f$ . Integrating over  $\mathbb{R}$  with respect to  $\omega$  we obtain

$$\|\psi(\cdot, t)\|_2 \leq C \|f\|_2 \exp \left\{ -\frac{1}{12} t^3 + \lambda_J t \right\},$$

where  $\|\cdot\|_2$  denotes the  $L^2(\mathbb{R}, \mathbb{C})$  norm.

□

The above superlinear convergence is in accordance with the result proved in the previous lemma by which the spectrum of  $\mathcal{L}$  is empty. We note that in [8] the inverse transform of (2.6) is obtained, but no decay proof is given.

## 2.2 Semi-infinite 1D domain

We now consider (2.1) on  $\mathbb{R}_+$ . We concentrate here on the Dirichlet boundary condition  $\psi(0) = 0$ , however, the same analysis apply to Neumann and mixed boundary conditions as well.

We start by proving the following result on the spectrum of the operator  $\mathcal{L} : \mathcal{D}_{\mathbb{R}_+}(\mathcal{L}) \rightarrow L^2(\mathbb{R}, \mathbb{C})$ , where

$$\mathcal{D}_{\mathbb{R}_+}(\mathcal{L}) = \{u \in L^2(\mathbb{R}_+, \mathbb{C}) \mid -u'' + i x u \in L^2(\mathbb{R}_+, \mathbb{C}), u \in H_0^1(\mathbb{R}_+, \mathbb{C})\}.$$

**Lemma 2.3.** *1. There exists a sequence of eigenvalues  $\{\lambda_n\}_{n=1}^\infty$  and eigenfunctions, with unity norm,  $\{u_n\}_{n=1}^\infty \subset \mathcal{D}_{\mathbb{R}_+}(\mathcal{L})$  of  $\mathcal{L}$ , i.e.,*

$$\mathcal{L}u_n = \lambda_n u_n.$$

*2.*

$$m = \max_{n \in \mathbb{N}} \Re \lambda_n > 0 \quad (2.7)$$

*3.  $\text{span}\{u_n\}_{n=1}^\infty = L^2(\mathbb{R}_+, \mathbb{C})$ .*

4. Suppose that  $u, v \in L^2(\mathbb{R}_+, \mathbb{C})$  can be represented in the form

$$v = \sum_{n=1}^{\infty} \alpha_n u_n \quad ; \quad w = \sum_{n=1}^{\infty} \beta_n u_n$$

where the convergence is in  $L^2$  sense. Let then

$$\langle v, w \rangle_U = \sum_{n=1}^{\infty} \alpha_n \bar{\beta}_n. \quad (2.8)$$

Then,

$$\inf_{v \in \mathcal{D}_{\mathbb{R}_+}(\mathcal{L})} \Re \langle v, \mathcal{L}v \rangle_U \geq m \|v\|_U^2, \quad (2.9)$$

where  $\|\cdot\|_U$  denotes the norm induced by (2.8).

*Proof:* Let

$$z = -ix + \lambda$$

Let  $u(x, \lambda) \in \mathcal{D}_{\mathbb{R}_+}(\mathcal{L})$  denote an eigenfunction of  $\mathcal{L}$ , i.e.,  $\mathcal{L}u = \lambda u$ . Let  $v(z, \lambda) = u(x, \lambda)$ . We have

$$\begin{cases} \frac{\partial^2 v}{\partial z^2} - zv = 0 & z \in \mathbb{C} \\ v(\lambda, \lambda) = 0 \end{cases}. \quad (2.10)$$

Since  $u \in L^2(\mathbb{R}_+, \mathbb{C})$ ,  $v$  must be subdominant (i.e., it decays exponentially fast [11]) in the sector

$$S_1 : -\pi < \arg z < -\frac{\pi}{3}.$$

The decaying solution of (2.10) in  $S_1$  is given by (cf. [11])

$$v = A_i(e^{2\pi i/3} z).$$

Since the zeroes of Airy's functions are eigenvalues of the self-adjoint operator  $d^2/dx^2 - x$  in  $\mathcal{D}_{\mathbb{R}_+}(\mathcal{L})$  they must all be real. Let  $\{\mu_n\}_{n=1}^{\infty} \subset \mathbb{R}$  denote the zeroes of Airy's function on the real axis. By the maximum principle they must all be strictly negative. We arrange them so that  $\mu_n \downarrow -\infty$ . As every eigenvalue of  $\mathcal{L}$  must satisfy  $v(\lambda, \lambda) = 0$ , the set  $\{\lambda_n\}_{n=1}^{\infty}$  where

$$\lambda_n = e^{-i2\pi/3} \mu_n \quad (2.11)$$

contains all the eigenvalues of  $\mathcal{L}$ . Since  $\mu_1 < 0$  we have that

$$m = \Re \lambda_1 = -\frac{\mu_1}{2} > 0.$$

The set  $\{\tilde{u}_n\}_{n=1}^\infty$  of eigenfunctions of  $\mathcal{L}$  in  $\mathcal{D}_{\mathbb{R}_+}(\mathcal{L})$  is given by

$$\tilde{u}_n = A_i(e^{i2\pi/3}(-ix + \lambda_n)) = A_i(e^{i\pi/6}x + \mu_n) \quad \forall n \in \mathbb{N}. \quad (2.12)$$

We then set

$$u_n = \frac{\tilde{u}_n}{\|\tilde{u}_n\|_{L^2(\mathbb{R}_+)}}. \quad (2.13)$$

To prove that  $\{u_n\}_{n=1}^\infty$  is complete in  $L^2(\mathbb{R}, \mathbb{C})$  we consider the resolvent  $\mathcal{L}_\lambda^{-1} = (\mathcal{L} - \lambda I)^{-1}$  (which is also the modified resolvent of  $\mathcal{L}^{-1}$ ). We have

$$\mathcal{L}_\lambda^{-1}f = \int_0^\infty \tilde{G}(x, \xi, \lambda)f(\xi)d\xi, \quad (2.14a)$$

in which

$$\tilde{G}(x, \xi, \lambda) = \begin{cases} \frac{\tilde{w}_2(\xi, \lambda)}{W(w_1, \tilde{w}_2)}w_1(x, \lambda) & x > \xi \\ \frac{w_1(\xi, \lambda)}{W(w_1, \tilde{w}_2)}\tilde{w}_2(x, \lambda) & x < \xi \end{cases}, \quad (2.14b)$$

where

$$\tilde{w}_2(x, \lambda) = \frac{w_1(0, \lambda)}{w_2(0, \lambda)}w_1(x, \lambda) - w_2(x, \lambda), \quad (2.15)$$

and  $w_1$  and  $w_2$  are given in (2.5). In the appendix, we prove that  $\tilde{G} \in L^2(\mathbb{R}_+ \times \mathbb{R}_+)$ , and that

$$\|\tilde{G}(\cdot, \cdot, \lambda)\|_{L^2(\mathbb{R}_+ \times \mathbb{R}_+)} \leq e^{M|\lambda|^{3/2}}, \quad (2.16)$$

as long as  $\lambda \notin \{\lambda_n\}_{n=1}^\infty$ .

Let  $u \in \mathcal{D}_{\mathbb{R}_+}(\mathcal{L})$ . We now multiply  $\mathcal{L}u$  by  $e^{i\theta}\bar{u}$  and integrate over  $\mathbb{R}_+$  to obtain

$$\Re \langle e^{i\theta}\mathcal{L}u, u \rangle = \int_0^\infty (\cos \theta |u'|^2 - \sin \theta x |u|^2) dx.$$

By theorem 12.8 in [2] we have that every direction  $e^{i \arg \lambda}$  with

$$\pi/2 < \arg \lambda < 3\pi/2$$

is a direction of minimal growth of the resolvent of  $e^{i\theta}\mathcal{L}$  for every  $-\pi/2 < \theta < 0$ . Consequently, every direction  $e^{i \arg \lambda}$  with

$$\pi/2 < \arg \lambda < 2\pi$$

is a direction of minimal growth of  $\mathcal{L}_\lambda^{-1}$ , i.e. (cf. [2])

$$\|\mathcal{L}_\lambda^{-1}\| \sim \mathcal{O}(|\lambda|^{-1}) \quad \pi/2 < \arg \lambda < 2\pi. \quad (2.17)$$

We now apply the same argument used in the proof of theorem 16.4 in [2]. Let  $f \in L^2(\mathbb{R}_+)$ , and  $g \in V^\perp$ , where  $V = \text{span}\{A_i(e^{i\pi/6}x + \mu_n)\}_{n=1}^\infty = \text{sp}'(\mathcal{L}^{-1})$ . Then,

$$F(\lambda) = \langle \mathcal{L}_\lambda^{-1} f, g \rangle$$

is an entire function (cf. [2]) of  $\lambda$  satisfying by (2.16) and (2.17)

$$\begin{cases} |F(\lambda)| \leq \frac{C}{|\lambda|} & \pi/2 < \arg \lambda < 2\pi \\ |F(\lambda)| \leq C e^{M|\lambda|^{3/2}} & \lambda \in \mathbb{C} \end{cases}.$$

By theorem 16.1 in [2] (or the Phragmen-Leindelöf theorem) and Liouville's theorem we must have  $F(\lambda) \equiv 0$ . Hence,  $\overline{V} = \overline{\text{range}(\mathcal{L}^{-1})} = \overline{\mathcal{D}_{\mathbb{R}_+}(\mathcal{L})}$ .

It remains still necessary to prove (2.9). Let  $w \in \mathcal{D}_{\mathbb{R}_+}(\mathcal{L})$ . Then,

$$w = \sum_{n=1}^{\infty} \alpha_n u_n$$

and

$$\mathcal{L}w = \sum_{n=1}^{\infty} \alpha_n \lambda_n u_n.$$

Hence

$$\Re \langle w, \mathcal{L}w \rangle_U = \sum_{n=1}^{\infty} \Re \lambda_n |\alpha_n|^2 \geq m \|w\|_U^2.$$

□

We note that the set  $\{u_n\}_{n=1}^\infty$  is not a basis in the usual sense in Banach spaces. In fact, it has been demonstrated in [5] that the system  $\{\tilde{u}_n, \overline{\tilde{u}_n}\}_{n=1}^\infty$ , which is a bi-orthogonal system after we appropriately normalize it, is wild. This means that  $\|u_n\|_2$  grows faster than any algebraic rate as  $n \rightarrow \infty$ .

We now prove the existence of a critical current  $J_c$  obtained in [8, 9].

**Lemma 2.4.** *Let  $\psi(x, t) \in H_0^2(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{C})$  denote a solution of the equation*

$$\frac{\partial \psi}{\partial t} - \frac{\partial^2 \psi}{\partial x^2} - \psi + iJx\psi = 0 \quad \text{in } \mathbb{R}_+ \times \mathbb{R}_+. \quad (2.18)$$

If

$$J > J_c = \left(-\frac{\mu_1}{2}\right)^{-3/2}, \quad (2.19)$$

in which  $\mu_1$  is the rightest zero of Airy's function, then  $\|\psi(\cdot, t)\|_U \xrightarrow{t \rightarrow \infty} 0$ . Otherwise, if  $J < J_c$ , then  $\psi \equiv 0$  is an unstable solution of (2.18).

*Proof:* We first apply (2.2) to obtain

$$\begin{cases} \frac{\partial \psi}{\partial t} + \mathcal{L}\psi - \lambda_J \psi = 0 \\ \psi(0, t) = 0 \\ \psi(x, 0) = \psi_0(x) \end{cases},$$

where  $\|\psi_0\|_U < \infty$ . Taking the inner product (2.8) of the above equation with  $\psi$  we obtain

$$\frac{1}{2} \frac{d}{dt} \|\psi\|_U^2 \leq \left( \lambda_J + \frac{\mu_1}{2} \right) \|u\|_U^2.$$

Hence, if (2.19) is satisfied we have  $\lambda_J + \mu_1/2 < 0$  and hence  $\psi$  tends to 0 exponentially fast as  $t \rightarrow \infty$ , in the  $\|\cdot\|_U$  sense.

If  $J < J_c$  let  $\psi(x, 0) = u_1(x)$  where  $u_1$  is given in (2.13). Then

$$\psi(x, t) = u_1(x) e^{(\lambda_J + \mu_1/2)t}.$$

Returning to the original variables by applying the inverse of (2.2) we obtain

$$\psi(x, t) = u_1(J^{1/3}x) e^{[1 - (J/J_c)^{2/3}]t},$$

and hence  $\psi \equiv 0$  is unstable.  $\square$

### 3 Unbounded domains in $\mathbb{R}^3$

In this section we consider several different problems: in  $\mathbb{R}^3$ , in  $\mathbb{R}_+^3$  and in a quarter-space. In contrast with the previous section we analyze these problems only to the extent needed in the next section, which is the existence of eigenvalues with non-positive real part of the elliptic operator on the right-hand-side of (1.4).

#### 3.1 Eigenfunctions in $\mathbb{R}^3$

We consider here (1.4) with  $\phi = Jx_1$ . This choice summarizes all possible electric potentials with constant gradients as the problem is invariant to translations and rotations. Thus, we have for every eigenfunction  $u$

$$-\Delta u - u + iJx_1 u = -\lambda u. \quad (3.1)$$

We shall assume here that  $\lambda \in \mathbb{R}$ , otherwise we can apply the transformation  $x_1 \rightarrow x_1 - \Im \lambda / J$ .

It is easy to find all  $L^2$  solutions of (3.1) in  $\mathbb{R}^3$ : apply the Fourier transform (2.4) in the  $x_2$  and  $x_3$  directions (using the respective Fourier coordinates  $\omega_2$  and  $\omega_3$ ) to obtain

$$\mathcal{L}\hat{u} = ((1 - \lambda)J^{-2/3} - \omega_2^2 - \omega_3^2)\hat{u}. \quad (3.2)$$

where assuming  $J > 0$  we have applied the transformation

$$x \rightarrow J^{1/3}x.$$

(We confine the discussion in the sequel to the case  $J > 0$ , as if  $J < 0$  we can consider the complex conjugate of (3.2) to obtain a new problem with  $J > 0$ .)

By lemma 2.1 we have that  $\hat{u} \equiv 0$  is the unique  $L^2$  solution of (3.2). However, for the blow-up arguments employed in the next section we need to obtain the above result for any uniformly bounded solution of (3.1) in  $\mathbb{R}^3$ . This is exactly what the next lemma states

**Lemma 3.1.** *Let  $u$  denote a uniformly bounded solution of (3.1), in  $\mathbb{R}^3$ . Then,  $u \equiv 0$ .*

*Proof:* We first show that  $u(\cdot, x_2, x_3) \in L^2(\mathbb{R}, \mathbb{C})$ . Let  $\chi_r \in C^\infty(\mathbb{R}_+, [0, 1])$  satisfy

$$\chi_r(x) = \begin{cases} 1 & x < r/2 \\ 0 & x > r \end{cases} \quad |\chi'| \leq \frac{C}{r}. \quad (3.3)$$

Multiplying (3.1) by  $\chi_r^2(|x - x_0|)\bar{u}$  we obtain, taking the real part of identity, that

$$\int_{B(x_0, r)} |\nabla(\chi_r u)|^2 \leq \int_{B(x_0, r)} [\chi_1^2 + |\nabla\chi_1|^2] |u|^2. \quad (3.4)$$

Consequently, since  $u$  is bounded in  $L^\infty(\mathbb{R}^3)$  we have

$$\int_{B(x_0, r/2)} |\nabla u|^2 \leq C \quad \forall x_0 \in \mathbb{R}^3. \quad (3.5)$$

From the imaginary part of the identity we obtain that

$$\int_{B(x_0, r)} \left( \nabla(\chi_r^2) \cdot \Im(\bar{u}\nabla u) + Jx_1\chi_r|u|^2 \right) = 0.$$

Let  $x_0 = (x_0^1, x_0^2, x_0^3)$ . Then,

$$\int_{B(x_0, r/2)} |x_1||u|^2 dx_1 \leq C \int_{B(x_0, r)} [|u|^2 + |\nabla u|^2].$$

Consequently, for  $|x_0^1| > r$ , since  $u$  is bounded and in view of (3.5) we have

$$\int_{B(x_0, r/2)} |u|^2 \leq \frac{C}{|x_0^1| - \frac{r}{2}} \leq \frac{C}{|x_0^1|}.$$

Repeating the above steps (from (3.4) to the above inequality)  $k$  times we obtain that

$$\int_{B(x_0, r/2^k)} |u|^2 \leq \frac{C_k}{|x_0^1|^k}$$

Hence,

$$\int_{B(x_0, r/2^k)} |x_1|^2 |u|^2 \leq \frac{C_k}{|x_0^1|^{k-2}},$$

which allows us to apply standard elliptic estimates [3] to obtain that

$$|u| \leq \frac{C_k}{(|x_1| + 1)^k} \quad \forall k \in \mathbb{N}.$$

In view of the above we have that  $x_1^k u(x_1, x') \in L^2(\mathbb{R}, \mathbb{C})$  for all fixed  $x' \in \mathbb{R}^2$ . Thus, one can apply to (3.1) the Fourier transform (2.4) to obtain

$$-\Delta_{\perp} \hat{u} + (\omega^2 - 1 + \lambda) \hat{u} - J \frac{\partial \hat{u}}{\partial \omega} = 0, \quad (3.6)$$

where

$$-\Delta_{\perp} = \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}.$$

Let

$$\tilde{x}_r(x) = \begin{cases} 1 & |x| < r \\ e^{-\frac{1}{r}(|x|-r)} & |x| > r \end{cases} \quad (3.7)$$

Multiplying (3.6) by  $\tilde{\chi}_r^2(|x' - x^0|) \bar{\hat{u}}$  and integrating over  $\mathbb{R}^2$  we obtain

$$-J \frac{dU_r}{d\omega} + (\omega^2 - 1 + \lambda) U_r = \int_{\mathbb{R}^2} (-|\nabla(\tilde{\chi}_r \hat{u})|^2 + |\nabla \tilde{\chi}_r|^2 |\hat{u}|^2) dx'.$$

where

$$U_r(\omega) = \int_{\mathbb{R}^2} \tilde{\chi}_r^2 |\hat{u}|^2 dx'.$$

For the last term we have

$$\int_{\mathbb{R}^2} |\nabla \tilde{\chi}_r|^2 |\hat{u}|^2 \leq \frac{1}{r^2} U_r.$$

Consequently,

$$-J \frac{dU_r}{d\omega} + (\omega^2 - 1 + \lambda - r^{-2}) U_r \leq 0,$$

and therefore, for every  $\omega_0 \in \mathbb{R}$  and  $\omega > \omega_0$  we have

$$U_r(\omega) \geq U_r(\omega_0) \exp \left\{ \frac{1}{3}(\omega^3 - \omega_0^3) - (1 + \lambda + r^{-2})(\omega - \omega_0) \right\}$$

Thus, since  $U_r$  is positive, it must diverge exponentially fast, unless

$$U_r \equiv 0.$$

From which the lemma easily follows.

□

### 3.1.1 Eigenfunctions in $\mathbb{R}_+^3$ : perpendicular current

Let

$$\mathbb{R}_+^3 = \{(x_1, x_2, x_3) \mid x_1 > 0\}.$$

We consider here solutions of (3.1) in  $\mathbb{R}_+^3$  satisfying a Dirichlet boundary condition on  $\partial\mathbb{R}_+^3$ . Instead of considering complex eigenvalues we consider only real ones and treat their imaginary part as part of the electric potential.

**Lemma 3.2.** *Let  $u \in H^2(\mathbb{R}_+^3)$  denote a uniformly bounded solution of*

$$\begin{cases} -\Delta u - u + iJ(x_1 - \mu)u = -\lambda u & \text{in } \mathbb{R}_+^3 \\ u = 0 & \text{on } \partial\mathbb{R}_+^3, \end{cases} \quad (3.8)$$

with  $\lambda \in \mathbb{R}_+$ . Then, if  $J > J_c$ , where  $J_c$  is defined in (2.19),  $u$  must vanish identically.

*Proof:* Let  $u_n(x_1)$  be defined in (2.12). Let further

$$a_n(x_2, x_3) = \int_0^\infty u_n(x_1)u(x_1, x_2, x_3)dx_1 = \langle u_n, \bar{u} \rangle,$$

where the inner product is in the regular  $L^2$  sense. Clearly,  $a_n$  is uniformly bounded in  $\mathbb{R}^2$  as  $u \in L^\infty(\mathbb{R}^3, \mathbb{C})$  and  $u_n \in L^1(\mathbb{R}, \mathbb{C})$ .

Applying the transformation

$$x \rightarrow J^{1/3}x, \quad (3.9)$$

multiplying (3.8) by  $u_n$ , and integrating over  $\mathbb{R}_+$  with respect to  $x_1$  we obtain, in view of the boundedness of  $u$  and the exponential rate of decay of  $u_n$  as  $x_1 \rightarrow \infty$ , that

$$-\Delta_\perp a_n + (\lambda_n - \tilde{\lambda}_J - i\mu)a_n = 0, \quad (3.10)$$

where  $\tilde{\lambda}_J = (1 - \lambda)J^{-2/3}$  and the definition of  $\lambda_n$  is given in lemma 2.3. Multiplying (3.10) by  $\tilde{\chi}_r(|x'|)$ , we obtain for the real part

$$\left(\frac{|\mu_n|}{2} - \tilde{\lambda}_J\right) \int_{\mathbb{R}^2} |\tilde{\chi}_r|^2 |a_n|^2 = - \int_{\mathbb{R}^2} |\nabla(\tilde{\chi}_r a_n)|^2 + \int_{\mathbb{R}^2} |\nabla \tilde{\chi}_r|^2 |a_n|^2 \leq \frac{C}{r^2} \int_{\mathbb{R}^2} |\tilde{\chi}_r|^2 |a_n|^2.$$

Since  $\tilde{\lambda}_J < |\mu_1|/2 \leq |\mu_n|/2$  by our assumption we obtain, that for sufficiently large  $r$  we must have

$$\int_{\mathbb{R}^2} |\tilde{\chi}_r|^2 |a_n|^2 = 0.$$

Hence,  $a_n \equiv 0$  in  $\mathbb{R}^2$ . Since by lemma 2.3  $\{u_n\}_{n=1}^\infty$  is a basis for  $L^2(\mathbb{R}_+, \mathbb{C})$  we must have  $u \equiv 0$ .

□

### 3.1.2 Steady solutions in $\mathbb{R}_+^3$ : non-perpendicular current

This problem is very similar to the problem in  $\mathbb{R}^3$ . Consider the equation

$$\begin{cases} -\Delta u - u + i(J_1 x_1 + J_2 x_2 - \mu)u = -\lambda u & \text{in } \mathbb{R}_+^3, \\ u = 0 & \text{on } \partial\mathbb{R}_+^3 \end{cases}, \quad (3.11)$$

with  $J_2 \neq 0$  and  $\lambda \in \mathbb{R}_+$ . Like the problem in  $\mathbb{R}^3$ , there is no need to consider  $\mu \neq 0$  here since the transformation

$$x_2 \rightarrow x_2 + \frac{\mu}{J_2}$$

sets  $\mu = 0$  in the transformed problem. Furthermore, we also obtain the following result, which is exactly the same as the result obtained in  $\mathbb{R}^3$

**Lemma 3.3.** *Let  $u$  denote a bounded solution of (3.11) with  $J_2 \neq 0$ . Then  $u \equiv 0$ .*

*Proof:* Consider first the case where  $J_1 \neq 0$ . We first apply the transformation (3.9) with  $J = J_1$  to obtain

$$-\Delta_\perp u + \mathcal{L}u - \tilde{\lambda}_{J_1} u + i\gamma x_2 u = 0, \quad (3.12)$$

where  $\tilde{\lambda}_{J_1} = (1 - \lambda)J_1^{-2/3}$  and  $\gamma = J_2/J_1$ . Multiplying (3.12) by  $u_n$  and integrating over  $\mathbb{R}_+$  with respect to  $x_1$  we obtain,

$$-\Delta_\perp a_n + (\lambda_n - \tilde{\lambda}_{J_1})a_n + i\gamma x_2 a_n = 0 \quad \text{in } \mathbb{R}^2,$$

where  $a_n = \langle u_n, \bar{u} \rangle$ . The above equation cannot have any non-trivial bounded solution in  $\mathbb{R}^2$ , otherwise it would also be a bounded solution in  $\mathbb{R}^3$ ,

which by lemma 3.1 must identically vanish. Consequently, since  $a_n$  must be bounded, we must have  $a_n \equiv 0$  for all  $n$ , from which the lemma easily follows in this case.

Consider now the case where  $J_1 = 0$ . Here we define

$$\tilde{u}(x_1, x_2, x_3) = \begin{cases} u(x_1, x_2, x_3) & x_1 > 0 \\ -u(-x_1, x_2, x_3) & x_1 < 0 \end{cases}.$$

Clearly,  $\tilde{u}$  is a bounded weak solution of (3.1) in  $\mathbb{R}^3$  and hence by lemma 3.1  $\tilde{u} \equiv 0$ .

□

For later reference we shall also need the following lemma:

**Lemma 3.4.** *Let  $u$  denote a bounded solution of*

$$\begin{cases} -\Delta u - u + iJ_2x_2u = -\lambda u & \text{in } \mathbb{R}_+^3 \\ \frac{\partial u}{\partial x_1} = 0 & \text{on } \partial\mathbb{R}_+^3 \end{cases},$$

with  $\lambda \in \mathbb{R}_+$ . Then  $u \equiv 0$ .

*Proof:* Once again we define, this time an even function,

$$\tilde{u}(x_1, x_2, x_3) = \begin{cases} u(x_1, x_2, x_3) & x_1 > 0 \\ u(-x_1, x_2, x_3) & x_1 < 0 \end{cases}.$$

Clearly,  $\tilde{u}$  is a bounded weak solution of (3.1) in  $\mathbb{R}^3$  and hence by lemma 3.1  $\tilde{u} \equiv 0$ .

□

The last result in this section is needed in the next section in order to deal with the interface between  $\partial\Omega^i$  and  $\partial\Omega_n$  (that are perpendicular by assumption).

**Lemma 3.5.** *Let*

$$Q = \{(x_1, x_2, x_3) \in \mathbb{R}_+^3 \mid x_2 > 0\}.$$

*Let  $u$  denote a bounded solution of*

$$\begin{cases} -\Delta u - u + i(J_2x_2 + J_3x_3)u = -\lambda u & \text{in } Q \\ \frac{\partial u}{\partial x_1}(0, x_2, x_3) = 0 & x_2 > 0, x_3 \in \mathbb{R} \\ u(x_1, 0, x_3) = 0 & x_1 > 0, x_3 \in \mathbb{R} \end{cases},$$

with  $\lambda \in \mathbb{R}_+$ . Then  $u \equiv 0$ .

*Proof:* Once again we define an even extension of  $u$ ,

$$\tilde{u}(x_1, x_2, x_3) = \begin{cases} u(x_1, x_2, x_3) & x_1 > 0 \\ u(-x_1, x_2, x_3) & x_1 < 0 \end{cases}.$$

By lemma 3.3  $\tilde{u} = 0$ .

□

## 4 Large bounded domains in $\mathbb{R}^3$

We consider here (1.6) in the limit  $R \rightarrow \infty$  which is the large domain limit. We first show that any eigenfunctions of the elliptic operator in (1.6) must decay exponentially fast, as  $R \rightarrow \infty$  away from the boundary. As in the previous section we insert the imaginary part of  $\lambda$  into the electric potential (and consequently consider a family of potentials) and then confine the discussion to real values of  $\lambda$ .

Before getting into the main discussion we repeat here the definition of  $\phi_R$  brought in § 1 and list some of its properties. Recall that  $\phi$  is the unique solution of (1.2), and that

$$\phi_R(x) = R\phi(x/R).$$

The following proposition lists some of the properties of  $\phi$  and  $\phi_R$ .

**Proposition 4.1.** *Let  $\phi$  denote a solution of (1.2) and  $\phi_R(x) = R\phi(x/R)$ . Let  $\mu \in \mathbb{R}$  and  $\mu_R = R\mu$ . Let further  $x_j \in \Omega_{R_j}$  where  $R_j \uparrow \infty$ , and  $\{\mu_j\}_{j=1}^\infty \subset \mathbb{R}$ . Then*

(i). *Either  $|\phi_{R_j}(x_j) - \mu_j|$  is unbounded, or else we must have, up to a subsequence,*

$$\exists(b, J) \in \mathbb{R} \times \mathbb{R}^3 : \|\phi_{R_j} - \mu_j - J \cdot x - b\|_{L^\infty(D_r(x_j))} \rightarrow 0 \quad \forall r > 0,$$

where  $D_r(x_j) = \Omega_{R_j} \cap B(x_j, r)$ .

(ii). *Let  $\Gamma_\mu$  denote the level set  $\phi = \mu$ . Let  $M = \max_{x \in \partial\Omega_c} \phi(x)$  and  $m = \min_{x \in \partial\Omega_c} \phi(x)$ . If  $\mu \notin [m, M]$ , then  $\Gamma_\mu$  is empty.*

(iii). *Assume that  $\partial\Omega_c$  is composed of exactly two connected sets as in Fig. 3, and that  $\Omega$  is diffeomorphic to a cylinder. If  $\mu \in [m, M]$ , but  $\mu \notin \phi(\partial\Omega_c)$ , then  $\Gamma_\mu \cap \partial\Omega_i$  is simple closed contour, separating  $\partial\Omega$  into two subsets, such that none of them is a subset of  $\partial\Omega_i$ . Furthermore,  $\nabla\phi \neq 0$  on  $\Gamma_\mu$ .*

*Proof.* Let  $b_j = \phi_{R_j}(x_j) - \mu_j$  and  $J_j = \nabla \phi_{R_j}(x_j)$ . To prove (i) we first choose a subsequence such that  $(b_j, J_j) \rightarrow (b, J)$ . The claim then follows from the Taylor expansion of  $\phi_{R_j}$  near  $x_j$ .

The proof of (ii) follows immediately from the maximum principle.

To prove (iii) we notice first that since  $\phi$  is real analytic  $\Gamma_\mu$  must be either closed or intersect  $\partial\Omega$ . If it is closed then  $\phi \equiv \mu$  inside  $\Gamma_\mu$  and hence also outside  $\Gamma_\mu$  (in view of its analyticity), which is clearly a contradiction. Thus  $\Gamma_\mu$  must intersect the boundary on  $\partial\Omega_i$ .

Since  $\phi$  is continuous and since on one the connected sets we must have  $\phi < \mu$  and on the other one  $\phi > \mu$ , the intersection of  $\partial\Omega_i$  with  $\Gamma_\mu$  must contain at least one closed contour. This contour, separates  $\partial\Omega$  into two disjoint subsets  $\partial\Omega_+$  and  $\partial\Omega_-$ , the first of them contains the connected subset of  $\partial\Omega_c$  over which  $\phi > \mu$  and a portion of  $\partial\Omega^i$ . Moreover, this contour is the boundary of a continuous subset of  $\Gamma_\mu$  which we denote by  $A$ .

To see this define cylindrical coordinates  $(r, \theta, z)$  in  $\Omega$ , where  $\theta \in [-\pi, \pi]$  and  $0 \leq r < R(\theta, z)$ . Then, for each  $(r, \theta)$  there exists a finite (as  $\phi$  is real analytic) set  $\{z_j\}$  such that  $(r, \theta, z_j) \in \Gamma_\mu$ . Denote the minimum in this set by  $z_m$ . Clearly,  $z = z_m(r, \theta)$  is continuous, and thus we can define

$$A = \{(r, \theta, z) \mid z = z_m(r, \theta)\}.$$

We now consider the problem for  $\phi$  in a subdomain of  $\Omega$  whose boundary consists of  $A$  and  $\partial\Omega_+$ . By the maximum principle and Hopf's lemma we have  $\phi > \mu$  on  $\partial\Omega_+$  and in the interior. In a similar manner we show that  $\phi < \mu$  on  $\partial\Omega_-$ , and in the interior of the subdomain surrounded by  $A$  and  $\partial\Omega_-$ . This shows that  $A = \Gamma_\mu$ . From Hopf's lemma it follows that  $\nabla\phi \neq 0$  on  $\Gamma_\mu$ .  $\square$

**Remark 4.1.** *While property 1 will be used extensively throughout this section, properties 2 and 3 are brought here to provide the reader with some intuition of the behaviour of  $\phi$  in the "wire-like" domain presented in figure 3.*

Since (1.6) contains the term  $i(\phi_R - \mu_R)\psi_R$  which might be unbounded as  $R \rightarrow \infty$  we must provide here the following elliptic estimate.

**Lemma 4.1.** *Let  $u_R$  denote a solution of*

$$\begin{cases} -\Delta u_R - u_R + i(\phi_R - \mu_R)u_R = -\lambda u_R & \text{in } \Omega_R \\ \frac{\partial \psi_R}{\partial \nu} = 0 & \text{on } \partial\Omega_R^i, \\ \psi_R = 0 & \text{on } \partial\Omega_R^c \end{cases} \quad (4.1)$$

*in which  $\lambda \in \mathbb{R}_+$ . Let  $D_r(x_0) = \Omega_R \cap B_r(x_0)$ , where  $x_0 \in \Omega_R$  is chosen such that either*

i.  $D_r(x_0) = B_r(x_0)$ .

or

ii.  $x_0 \in \partial\Omega_R$  and either  $(\partial D_r(x_0) \cap \partial\Omega_R) \subset \partial\Omega_R^c$  or  $(\partial D_r(x_0) \cap \partial\Omega_R) \subset \partial\Omega_R^i$ .

Then,

$$\exists \tilde{r} > 0 : \|u_R\|_{L^\infty(D_r(x_0))} \leq C_r \|u_R\|_{L^2(D_{2r}(x_0))} \quad \forall x_0 \in \Omega_R, r < \tilde{r}, \quad (4.2)$$

where  $C_r$  is independent of  $R$  and  $x_0$ .

*Proof.* Let  $\rho_R = |u_R|$ . By (4.1) we have that

$$-\Delta\rho_R - \rho_R \leq 0$$

in  $\Omega_R$ . Let  $x_0 \in \Omega_R$ . We set  $U_r$  to be the solution (we discuss its existence below) of

$$\begin{cases} -\Delta U_r - U_r = 0 & \text{in } D_r(x_0) \\ U_r = 0 & \text{on } \partial D_r^c(x_0) \\ \frac{\partial U_r}{\partial \nu} = 0 & \text{on } \partial D_r^i(x_0) \\ U_r = \rho_R & \text{on } \partial D_r^s(x_0) \end{cases}, \quad (4.3)$$

where  $\partial D_r^c(x_0) = \partial D_r(x_0) \cap \partial\Omega_R^c$ ,  $\partial D_r^i(x_0) = \partial D_r(x_0) \cap \partial\Omega_R^i$ , and  $\partial D_r^s(x_0) = \partial D_r(x_0) \setminus (\partial D_r(x_0) \cap \partial\Omega_R)$ . Note that either  $\partial D_r^c(x_0) = \emptyset$  or  $\partial D_r^i(x_0) = \emptyset$ . Clearly, there exists  $\tilde{r}$ , independent of  $x_0$  and  $R$ , such that for every  $r < \tilde{r}$  we have

$$\inf_{u \in \mathcal{D}} \frac{\int_{D_r(x_0)} |\nabla u|^2}{\int_{D_r(x_0)} |u|^2} > 1, \quad (4.4)$$

where

$$\mathcal{D} = \{u \in H^1(D_r) \mid u = 0 \text{ on } \partial D_r(x_0) \setminus \partial D_r^i(x_0)\}.$$

For  $r < \tilde{r}$  the elliptic operator in (4.3) is invertible, and hence a unique  $U_r$  exists.

Let then  $V = \rho_R - U_r$ . Clearly

$$\begin{cases} -\Delta V - V \leq 0 & \text{in } D_r(x_0) \\ \frac{\partial V}{\partial \nu} = 0 & \text{on } \partial D_r^i(x_0) \\ V = 0 & \text{on } \partial D_r(x_0) \setminus \partial D_r^i(x_0) \end{cases}. \quad (4.5)$$

Let further

$$V_+ = \begin{cases} V & V \geq 0 \\ 0 & V < 0 \end{cases}.$$

Multiplying (4.5) by  $V_+$  and integrating over  $D_r(x_0)$  we obtain that

$$\int_{D_r} |\nabla V_+|^2 - |V_+|^2 \leq 0,$$

and since  $V \in \mathcal{D}$ , we obtain by (4.4) that  $V_+ = 0$ . Consequently we have

$$\rho_R \leq U_r \quad \text{in } D_r(x_0) \quad (4.6)$$

To complete the proof of (4.2) it is necessary, thus, to obtain an estimate of  $\|U_r\|_{L^\infty(D_r)}$  in terms of  $\|\rho_R\|_{L^2(D_r)}$ . In case (i) above, since  $U_r$  is unique, we can use theorem 10.5 in [3], together with Sobolev embeddings to obtain

$$\|U_r\|_{L^\infty(D_r(x_0))} \leq C_r \|\rho_R\|_{H^{1/2}(\partial D_r^s)}, \quad (4.7)$$

where  $C_r$  is independent of  $x_0$  and  $R$ .

In case (ii)  $D_r(x_0)$  is diffeomorphic to a hemisphere with radius  $r$ . Denote the diffeomorphism by  $T_{R,x_0}$ . Note that  $T_{R,x_0} \rightarrow I$  as  $R \rightarrow \infty$  uniformly in  $x_0$  (as long as the assumption in (ii) hold), in view of the transformation (1.5). Let  $B_+ = T_{R,x_0}(D_r)$ . On the flat surface of  $B_+$  we have either  $\partial U_r / \partial \nu = 0$  or  $U_r = 0$ . In the first case one can extend  $U_r$  evenly to a sphere  $B$  whereas in the second case we use an odd extension for that matter. In both cases  $U_r$  satisfies

$$A : \nabla(A^t \nabla U_r) + U_r = 0 \quad \text{in } B,$$

Where  $A = DT_{R,x_0}$  is smooth in  $B$  and satisfies  $A \rightarrow I$  uniformly in  $B$  as  $R \rightarrow \infty$ . On  $\partial B$   $U_r$  is equal either to the even extension or the odd extension of  $\rho_R$ . Hence, for sufficiently large  $R$ ,  $U_r$  must satisfy (4.7) in case (ii) as well.

Clearly,

$$\|\rho_R\|_{H^{1/2}(\partial D_r^s)} \leq \|\rho_R\|_{H^1(D_r(x_0))}. \quad (4.8)$$

Multiplying (4.1) by  $\chi_{2r}^2(|x - x_0|)\bar{u}_R$  and integrating over  $\Omega_R$  we obtain from the real part of the identity

$$\int_{D_{2r}(x_0)} |\nabla(\chi_{2r} u_R)|^2 - (|\nabla \chi_{2r}|^2 + (1 - \lambda)\chi_{2r}^2) |u_R|^2 = 0.$$

Hence,

$$\int_{D_r(x_0)} |\nabla \rho_R|^2 \leq C_r \int_{D_{2r}(x_0)} |\rho_R|^2. \quad (4.9)$$

Combining the above with (4.6-4.8) yields (4.2).  $\square$

As an immediate conclusion of lemma 4.1 we prove the following lemma

**Lemma 4.2.** *Let  $u_R$  denote a solution of (4.1) with  $\lambda \in \mathbb{R}_+$ . Let  $X_R$  denote a maximum point of  $|u_R|$  in  $\Omega_R$ . Then,  $|\phi_R(x_R) - \mu_R|$  is bounded as  $R \rightarrow \infty$ .*

*Proof.* We first normalize  $u_R$  by  $|u_R(x_R)|$  so that  $\|u_R\|_\infty = 1$ . Let  $D_r(x_R) = B(x_R, r) \cap \Omega_R$ . Multiplying (4.1) by  $\chi_r^2(|x - x_R|)\bar{u}_R$  and integrating over  $\Omega_R$  we obtain from the imaginary part of the identity

$$\int_{D_r} \nabla(\chi_r^2) \cdot \frac{1}{2i}(\bar{u}_{Rj} \nabla u_R - \nabla \bar{u}_R) + \int_{D_r} \chi_r^2(\phi_R - \mu_R)|u_R|^2 = 0.$$

Let  $b_R = \phi_R(x_R) - \mu_R$ . Since  $\nabla \phi_R$  is bounded in  $\Omega_R$ , and since  $r$  is fixed, we have

$$\inf_{x \in D_r} |\phi_R - \mu_R| \geq \frac{1}{2}b_R.$$

Thus,

$$b_R \int_{D_{r/2}} |u_R|^2 \leq C_r \int_{D_r} |u_R|^2 + |\nabla u_R|^2.$$

Using (4.9) we obtain, and the fact that  $|u_R| \leq 1$  we obtain that

$$\int_{D_{r/2}} |u_R|^2 \leq \frac{C}{b_R}.$$

By lemma 4.1 we then have that

$$1 = |u_R(x_R)| \leq \frac{C_r}{b_R}.$$

From which the lemma immediately follows.  $\square$

**Remark 4.2.** *Note that if  $\phi_R \neq \mu_R$  for all  $x \in \Omega_R$  then  $u_R \equiv 0$  must be the unique solution of (4.1). To see this multiply (4.1) by  $\bar{u}_R$  and integrate over  $\Omega_R$  to obtain from the imaginary part:*

$$\int_{\Omega_R} (\phi_R - \mu_R)|u_R|^2 = 0.$$

*Since  $\phi_R - \mu_R$  is either positive or negative throughout  $\Omega_R$ ,  $u_R$  must vanish everywhere.*

Denote the curve in  $\Omega$  along which we have  $\phi = \mu$  by  $\Gamma_\mu$ . Denote its image under the mapping (1.5) by  $\Gamma_R$ . By the previous lemma we have that  $d(x_R, \Gamma_R)$  is bounded as  $R \rightarrow \infty$ . We now prove that  $u_R$  must decay exponentially fast away from  $\Gamma_R$ .

**Lemma 4.3.** *Let  $u_R$  denote a solution of (4.1) with  $\lambda \in \mathbb{R}_+$ . Then, there exists  $\alpha > 0$  such that*

$$\int_{\Omega_R} |u_R|^2 e^{2\alpha s} \leq C, \quad (4.10)$$

where  $s = d(x, \Gamma_R)$  and  $C$  is independent of  $R$ .

Recall that by proposition 4.1, for domains that are diffeomorphic to a cylinder, when  $\phi \neq \mu$  for every  $x \in \partial\Omega^c$ ,  $\Gamma_1$  must be a surface whose boundary is a closed simple contour on  $\partial\Omega^i$ . We also have  $\nabla\phi \neq 0$  on  $\Gamma_\mu$ . Note also that by the above remark, if  $\Gamma_R$  is empty, then  $u_R \equiv 0$ .

*Proof:* It is convenient to consider here  $u_R$  for which  $\|u_R\|_{L^2(\Omega_R)} = 1$ . Let  $\Omega_\beta^+$  and  $\Omega_\beta^-$  be respectively defined by

$$\begin{aligned} \Omega_\beta^+ &= \{x \in \Omega_R \mid \phi_R - \mu_R > \beta\} \\ \Omega_\beta^- &= \{x \in \Omega_R \mid \phi_R - \mu_R < -\beta\}. \end{aligned}$$

for some  $\beta > 0$  which is independent of  $R$ .

Let  $\eta_\beta^+ \in C^\infty(\Omega_R, [0, 1])$  and  $\eta_\beta^-$  respectively be defined by

$$\eta_\beta^+ = \begin{cases} 1 & x \in \Omega_\beta^+ \\ 0 & x \in \Omega_\beta^- \end{cases}$$

and

$$\eta_\beta^- = \begin{cases} 1 & x \in \Omega_\beta^- \\ 0 & x \in \Omega_\beta^+ \end{cases}.$$

Let  $\mathcal{C}_\beta^\pm$  respectively denote the portion of  $\partial\Omega_\beta^\pm$  which is not on  $\partial\Omega$ . As  $\nabla\phi_R(x) = \nabla\phi(x/R)$  and since  $\nabla\phi$  is bounded in  $\Omega$  we have that

$$d(\mathcal{C}_\beta^\pm, \Gamma_\mu) \geq \frac{\beta}{\|\nabla\phi\|_{L^\infty(\Omega)}}.$$

Hence, we can choose  $\eta_\beta^\pm$  such that  $|\nabla\eta_\beta^\pm| < C$ . Let

$$D_\beta^+ = \{x \in \Omega_R \mid 0 < \eta_\beta^+ < 1\}.$$

We choose  $\eta_\beta^\pm$  such that

$$\sup_{x \in D_\beta^+} s \leq 1.$$

Multiplying (4.1) by  $(\eta_\beta^+)^2 e^{2\alpha s} \bar{u}_R$  and integrating over  $\Omega_R$  we obtain, for the the imaginary part and the real part respectively

$$\int_{\Omega_R} \nabla((\eta_\beta^+)^2 e^{2\alpha s}) \cdot \frac{1}{2i} (\bar{u}_R \nabla u_R - u_R \nabla \bar{u}_R) + \int_{\Omega_R} (\phi_R - \mu_R) (\eta_\beta^+)^2 |u_R|^2 e^{2\alpha s} = 0, \quad (4.11a)$$

$$\int_{\Omega_R} (\eta_\beta^+)^2 |\nabla u_R|^2 e^{2\alpha s} = (1 - \lambda) \int_{\Omega_R} (\eta_\beta^+)^2 |u_R|^2 e^{2\alpha s} - \frac{1}{2} \int_{\Omega_R} \nabla((\eta_\beta^+)^2 e^{2\alpha s}) \cdot \nabla |u_R|^2. \quad (4.11b)$$

From the real part, (4.11b), we obtain, that for every  $\epsilon > 0$  we have

$$(1 - 2\alpha\epsilon) \int_{\Omega_R} (\eta_\beta^+)^2 e^{2\alpha s} |\nabla u_R|^2 \leq \left(1 + \frac{\alpha}{2\epsilon}\right) \int_{\Omega_R} (\eta_\beta^+)^2 e^{2\alpha s} |u_R|^2 + C \int_{D_\beta^+} e^{2\alpha s} |\nabla u_R|^2. \quad (4.12)$$

From (4.11a), or the imaginary part, we obtain

$$\beta \int_{\Omega_\beta^+} |u_R|^2 e^{2\alpha s} \leq \alpha \int_{\Omega_R} (\eta_\beta^+)^2 e^{2\alpha s} [|u_R|^2 + |\nabla u_R|^2] + C \int_{D_\beta^+} e^{2\alpha s} [|u_R|^2 + |\nabla u_R|^2]$$

Combining the above with (4.12) for  $\epsilon = 4\alpha^{-1}$  we obtain

$$(\beta - 2\alpha - 4\alpha^3) \int_{\Omega_\beta^+} |u_R|^2 e^{2\alpha s} \leq C \int_{D_\beta^+} e^{2\alpha s} [|u_R|^2 + |\nabla u_R|^2]. \quad (4.13)$$

Multiplying (4.1) by  $\bar{u}_R$  and integrating over  $\Omega_R$  we obtain (for the real part)

$$\int_{\Omega_R} |\nabla u_R|^2 = (1 - \lambda) \int_{\Omega_R} |u_R|^2 = 1 - \lambda.$$

Consequently, we obtain from (4.13), that for any given  $\alpha$  we may choose  $\beta$  to be sufficiently large (but still independent of  $R$ ) so that  $\beta > 2\alpha + 4\alpha^3$  and hence

$$\int_{\Omega_\beta^+} |u_R|^2 e^{2\alpha s} \leq C e^{2\alpha} \int_{D_\beta^+} [|u_R|^2 + |\nabla u_R|^2] \leq 2C e^{2\alpha}.$$

□

**Remark 4.3.** *By lemma 4.1 it follows that  $|u_R|$  decays exponentially fast away from  $\Gamma_R$  also in a pointwise sense.*

We now prove that any eigenfunction corresponding to a non-positive eigenvalue must decay exponentially fast away from the boundary

**Lemma 4.4.** *Let  $u_R$  denote a solution of (4.1). Then*

$$\exists \alpha > 0 : |u_R| \leq C e^{-\alpha d(x, \partial\Omega)} \quad \forall \mu_R \in \mathbb{R}, \quad (4.14)$$

where  $\alpha$  is independent of  $R$ ,  $\mu_R$ , and  $\lambda$ . Furthermore, denote by  $x_R$  the maximum point of  $|u_R|$ . Then  $d(x_R, \partial\Omega_R)$  is bounded as  $R \rightarrow \infty$ .

*Proof:* We apply standard blow up arguments to prove the lemma. Let

$$\Omega(R, k, s) = \{x \in \Omega_R \mid d(x, \partial\Omega_R) \geq ks\}.$$

We prove exponential rate of decay by showing first that

$$\exists R_0, s_0 : \|u_R\|_{L^\infty(\Omega(R, k+1, s))} \leq \frac{1}{2} \|u_R\|_{L^\infty(\Omega(R, k, s))} \quad \forall s > s_0, R > R_0, k \in \mathbb{N} \quad (4.15)$$

Suppose, for a contradiction, that (4.15) does not hold. Then, there exist sequences  $\{R_j\}_{j=1}^\infty$ ,  $\{s_j\}_{j=1}^\infty$ , and  $\{k_j\}_{j=1}^\infty$  satisfying  $R_j \uparrow \infty$ ,  $s_j \uparrow \infty$ ,  $k_j \in \mathbb{N}$ , and

$$\|u_{R_j}\|_{L^\infty(\Omega(R_j, k_j+1, s_j))} \geq \frac{1}{2} \|u_{R_j}\|_{L^\infty(\Omega(R_j, k_j, s_j))} \stackrel{def}{=} \frac{1}{2} m_j \quad (4.16)$$

Let

$$\tilde{u}_{R_j} = \frac{u_{R_j}}{m_j}.$$

By (4.15) there exists  $x_j \in \Omega(R_j, k_j + 1, s_j)$  such that

$$|\tilde{u}_{R_j}(x_j)| \geq \frac{1}{2}. \quad (4.17)$$

For notation convenience we also let  $f_j(x) = \tilde{u}_{R_j}(x_j + x)$ .

We now distinguish between two different cases:

*case 1:*

$$b_j = \inf_{x \in B(x_j, s_j)} |\phi_{R_j} - \mu_{R_j}| \rightarrow \infty$$

up to a subsequence.

Let  $\chi_r \in C^\infty(\mathbb{R}_+, [0, 1])$  be defined by (3.3). Since  $f_j$  satisfies (4.1) we multiply it by  $\chi_r^2(0) \bar{f}_j$  and integrate over  $B(0, r)$  to obtain, from the imaginary part

$$\int_{B(0, r)} \nabla(\chi_r^2) \cdot \frac{1}{2i} (\bar{f}_j \nabla f_j - f_j \nabla \bar{f}_j) + \int_{B(0, r)} \chi_r^2 (\phi_{R_j} - \mu_{R_j}) |f_j|^2 = 0.$$

Yielding

$$b_j \int_{B(0, r/2)} |f_j|^2 \leq C_r \int_{B(0, r)} |f_j|^2 + |\nabla f_j|^2, \quad (4.18)$$

for all  $r < s_j$ . From the real part we obtain that

$$\int_{B(0,r)} |\nabla(\chi_r f_j)|^2 - (|\nabla\chi_r|^2 + (1-\lambda)\chi_r^2) |f_j|^2 = 0,$$

and hence

$$\int_{B(0,r/2)} |\nabla f_j|^2 \leq C \int_{B(0,r)} |f_j|^2. \quad (4.19)$$

Combining (4.18) with (4.19) we obtain

$$\int_{B(0,r/2)} |f_j|^2 \leq \frac{C_r}{b_j} \int_{B(0,2r)} |f_j|^2.$$

As  $|f_j| \leq 1$  we obtain that

$$\int_{B(0,r/2)} |f_j|^2 \rightarrow 0$$

as  $j \rightarrow \infty$ , and by lemma 4.1 also that  $f_j(0) \rightarrow 0$ . A contradiction.

*Case 2:*  $\limsup_{j \rightarrow \infty} b_j < \infty$

Let  $J = |\nabla\phi|$ . We choose a coordinate system where  $\nabla\phi(x_j)$  is parallel to the  $x_1$  axis. Then, by lemma 4.2 we have a subsequence for which

$$\phi_{R_j} - \mu_{R_j} \rightarrow Jx_1 + b,$$

uniformly in  $B(0,r)$  for all  $r > 0$ , where  $b$  is a constant. Thus, by standard elliptic estimates and Sobolev embeddings, there exists a subsequence  $\{f_{J_k}\}_{k=1}^\infty$  such that  $f_{J_k} \rightarrow f_\infty$  uniformly on every compact set in  $\mathbb{R}^3$  and such that  $f_\infty$  is a bounded solution of

$$-\Delta f_\infty - f_\infty + i(Jx_1 + b)f_\infty = -\lambda f_\infty \quad \text{in } \mathbb{R}^3.$$

By lemma 3.1 we must have  $f_\infty \equiv 0$ . A contradiction.

Thus, we have proved (4.15), and hence also (4.14). The boundedness of  $d(x_R, \partial\Omega_R)$  follows from (4.14) as well.

□

The following lemma provides the basis for our main stability result

**Lemma 4.5.** *Let  $\partial\Omega_n$  denote the subset of  $\partial\Omega_c$  where  $\nabla\phi$  is perpendicular to  $\partial\Omega$ . Suppose that either  $|\nabla\phi| > J_c$  for all  $x \in \partial\Omega_n$  or that  $\partial\Omega_n$  is empty. Then, for sufficiently large  $R$ ,  $u_R \equiv 0$  is the unique solution of (4.1) for all  $\mu_R \in \mathbb{R}$ .*

*Proof:* Let  $x_R$  be the point where  $u_R$  obtains its maximum in  $\Omega_R$ . Let  $x_0^R$  denote its projection on  $\partial\Omega_R$ . Recall that by lemma 4.4  $|x_R - x_0^R|$  is bounded as  $R \rightarrow \infty$ . Note that by lemma 4.2  $\phi_R$  converges uniformly to a linear function in  $D_r(x_0^R)$  for all fixed  $r > 0$  as  $R \rightarrow \infty$ . Suppose first that  $x_0^R \in \partial\Omega_i^R$ . Following [10], let  $(t_1, t_2, t_3)$  denote a local curvilinear coordinate system, whose origin lies at  $x_0^R$ , such that  $t_3 = d(x, \partial\Omega_R)$  when  $x \in \Omega_R$  and such that the  $t_1$  and  $t_2$  curves on  $\partial\Omega$  are the lines of curvature. Let further  $\kappa_1^R$  and  $\kappa_2^R$  denote the respective principal curvature on  $\partial\Omega_R$ . Clearly,  $\kappa_i^R = \kappa_i/R$  ( $i = 1, 2$ ) where  $\kappa_i$  is the corresponding principal curvature on  $\partial\Omega$ , at  $x_0 = x_0^R/R$ .

Since  $\partial\Omega$  is smooth near  $x_0^R$ , this curvilinear coordinate system is properly defined in some neighborhood of  $x_0^R$ . Let

$$B^+(0, r) = \{(t_1, t_2, t_3) \in B(0, r) \mid t_3 > 0\}.$$

Then, the above coordinate system is well-defined in  $B^+(0, \delta R)$  for some  $\delta > 0$ . We can now present any  $x$  in this neighborhood by

$$x = r(t_1, t_2) - t_3\nu,$$

where  $\nu$  is the outward normal at  $(t_1, t_2, 0)$ . Let

$$g_{ij}(t_1, t_2) = \frac{\partial r}{\partial t_i} \cdot \frac{\partial r}{\partial t_j} \quad i, j = 1, 2$$

and

$$G_{ij} = [1 - \kappa_i t_3/R]g_{ij} \quad i, j = 1, 2.$$

Since our coordinate system is orthogonal we have

$$g_{12} = 0.$$

Furthermore, we can scale  $t_1$  and  $t_2$  so that

$$g_{11} = g_{22} = 1 + O(1/R) \quad \text{as } R \rightarrow \infty,$$

uniformly in  $B^+(0, r)$  for every fixed  $r > 0$ . Finally, we define

$$\begin{cases} G = \sqrt{G_{11}G_{22}} \\ \alpha_j = \frac{G}{G_{jj}} & j = 1, 2 \\ \alpha_3 = G \end{cases}.$$

Let  $w_R(x) = u_R(x)/|u_R(x_R)|$ . In the new coordinates (4.1) takes the form

$$-\sum_{j=1}^3 \frac{1}{G} \frac{\partial}{\partial t_j} \left( \alpha_j \frac{\partial w_R}{\partial t_j} \right) - w_R + i(\phi_R - \mu_R)w_R = -\lambda w_R,$$

in  $B^+(0, \delta R)$ . Standard elliptic estimates then prove the existence of a sequence  $\{w_{R_j}\}_{j=1}^\infty$  such that  $w_{R_j} \rightarrow w_\infty$  uniformly on every compact set in  $\mathbb{R}_+^3$  where here

$$\mathbb{R}_+^3 = \{(t_1, t_2, t_3) \mid t_3 > 0\}.$$

By standard elliptic estimates again we have that  $w_\infty$  satisfies the following problem

$$\begin{cases} -\Delta w_\infty - w_\infty + i(J_1 t_1 + J_2 t_2 + b)w_\infty = -\lambda w_\infty & \text{in } \mathbb{R}_+^3 \\ \frac{\partial w_\infty}{\partial \nu} = 0 & \text{on } \partial \mathbb{R}_+^3 \end{cases}.$$

By lemma 3.4 we have  $w_\infty \equiv 0$  in  $\mathbb{R}_+^3$  - a contradiction since  $|w_R(x_R)| = 1$  and  $|x_R - x_0^R|$  is bounded.

Consider now the case where  $x_0^R \in \partial \Omega_c^R$ . Following the same procedure as before we obtain that  $w_R \rightarrow w_\infty$  uniformly on every compact set in  $\mathbb{R}_+^3$  where  $w_\infty$  must satisfy

$$\begin{cases} -\Delta w_\infty - w_\infty + i(J_1 t_1 + J_2 t_2 + J_3 t_3 + b)w_\infty = -\lambda w_\infty & \text{in } \mathbb{R}_+^3 \\ w_\infty = 0 & \text{on } \partial \mathbb{R}_+^3 \end{cases}.$$

If  $\partial \Omega_n$  is empty we have  $J_1^2 + J_2^2 > 0$ . By lemma 3.3 we then have  $w_\infty \equiv 0$ . Otherwise, if  $x_0 \in \partial \Omega_n$  we must have  $w_\infty \equiv 0$  by lemma 3.2 since  $J_3 > J_c$ .

Finally, if  $x_0$  lies on the interface between  $\partial \Omega_i$  and  $\partial \Omega_c$  we obtain that  $w_\infty$  must satisfy, since the two surfaces are perpendicular to each other at the interface, a problem in  $Q$  where

$$Q = \{(t_1, t_2, t_3) \mid t_3 > 0, t_1 > 0\}.$$

We have

$$\begin{cases} -\Delta w_\infty - w_\infty + i(J_1 t_1 + J_2 t_2 + b)w_\infty = 0 & \text{in } Q \\ \frac{\partial w_\infty}{\partial \nu} = 0 & x \in \partial Q \ t_3 = 0 \\ w_\infty = 0 & x \in \partial Q : t_1 = 0 \end{cases}.$$

By lemma 3.5 we have  $w_\infty \equiv 0$ . (Note that some modification of the local coordinate system is necessary in this case.)

□

*Proof of Theorem 1.1:* Since the principal part of the differential operator on the left-hand-side of (4.1) is the Laplacian, it can be regarded as a perturbation of a self-adjoint operator. Thus, it follows from the discussion below theorem 15.2 in [2] regarding such perturbations that it has exactly

one direction which is not a direction of minimal growth:  $\arg \lambda = 0$ . Hence, it follows from theorem 15.1 there, that the spectrum of this differential operator must be discrete and that all its eigenvalues must have finite multiplicity. Furthermore, by theorem 16.5 in [2], the eigenfunctions span  $L^2(\Omega_R, \mathbb{C})$ . Hence, for  $J > J_c$ , since all the eigenvalues of the above operator must have positive real part, the normal state must be stable.

Consider now the case where a point  $x \in \partial\Omega_c$  exists where

$$\left| \frac{\partial \phi}{\partial \nu} \right| = |\nabla \phi| = J < J_c.$$

To prove the short time instability we look at the solution of (1.6), after applying to it the transformation (2.2), with the initial condition

$$\psi(x, 0) = u_1(t_3) \chi_{R^{1/2}}(t_1, t_2) \eta_R(t_3),$$

where  $(t_1, t_2, t_3)$  are the above-defined system of local curvilinear coordinates,  $\chi_r$  is defined in (3.3) and

$$\eta_R(x) = \begin{cases} 1 & x < \frac{1}{2} \delta R \\ 0 & x > \delta R \end{cases},$$

is a smooth cutoff function.

Let  $\tilde{\beta} = \lambda_J - \lambda_{J_c}$ . We write

$$\psi_R = v + \psi_0(x) e^{\tilde{\beta} t}, \quad (4.20)$$

to obtain

$$\frac{\partial v}{\partial t} - \Delta v - \lambda_J v = f. \quad (4.21)$$

The precise form of  $f$  need not concern us except for the fact that

$$\|f\|_2 \leq \frac{C_\alpha}{R^\alpha} \|\psi_0\|_2 e^{\tilde{\beta} t} \quad \forall \alpha < 1. \quad (4.22)$$

Multiplying (4.21) by  $\bar{v}$  and integrating by parts we obtain for the real part

$$\begin{cases} \frac{\partial \|v\|_2}{\partial t} - \lambda_J \|v\|_2 = \|f\|_2 \\ \|v\|_2(0) = 0 \end{cases}.$$

Consequently,

$$\|v\|_2(t) \leq \int_0^t e^{\lambda(t-\tau)} \|f\|_2(\tau) d\tau,$$

and hence,

$$\|v\|_2 \leq \frac{C_\alpha}{R^\alpha} \|\psi_0\|_2 e^{(\lambda_J + \tilde{\beta})t}.$$

Clearly, there exist  $T_R \sim \mathcal{O}(\ln R)$ , as  $R \rightarrow \infty$ , such that

$$t < T_R \Rightarrow \frac{C_\alpha}{R^\alpha} e^{\lambda_J t} < \frac{1}{2},$$

and thus

$$t < T_R \Rightarrow \|\psi\|_2 \geq \frac{1}{2} \|\psi_0\|_2 e^{\tilde{\beta}t}.$$

Applying the inverse of (2.2) we obtain (1.7).

□

The above instability result is valid, of course, only for  $T < T_R$ . Proving long time instability appears to be a much more difficult problem. The stability proof presented above relies on the convergence of any solution of (4.1) to a solution of (3.8) uniformly on every compact set near the point on the boundary where  $J$  is perpendicular to it. This however does not prove convergence of the spectrum and not even of its bottom. What has been demonstrated is only the upper semi-continuity of the spectrum of the differential operator on the left-hand-side of (4.1). Lower semi-continuity of the spectrum appears to be much harder to prove especially since the operator is not self-adjoint. Nevertheless, it does seem reasonable to conjecture that the solution would continue to grow exponentially fast as  $t \rightarrow \infty$ , in view of the above short-time instability result. Further research is necessary in order to establish that result.

## 5 Concluding remarks

In the previous section we proved that the normal state remains stable in the large domain limit, as long as the current on the boundary, at points where it is perpendicular to it, is greater than  $J_c$ . If the current is nowhere perpendicular to the boundary, then as long as it doesn't vanish there, the normal state must be stable. We also demonstrate short time instability when  $J_m < J_c$ .

In the following we provide a short list of interesting problems that are waiting to be resolved:

1. Proving long time instability when  $J > J_c$ . We have elaborated on this matter in the preceding section end.

2. Adding the effect of magnetic fields. This magnetic field can be either induced by the electric current [via (1.3)], or else be applied externally (or both). It has been verified experimentally that an induced magnetic field can generate vortices [12] if the current is sufficiently large, and the material is closed to the wholly superconducting state. However its effect on the critical current  $J_c$  has not been investigated. It is reasonable to believe that  $J_c$  would become smaller if we combine that effect.
3. Adding the effect of temperature: since electric currents have the tendency to heat the sample, thereby creating vortices [12]. Incorporating this effect requires modification of (1.1), and the use of different non-dimensionalization, otherwise the domain would become temperature-dependent
4. Proving that the bifurcating branch (at  $J = J_c$ ) is unstable.

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## A The Hilbert-Schmidt norm of $\mathcal{L}_\lambda^{-1}$

**Lemma A.1.** *Let  $\tilde{G}$  be given by (2.14b). Then, for any  $\lambda \in \mathbb{C} \setminus \{\lambda_n\}_{n=1}^\infty$ , where  $\{\lambda_n\}_{n=1}^\infty$  are given in (2.11)*

$$\exists C, M > 0 : \|\tilde{G}\| \leq Ce^{M\lambda^{3/2}}. \quad (\text{A.1})$$

*Proof:* We first note that

$$W(w_1, \tilde{w}_2)(x, \lambda) = w_2'(x, \lambda)w_1(x, \lambda) - w_1'(x, \lambda)w_2(x, \lambda),$$

is independent of  $x$  by Abel's formula. Furthermore, for  $y = x + i\lambda$  we have

$$W(w_1, \tilde{w}_2)(x, \lambda) = W(w_1, \tilde{w}_2)(y, 0) = W(w_1, \tilde{w}_2)(0, 0).$$

Therefore,  $W$  is independent of both  $x$  and  $\lambda$ .

Since  $W$  is constant, it follows that  $\tilde{G}$  is symmetric, i.e.,

$$\tilde{G}(x, \xi, \lambda) = \tilde{G}(\xi, x, \lambda).$$

Consequently, suffices it to prove that

$$\int_0^\infty d\xi \int_\xi^\infty dx |\tilde{G}(x, \xi)|^2 = C_1 \int_0^\infty d\xi |\tilde{w}_2(\xi)|^2 \int_\xi^\infty dx |\tilde{w}_1(x)|^2 \leq C e^{M\lambda^{3/2}},$$

where  $C$ ,  $C_1$ , and  $M$  are all independent of  $\lambda$ .

To obtain the above estimate we use asymptotic properties of Airy's functions [1, 11], from which it follows that

$$|A_i(z)| \leq \frac{C}{|z|^{1/4}} \left| e^{-\frac{2}{3}z^{3/2}} \right|. \quad (\text{A.2})$$

Consider then first the domain  $\xi > M_0\lambda$  for sufficiently large  $M_0 > 0$ . We have

$$\int_\xi^\infty |w_1(x)|^2 dx \leq \int_\xi^\infty \frac{dx}{|x+i\lambda|^{1/2}} e^{-\beta(x)|x+i\lambda|^{3/2}},$$

where

$$\beta(x) = \frac{4}{3} \cos \left( \frac{3}{2} \arg(x+i\lambda) + \frac{\pi}{4} \right).$$

It is easy to show that

$$\begin{cases} |\beta'(x)| \leq C \frac{|\lambda|}{|x+i\lambda|^2} \\ |\beta''(x)| \leq C \frac{|\lambda|}{|x+i\lambda|^3} \end{cases}. \quad (\text{A.3})$$

Let then

$$\gamma(x) = \beta(x)|x+i\lambda|^{3/2}.$$

By (A.3) we have

$$\begin{cases} |\gamma'(x)| \geq \frac{3}{2}\beta(x)|x+i\lambda|^{1/2} \left[ 1 - C \frac{|\lambda|}{|x+i\lambda|} \right] \\ |\gamma''(x)| \leq \frac{3}{4}\beta(x)|x+i\lambda|^{-1/2} \left[ 1 + C \frac{|\lambda|}{|x+i\lambda|} \right] \end{cases}. \quad (\text{A.4})$$

Using (A.4) and integration by parts we obtain

$$\begin{aligned} \int_\xi^\infty e^{-\gamma(x)} dx &\leq \frac{1}{|\gamma'(\xi)|} e^{-\gamma(\xi)} + \int_\xi^\infty \frac{|\gamma''(x)|}{|\gamma'(x)|} e^{-\gamma(x)} dx \\ &\leq \frac{C}{|\xi+i\lambda|^{1/2}} e^{-\gamma(\xi)} + \frac{C}{|\xi+i\lambda|^{3/2}} \int_\xi^\infty e^{-\gamma(x)} dx. \end{aligned}$$

Hence,

$$\int_\xi^\infty e^{-\gamma(x)} dx \leq \frac{C}{|\xi+i\lambda|^{1/2}} e^{-\gamma(\xi)},$$

from which we easily obtain that

$$\int_{\xi}^{\infty} |w_1(x)|^2 dx \leq \frac{C}{|\xi + i\lambda|} e^{-\gamma(\xi)}. \quad (\text{A.5})$$

From the asymptotic behaviour of Airy's function (A.2) we obtain again

$$|\tilde{w}_2(\xi)|^2 \leq \frac{C}{|\xi + i\lambda|^{1/2}} e^{\gamma(\xi)}.$$

Thus,

$$\int_{M_0\lambda}^{\infty} d\xi |\tilde{w}_2(\xi)|^2 \int_{\xi}^{\infty} dx |\tilde{w}_1(x)|^2 \leq \frac{C}{\lambda^{1/2}}.$$

To complete the proof we need to bound the norm for  $0 < \xi < M_0\lambda$ . By (A.2) and (2.14b) we have

$$|\tilde{G}(x, \xi, \lambda)| \leq C \exp \left\{ \frac{2}{3} (M_0 + 1)^{3/2} |\lambda|^{3/2} \right\}.$$

Consequently, from the above and (A.5) we obtain

$$\begin{aligned} & \int_0^{M_0\lambda} d\xi |\tilde{w}_2(\xi)|^2 \int_{\xi}^{\infty} dx |\tilde{w}_1(x)|^2 = \int_0^{M_0\lambda} d\xi |\tilde{w}_2(\xi)|^2 \int_{\xi}^{M_0\lambda} dx |\tilde{w}_1(x)|^2 \\ & + \int_0^{M_0\lambda} d\xi |\tilde{w}_2(\xi)|^2 \int_{M_0\lambda}^{\infty} dx |\tilde{w}_1(x)|^2 \leq CM_0^2 \lambda^2 \exp \left\{ \frac{2}{3} (M_0 + 1)^{3/2} |\lambda|^{3/2} \right\}. \end{aligned}$$

from which (A.1) easily follows.  $\square$

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