SUPERCONDUCTIVITY NEAR THE NORMAL STATE UNDER THE ACTION OF ELECTRIC CURRENTS AND INDUCED MAGNETIC FIELDS IN $\mathbb{R}^2$

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Abstract. We consider the linearization of the time-dependent Ginzburg-Landau system near the normal state. We assume that an electric current is applied through the sample, which captures the whole plane, inducing thereby, a magnetic field. We show that independently of the current, the normal state is always stable. Using Fourier analysis the detailed behaviour of solutions is obtained as well. Relying on semi-group theory we then obtain the spectral properties of the steady-state elliptic operator.

Contents

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1. Introduction

Consider a superconductor placed at a temperature lower than the critical one. If an electric current is applied through the sample it will induce a magnetic field. It is well-understood that if the electric current is sufficiently strong, then the sample must be at a normal state. If the current is then lowered, the normal state would loose stability and the sample would become superconducting again. Such a pattern of behaviour has been observed in numerous experiments [28]. It has also been obtained theoretically by analyzing the stability of the normal state for the time dependent Ginzburg-Landau system, but with the induced magnetic field neglected [20, 3]. Our interest here is the joint effect of the current and the magnetic field it induces on the stability of the normal state. From a mathematical point of view we thus need to consider the linearization of the Ginzburg-Landau system near the normal state, this time with the effect of induced magnetic field included.

In contrast with the simplified model, for which the induced magnetic field is neglected, one cannot present the full model in a one-dimensional setting. In the present contribution we thus consider a two-dimensional superconducting sample capturing the entire xy-plane. For a complete analysis of the effect of induced magnetic fields, we need, of course, to include boundaries and three-dimensional effects. However, the present contribution appears to be a necessary first step before one moves on to include these effects. Assuming that a magnetic field of magnitude $\mathcal{H}^e$ is perpendicularly applied to the sample the time-dependent Ginzburg-Landau system can be written as follows (see for instance [4, 6, 9, 10, 20, 27, 29]):

$$\begin{cases} \partial_t\psi + i\kappa\Phi\psi = \nabla^2_{\kappa\mathbf{A}}\psi + \kappa^2(1-|\psi|^2)\psi & \text{in } (0,T)\times\mathbb{R}^2, \\ \kappa^2\mathrm{curl}^2\mathbf{A} + \sigma(\partial_t\mathbf{A}+\nabla\Phi) = \kappa\,\mathrm{Im}\,(\bar\psi\nabla_{\kappa\mathbf{A}}\psi) + \kappa^2\mathrm{curl}\mathcal{H}^e & \text{in } (0,T)\times\mathbb{R}^2, \end{cases} \tag{1.1}$$

where ψ is the order parameter, $\mathbf{A}$ is the magnetic potential, Φ is the electric potential, the Ginzburg-Landau parameter of the superconductor is denoted by κ and the normal conductivity of the sample by σ. The triplet $(\psi,\mathbf{A},\Phi)$ should also satisfy an initial condition at $t=0$. A solution $(\psi,\mathbf{A},\Phi)$ is called a normal state solution if $\psi\equiv 0$. From (1.1) we see that if $(0,\mathbf{A},\Phi)$ is a time-independent normal state solution then $(\mathbf{A},\Phi)$ satisfies the equality

$$\kappa^2\mathrm{curl}^2\mathbf{A} + \sigma\nabla\Phi = \kappa^2\mathrm{curl}\mathcal{H}^e \quad \text{in } \mathbb{R}^2. \tag{1.2}$$

In the following we further assume that a current of constant magnitude J is flown through the sample in the y-axis direction, and that the applied magnetic field is of constant magnitude, hence $\mathcal{H}^e = h\mathbf{i}_z$, throughout the entire sample. Here $\mathbf{i}_x, \mathbf{i}_y$ and $\mathbf{i}_z$ denote the canonical basis in $\mathbb{R}^3$. Under these additional assumptions (1.2) admits the following solution, which is also a normal state solution of (1.1),

$$\mathbf{A} = \frac{1}{2J}(Jx+h)^2\mathbf{i}_y, \quad \Phi = \frac{\kappa^2 J}{\sigma}y. \tag{1.3}$$

Note that the magnetic field

$$\mathbf{H} = \mathrm{curl}\mathbf{A} = (Jx+h)\mathbf{i}_z,$$

is the sum of the constant applied magnetic field $h\mathbf{i}_z$ and a linear term produced by the electric current. The linearization of (1.1) near the normal state solution (1.3) is

$$\partial_t\psi + \frac{i\kappa^3 Jy}{\sigma}\psi = \Delta\psi - \frac{i\kappa}{J}(Jx+h)^2\partial_y\psi - (\frac{\kappa}{2J})^2(Jx+h)^4\psi + \kappa^2\psi \quad \text{in } (0,T)\times\mathbb{R}^2\,. \tag{1.4}$$

Applying the transformation

$$(x,y) \to (x-\frac{h}{J}, y-\frac{h}{J})\,,$$

we obtain

$$\partial_t\psi + \frac{i\kappa^3 Jy}{\sigma}\psi = \Delta\psi - i\kappa Jx^2\partial_y\psi - \Big((\frac{\kappa J}{2})^2x^4 - \kappa^2\Big)\psi. \tag{1.5}$$

In this work we shall analyze the asymptotic behavior of the solutions of (1.5) for large t.

We assume $J > 0$ in the sequel. Otherwise we may either consider the complex conjugate of (1.4) or apply the transformation $y \to -y$. Hence, we can rescale x and t by applying

$$t \to (\kappa J)^{2/3}t, \quad (x,y) \to (\kappa J)^{1/3}(x,y)\,, \tag{1.6}$$

yielding

$$\partial_t u = -(\mathcal{A}_{0,c} - \lambda)u\,, \tag{1.7}$$

where $\mathcal{A}_{0,c}$ is the differential operator defined by

$$\mathcal{A}_{0,c} := -\partial_{xx} - (\partial_y + \frac{i}{2}x^2)^2 + icy\,, \tag{1.8}$$

and

$$c = \frac{\kappa^2}{\sigma}, \quad \lambda = \frac{\kappa^{4/3}}{J^{2/3}}, \quad u(x,y,t) = \psi((\kappa J)^{-1/3}x, (\kappa J)^{-1/3}y, (\kappa J)^{-2/3}t)\,.$$

Our main objective in this work is to analyze the long time behavior of the semi-group associated with $\mathcal{A}_{0,c}$. If one assumes that the magnetic field induced by the current is negligible, the following simplified elliptic operator is obtained from (1.8)

$$\mathcal{B} = -\Delta + icy\,.$$

In [3] it is shown that one can obtain the spectral properties of $\mathcal{B}$ in large two or three-dimensional bounded domains, by analyzing first a pair of one-dimensional problems for functions which depend on y only and involve the so called complex Airy operator

$$D_y^2 + icy$$

on $\mathbb{R}$ or $\mathbb{R}_+$. Obviously, for the present operator (1.8), it is impossible to find a meaningful one-dimensional setting which could teach us anything about the properties of $\mathcal{A}_{0,c}$. Thus, we have to discuss two-dimensional settings as our basic problems, imposing a significant complication on the spectral analysis. We shall therefore confine the present discussion to samples capturing the whole plane. We leave the analysis of the effect of boundaries to future researches.

Applying to (1.7) the transformation

$$u \to u\, e^{icyt}$$

yields

$$\partial_t u = \partial_{xx} u + \left(\partial_y - i(\frac{1}{2}x^2 + ct)\right)^2 u + \lambda u\,. \tag{1.9}$$

Note that by applying the partial Fourier transform in y,

$$\hat{u}(x,\omega,t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\omega y} u(x,y,t)\, dy\,, \tag{1.10}$$

to (1.9) we obtain

$$\partial_t \hat{u} = \partial_{xx} \hat{u} - \left[\left(\frac{1}{2}x^2 + (ct - \omega) \right)^2 - \lambda \right] \hat{u}\,. \tag{1.11}$$

The above can in turn be rewritten as a family (depending on $\omega \in \mathbb{R}$) of time-dependent problems on $\mathbb{R}$

$$\partial_t \hat{u} = -\mathcal{L}_{\beta(t,\omega)} \hat{u} + \lambda \hat{u}\,, \tag{1.12}$$

with $\mathcal{L}_\beta$ being the well-known anharmonic oscillator [22] :

$$\mathcal{L}_\beta = -\partial_{xx} + (\frac{1}{2}x^2 + \beta)^2\,, \tag{1.13}$$

and

$$\beta(t,\omega) = ct - \omega\,.$$

From this point we may proceed by translating t by $-\omega/c$, i.e. by setting

$$\tau = t - \frac{\omega}{c}, \qquad v(x,\tau) = \hat{u}(x,t) \tag{1.14}$$

to obtain from (1.12) the following

$$\partial_\tau v(x,\tau) = -(\mathcal{L}_{c\tau} v)(x,\tau) + \lambda v(x,\tau)\,. \tag{1.15}$$

The initial condition at $t = 0$ is then prescribed at

$$\tau = -\frac{\omega}{c}.$$

Hence the dependence on ω appears only through the time at which the initial condition is set up.

The main result of this work is that all solutions of (1.5) decay exponentially fast as $t \to \infty$, for every $J \neq 0$. This means that the normal state $\psi \equiv 0$ is stable even for very weak currents. The simplest form in which this fact is being displayed in this contribution is (4.2). We bring it here in terms of the physical variables and parameters in (1.5):

$$\|\psi(t)\|_{L^2(\mathbb{R}^2)} \leq \|\psi(t=0)\|_{L^2(\mathbb{R}^2)} \exp\Big(-\frac{2\sqrt{2}}{3} \frac{\kappa^2 J}{\sigma^{1/2}} t^{3/2} + \kappa^2 t + C t^{3/4} \Big)\,, \tag{1.16}$$

where C is an appropriately chosen positive constant.

The rest of this contribution is arranged as follows. In Section 2, we derive some of the basic properties of the non-selfadjoint operator introduced in (1.8). In Section 3 we review some semi-classical properties of the anharmonic operator (1.13) for large β, which will play a key role in our analysis. In Section 4 we obtain the long-time asymptotic behaviour of solutions of (1.7) as shown in (1.16), and then apply the estimates obtained there to control the norm of the resolvent. In Section 5, we propose a finer analysis of (1.7) which allows us in particular to obtain a lower bound for the norm of the resolvent.

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2. Basic Properties of $\mathcal{A}_{0,c}$

2.1. **Preliminaries and maximal accretiveness.** Since some of the methods applied in this section can deal with classes of operators more general than (1.8), we state our results more generally, and then apply them to $\mathcal{A}_{0,c}$. Denote then by $L^2(\mathbb{R}^n)$ the space of real valued functions with finite L^2 norm, and $L^2(\mathbb{R}^n,\mathbb{C})$ the complex valued functions. To simplify notation we use $\|u\|_{L^2(\mathbb{R}^n)}$ to denote the norms in both spaces and

$$\langle u,v\rangle = \int_{\mathbb{R}^n} u\bar{v}dx$$

to denote the inner product in them. Let $\mathbf{A}=(A_1,\cdots,A_n)\in C^\infty(\mathbb{R}^n,\mathbb{R}^n)$ and $V\in C^\infty(\mathbb{R}^n,\mathbb{C})$. Define

$$\nabla_{\mathbf{A}}^2=\sum_{j=1}^n(\partial_{x_j}-iA_j)^2,\qquad P_{\mathbf{A},V}=-\nabla_{\mathbf{A}}^2+V(x). \tag{2.1}$$

Then $\nabla_{\mathbf{A},}^2$ and $P_{\mathbf{A},V}$ are well-defined in $C_c^\infty(\mathbb{R}^n,\mathbb{C})$. The operator $P_{\mathbf{A},V}$ with domain $C_0^\infty(\mathbb{R}^n)$ is closable and hence we can define

$$\mathcal{P}=\overline{P_{\mathbf{A},V}} \tag{2.2}$$

as its closure. By the definition, the domain of $\mathcal{P}$, $D(\mathcal{P})$, is the closure of $C_0^\infty(\mathbb{R}^n,\mathbb{C})$ under the graph norm

$$u\mapsto(\|u\|^2_{L^2(\mathbb{R}^n)}+\|\mathcal{P}u\|^2_{L^2(\mathbb{R}^n)})^{1/2}.$$

We study here the mapping properties of this operator and its spectrum. The operator $\mathcal{A}_{0,c}$, introduced in (1.8), clearly belongs to the class (2.1) via the particular choice

$$n=2\,,\ A_1(x,y)=0\,,\ A_2(x,y)=-\frac{x^2}{2}\,,\ V(x,y)=icy\,,\ c\in\mathbb{R}\setminus\{0\}\,. \tag{2.3}$$

Hence, we can derive from the spectral properties of $\mathcal{P}$ the spectral properties of

$$\mathcal{A}=\overline{\mathcal{A}_{0,c}}\,. \tag{2.4}$$

Semi-boundedness.

Clearly, for all $u\in C_0^\infty(\mathbb{R}^n,\mathbb{C})$, we have

$$\langle P_{\mathbf{A},V}u,u\rangle=\langle-\nabla_{\mathbf{A}}^2u,u\rangle+\langle Vu,u\rangle=\|\nabla_{\mathbf{A}}u\|^2_{L^2(\mathbb{R}^n)}+\int_{\mathbb{R}^n}V|u|^2dx\,. \tag{2.5}$$

Assuming on $\operatorname{Re}V\geq 0$, we can introduce the bottom of the numerical range

$$E^*=\inf_{u\in C_c^\infty(\mathbb{R}^n,\mathbb{C})\setminus\{0\}}\frac{\operatorname{Re}\langle P_{\mathbf{A},V}u,u\rangle}{\|u\|^2_{L^2(\mathbb{R}^n)}}. \tag{2.6}$$

Note that E^* is the bottom of the spectrum of $\frac{1}{2}(\mathcal{P}+\mathcal{P}^*)$ (where $\mathcal{P}$ is given in (2.2) and $\mathcal{P}^*$ is the conjugate of $\mathcal{P}$), which is the bottom of the spectrum of the (unique) selfadjoint realization of $-\nabla^2_{\mathbf{A}}+\operatorname{Re}V$:

$$E^* := \inf \sigma(\overline{P_{\mathbf{A},\operatorname{Re}V}}) \geq 0\,. \tag{2.7}$$

In the particular case where (2.3) is satisfied, E^* is the bottom of the selfadjoint operator $D_x^2 + (D_y + \frac{x^2}{2})^2$. As we shall see in the next section, we have in this case

$$E^* = \inf_{\beta\in\mathbb{R}} \inf \sigma\Big(D_x^2 + (\beta + \frac{x^2}{2})^2\Big)\,. \tag{2.8}$$

Theorem 2.1. *Consider the magnetic Schrödinger operator $\mathcal{P}$ defined in (2.2) on $\mathbb{R}^n$ with $\mathbf{A}\in C^\infty(\mathbb{R}^n,\mathbb{R}^n)$ and $V\in C^\infty(\mathbb{R}^n,\mathbb{C})$ such that*

$$\operatorname{Re}V(x)\geq 0\,. \tag{2.9}$$

Then the operator $\mathcal{P}-E^*$ is maximally accretive. Moreover

$$\mathcal{P} = (P_{\mathbf{A},\bar V})^*\,. \tag{2.10}$$

The proof of this theorem is given in the next subsection.

2.2. **Proof of maximal accretiveness.** By the definition of E^*, $P_{\mathbf{A},V}-E^*$ is positive on $C_0^\infty(\mathbb{R}^n)$. Hence the main point for the proof of Theorem 2.1 is to apply Lumer-Phillips theorem for closable operators (see [8, theorem 8.3.5]) to $P_{\mathbf{A},V}-E^*$ and $P_{\mathbf{A},\bar V}-E^*$. By this theorem, if the range of $(P_{\mathbf{A},V}-E^*+\gamma)$ is dense in $L^2(\mathbb{R}^n,\mathbb{C})$ for some $\gamma>0$, then $\mathcal{P}-E^*$ is the generator of a one-parameter contracting semigroup, and hence also maximally accretive. This reduces the proof of Theorem 2.1 to the following proposition:

Proposition 2.2. *Under the condition* (2.9)*, for any $\gamma>0$, $(P_{\mathbf{A},V}-E^*+\gamma)\,(C_0^\infty(\mathbb{R}^n,\mathbb{C}))$ is dense in $L^2(\mathbb{R}^n,\mathbb{C})$.*

Proof. Changing V into $V-E^*$ if necessary, we can assume that $E^*=0$. We rely on a proof given in [12] to a similar statement appearing in the proof of essential selfadjointness of $P_{\mathbf{A},V}$ in the case when V is real. Suppose that $f\in L^2(\mathbb{R}^n,\mathbb{C})$ is such that

$$\langle f,(P_{\mathbf{A},V}+\gamma)u\rangle = 0 \quad \text{for all } u\in C_0^2(\mathbb{R}^n,\mathbb{C}). \tag{2.11}$$

Proving that $f\equiv 0$ would achieve our goal.

We first observe that (2.11) implies that

$$(-\nabla^2_{\mathbf{A}}+\bar V+\gamma)f=0$$

in the sense of distributions. Standard elliptic regularity theory for the Laplacian (with our assumptions on V and $\mathbf{A}$ in mind) implies then that $f\in H^2_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{C})$. We now introduce a family of cut-off functions, ζ_k , by

$$\zeta_k(x) := \zeta(\frac{x}{k}), \quad \text{for all } k\in\mathbb{N},$$

where $\zeta\in C_0^\infty(\mathbb{R}^n,\mathbb{C})$ satisfies $0\leq\zeta\leq 1$, $\zeta=1$ on the unit ball $B_1(0)$ and $\operatorname{Supp}\zeta\subset \mathrm{B}_2(0)$, where $B_R(x_0)$ denotes the ball with center at x_0 and radius R. For any $u\in$

$C_0^\infty(\mathbb{R}^n,\mathbb{C})$ we have the identity

$$\begin{aligned}&\int_{\mathbb{R}^n}\{\nabla_{\mathbf{A}}(\zeta_k f)\cdot\overline{\nabla_{\mathbf{A}}(\zeta_k u)}+\zeta_k^2\bar{V}f\bar{u}\}dx\\ =&\langle f,(P_{\mathbf{A},V}+\gamma)(\zeta_k^2u)\rangle+\int_{\mathbb{R}^n}\{|\nabla\zeta_k|^2f\bar{u}+\zeta_k\nabla\zeta_k\cdot[f\overline{\nabla_{\mathbf{A}}u}-\bar{u}\nabla_{\mathbf{A}}f]\}dx.\end{aligned}\tag{2.12}$$

Since f satisfies (2.11) we obtain

$$\begin{aligned}&\int_{\mathbb{R}^n}\{\nabla_{\mathbf{A}}(\zeta_k f)\cdot\overline{\nabla_{\mathbf{A}}(\zeta_k u)}+\zeta_k^2(\bar{V}+\gamma)f\bar{u}\}dx\\ =&\int_{\mathbb{R}^n}\{|\nabla\zeta_k|^2f\bar{u}+\zeta_k\nabla\zeta_k\cdot[f\overline{\nabla_{\mathbf{A}}u}-\bar{u}\nabla_{\mathbf{A}}f]\}dx\,.\end{aligned}$$

This formula can be extended by continuity to all functions $u\in H^1_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{C})$. In particular, letting $u=f$ and taking the real part of the above identity we obtain

$$\|\nabla_{\mathbf{A}}(\zeta_k f)\|^2_{L^2(\mathbb{R}^n)}+\int_{\mathbb{R}^n}\zeta_k^2(\operatorname{Re}V+\gamma)|f|^2\,dx=\int_{\mathbb{R}^n}|\nabla\zeta_k|^2|f|^2\,dx\,,$$

hence, by (2.9),

$$\gamma\int_{\mathbb{R}^n}\zeta_k^2|f|^2\,dx\le\int_{\mathbb{R}^n}|\nabla\zeta_k|^2|f|^2\,dx\,.$$

Using this and the definition of ζ_k, and taking the limit $k\to\infty$, we obtain

$$\gamma\|f\|^2_{L^2(\mathbb{R}^n)}=\gamma\lim_{k\to\infty}\|\zeta_k f\|^2\le\limsup_{k\to\infty}\int_{\mathbb{R}^n}|\nabla\zeta_k|^2|f|^2\,dx=0\,,$$

furnishing, thereby, the density of the range of $P_{\mathbf{A},V}+\gamma$ in $L^2(\mathbb{R}^n,\mathbb{R}^n)$. ∎

Theorem 2.1 permits us to apply some results of semi-group theory and of the theory of maximally accretive operators. We refer to [8] for a recent presentation of the first theory (and particularly Theorem 8.3.5) and to [18, Theorem 5.4] for the second. The following proposition gives a simple description of the domain $D(\mathcal{P})$ of $\mathcal{P}$ (see the definition of $D(\mathcal{P})$ given at the beginning of Section 2).

Proposition 2.3. *Let $\mathcal{P}$ be the operator defined in* (2.2) *and $D(\mathcal{P})$ be the domain of $\mathcal{P}$. Then,*

$$D(\mathcal{P})=\{u\in L^2(\mathbb{R}^n,\mathbb{C}):\ P_{\mathbf{A},V}u\in L^2(\mathbb{R}^n,\mathbb{C})\}.\tag{2.13}$$

We now observe that (2.6) implies by Hille-Yosida Theorem (or Theorem 12.8 in [2]) that, for λ such that $\operatorname{Re}\lambda>-E^*$, the operator $\mathcal{P}+\lambda:D(\mathcal{P})\to L^2(\mathbb{R}^n,\mathbb{C})$ is an isomorphism and

$$\|(\mathcal{P}+\lambda)^{-1}\|\le(\operatorname{Re}\lambda+E^*)^{-1}.\tag{2.14}$$

2.3. **Compactness of the resolvent.** Although more general results have been obtained for example in [17] for the selfadjoint case, in [19] for the polynomial case, or in [18] for the case of Fokker-Planck operators, the next criterion is sufficient for all purpose we have in mind.

Proposition 2.4. *Let* $\mathbf{A} \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ *and let* $V \in C^\infty(\mathbb{R}^n, \mathbb{C})$ *satisfy* (2.9). *Let* $\mathbf{B} = \mathrm{curl}\mathbf{A}$ *and*

$$m_{\mathbf{B},V}(x) := \sqrt{|\mathbf{B}(x)|^2 + |V(x)|^2 + 1}. \tag{2.15}$$

Suppose that

$$\lim_{|x|\to\infty} m_{\mathbf{B},V}(x) = +\infty, \tag{2.16}$$

and that either

(i) $\mathrm{Im}\, V$, *and* $B_{k\ell}$ *do not change sign in* $\mathbb{R}^n$ *for all* $1 \leq k, \ell \leq n$.

or

(ii) *there exists a constant* C_0 *such that for all* $x \in \mathbb{R}^n$

$$|\nabla V(x)| + \sum_{i,k,\ell} |\partial_i B_{k\ell}(x)| \leq C_0\, m_{\mathbf{B},V}(x)\,. \tag{2.17}$$

Then, $\mathcal{P}$ *has a compact resolvent.*

Proof. It is enough to show the existence of a constant C such that, for all $u \in C_0^\infty(\mathbb{R}^n, \mathbb{C})$,

$$\int_{\mathbb{R}^n} m_{\mathbf{B},V}(x)|u(x)|^2 dx \leq C\{\|P_{\mathbf{A},V}u\|^2_{L^2(\mathbb{R}^n)} + \|u\|^2_{L^2(\mathbb{R}^n)}\}. \tag{2.18}$$

In the following we write $m(x)$ for $m_{\mathbf{B},V}(x)$ and write $\mathrm{Re}\, V = V_1$, $\mathrm{Im}\, V = V_2$ in order to simplify the notation. Using the condition (2.9) we have, for all $u \in C_0^\infty(\mathbb{R}^n, \mathbb{C})$,

$$\int_{\mathbb{R}^n} \{|\nabla_{\mathbf{A}} u|^2 + |V_1||u|^2\} dx = \mathrm{Re}\, \langle P_{\mathbf{A},V} u\,,\, u\rangle \leq \frac{1}{2}\{\|P_{\mathbf{A},V}u\|^2_{L^2(\mathbb{R}^n)} + \|u\|^2_{L^2(\mathbb{R}^n)}\}. \tag{2.19}$$

In the following we estimate the integrals

$$\int_{\mathbb{R}^n} |V_2||u|^2 dx \quad \text{and} \quad \int_{\mathbb{R}^n} |B_{k\ell}||u|^2 dx\,,\ \forall (k,\ell) \in \{1,\dots,n\} \times \{1,\dots,n\}\,.$$

Suppose first that V_2 has constant sign. In this case we immediately have, for all $u \in C_0^\infty(\mathbb{R}^n, \mathbb{C})$,

$$\int_{\mathbb{R}^n} |V_2||u|^2 dx \leq \big|\langle P_{\mathbf{A},V} u\,,\, u\rangle\big| \leq \frac{1}{2}\{\|P_{\mathbf{A},V}u\|^2_{L^2(\mathbb{R}^n)} + \|u\|^2_{L^2(\mathbb{R}^n)}\}. \tag{2.20}$$

Without this sign assumption, we use integration by parts to obtain

$$\mathrm{Im}\, \langle P_{\mathbf{A},V} u\,,\, m^{-1}V_2 u\rangle = \mathrm{Im} \int_{\mathbb{R}^n} (\nabla - i\mathbf{A})u \cdot \nabla(m^{-1}V_2)\bar{u}\, dx + \int_{\mathbb{R}^n} m^{-1}V_2^2 |u|^2 dx,$$

and hence

$$\begin{aligned} &\mathrm{Im}\, \langle P_{\mathbf{A},V} u\,,\, m^{-1}V_2 u\rangle \\ &\geq \int_{\mathbb{R}^n} m^{-1}V_2^2|u|^2 dx - \sup_{x\in\mathbb{R}^n} |\nabla(m^{-1}V_2)| \|u\|_{L^2(\mathbb{R}^n)} \|\nabla_{\mathbf{A}} u\|_{L^2(\mathbb{R}^n)}\,. \end{aligned}$$

Using (2.17) and (2.19), we then obtain, for all $u \in C_0^\infty(\mathbb{R}^n, \mathbb{C})$,

$$\int_{\mathbb{R}^n} |V_2||u|^2 dx \leq \int_{\mathbb{R}^n} m^{-1} V_2^2 |u|^2 dx \leq C\{\|P_{\mathbf{A},V} u\|^2_{L^2(\mathbb{R}^n)} + \|u\|^2_{L^2(\mathbb{R}^n)}\}, \tag{2.21}$$

where C depends on the constant C_0 in (2.17).

To control the magnetic field part, suppose again first that a component $B_{k\ell}$ has a constant sign. In this case we use the operator identity ($[\cdot,\cdot]$ being the Poisson bracket)

$$B_{k\ell} = i[\partial_{x_k} - iA_k, \partial_{x_\ell} - iA_\ell]\,, \tag{2.22}$$

to obtain that

$$\begin{aligned}\int_{\mathbb{R}^n} \big|B_{k\ell}\big||u|^2\, dx \leq & 2\|(\partial_{x_\ell} - iA_\ell)u\|_{L^2(\mathbb{R}^n)} \cdot \|(\partial_{x_k} - iA_k)u\|_{L^2(\mathbb{R}^n)} \\ & \leq |\,\langle P_{\mathbf{A},V} u\,,\, u\rangle\,| \leq \frac{1}{2}(\|P_{\mathbf{A},V} u\|^2_{L^2(\mathbb{R}^n)} + \|u\|^2_{L^2(\mathbb{R}^n)}).\end{aligned} \tag{2.23}$$

In case *(ii)*, we use (2.22) once again to obtain

$$\begin{aligned}\int_{\mathbb{R}^n} & m^{-1} B_{k\ell}^2 |u|^2\, dx = i \int_{\mathbb{R}^n} \{[\partial_{x_k} - iA_k, \partial_{x_\ell} - iA_\ell]u\} \cdot (m^{-1} B_{k\ell} \bar{u})\, dx \\ =& i \int_{\mathbb{R}^n} m^{-1} B_{kl} \{[(\partial_{x_l} - iA_l)u]\overline{[(\partial_{x_k} - iA_k)u]} - [(\partial_{x_k} - iA_k)u]\overline{[(\partial_{x_l} - iA_l)u]}\} dx \\ & + i \int_{\mathbb{R}^n} \bar{u}\{[(\partial_{x_l} - iA_l)u]\partial_{x_k}(m^{-1}B_{kl}) - [(\partial_{x_k} - iA_k)u]\partial_{x_l}(m^{-1}B_{kl})\} dx \\ & \leq C\Big\{\|(\partial_{x_\ell} - iA_\ell)u\|_{L^2(\mathbb{R}^n)} \|(\partial_{x_k} - iA_k)u\|_{L^2(\mathbb{R}^n)} \\ & \quad + (\|(\partial_{x_\ell} - iA_\ell)u\|_{L^2(\mathbb{R}^n)} + \|(\partial_{x_k} - iA_k)u\|_{L^2(\mathbb{R}^n)})\|u\|_{L^2(\mathbb{R}^n)} \sup_{x \in \mathbb{R}^n} |\nabla(m^{-1} B_{k\ell})|\Big\}.\end{aligned}$$

As before this and the condition (2.17) lead to

$$\int_{\mathbb{R}^n} |B_{k\ell}||u|^2\, dx \leq \int_{\mathbb{R}^n} m^{-1} B_{k\ell}^2 |u|^2\, dx \leq C\{\|P_{\mathbf{A},V} u\|^2_{L^2(\mathbb{R}^n)} + \|u\|^2_{L^2(\mathbb{R}^n)})\,, \tag{2.24}$$

for all $u \in C_0^\infty(\mathbb{R}^n)$. Combining (2.19), (2.20), (2.21), (2.23), (2.24), we get (2.18). ∎

Note that the same proof gives

Proposition 2.5. *Assume* $\mathbf{A} \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ *and* $V \in C^\infty(\mathbb{R}^n, \mathbb{C})$ *and the conditions* (2.9) *and* (2.17) *hold. Let* $\mathcal{P} = \overline{P_{\mathbf{A},V}}$. *Then we have*

$$D(\mathcal{P}) \subset H^1_{\mathbf{A},V}(\mathbb{R}^n, \mathbb{C}) := \{u \in L^2(\mathbb{R}^n, \mathbb{C})\,,\, \nabla_{\mathbf{A}} u \in L^2(\mathbb{R}^n, \mathbb{C}^n)\,, |V|^{1/2} u \in L^2(\mathbb{R}^n, \mathbb{C})\}\,, \tag{2.25}$$

and there exists a constant C *depending on the constant* C_0 *in* (2.17) *such that*

$$\|\nabla_{\mathbf{A}} u\|^2_{L^2(\mathbb{R}^n)} + \int_{\mathbb{R}^n} |V(x)||u(x)|^2\, dx \leq C\{\|P_{\mathbf{A},V} u\|^2_{L^2(\mathbb{R}^n)} + \|u\|^2_{L^2(\mathbb{R}^n)}\}, \quad \forall u \in D(\mathcal{P})\,. \tag{2.26}$$

Corollary 2.6. *Let* $\mathcal{A} = \overline{\mathcal{A}_{0,c}}$ *be the operator on* $\mathbb{R}^2$ *defined in* (2.4) *where* $c \neq 0$. *Then* $\sigma(\mathcal{A}) = \emptyset$.

Proof. From Proposition 2.4 we see that $\mathcal{A}$ has a compact resolvent. Hence it has a discrete spectrum. For some $a \in \mathbb{R}$, we now introduce the translation operator T_a defined by

$$(T_a u)(x,y) = u(x, y-a),$$

to obtain :

$$T_a \mathcal{A} = (\mathcal{A} + ia) T_a. \tag{2.27}$$

Consequently, if $\sigma(\mathcal{A}) \neq \emptyset$ then $\sigma(\mathcal{A})$ is not discrete. ∎

Note that the same result holds for the complex Airy operator $D_x^2 + ix$ on $\mathbb{R}$, for which the emptiness of the spectrum has been established, using various techniques, in several other contributions including [3, 8, 21].

2.4. $L^\infty(\mathbb{R}^n)$ **spectral pairs.** We now move to consider bounded generalized eigenfunctions of the operator $P_{\mathbf{A},V}$. We say that (ψ, λ) is an L^∞-spectral pair for the operator $P_{\mathbf{A},V}$ if $\lambda \in \mathbb{C}$ and $\psi \in L^\infty(\mathbb{R}^n, \mathbb{C}) \setminus \{0\}$ is a solution, in the sense of distributions, of

$$(P_{\mathbf{A},V} - \lambda)\psi = 0\,. \tag{2.28}$$

Theorem 2.7. *Assume $\mathbf{A} \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ and $V \in C^\infty(\mathbb{R}^n, \mathbb{C})$ and the conditions* (2.9) *and* (2.17) *hold. If (ψ, λ) is an L^∞-spectral pair of the operator $P_{\mathbf{A},V}$, then $\lambda \in \sigma(\mathcal{P})$, where $\mathcal{P} = \overline{P_{\mathbf{A},V}}$.*

Proof. Let (ψ, λ) denote an L^∞-spectral pair of the operator $P_{\mathbf{A},V}$, and suppose $\lambda \notin \sigma(\mathcal{P})$. We shall derive a contradiction.

As the operator $P_{\mathbf{A},V}$ is elliptic and $\mathbf{A}$ and V are of class of C^∞, it is clear that ψ is a C^∞-function. The proof is reminiscent of the so-called Schnol's theorem [7]. Consider, for $R \geq 1$, a family of cut-off functions

$$\chi_R = \chi(\frac{x}{R})\,,$$

with χ being a non-negative smooth function satisfying $\chi = 1$ on the ball of radius 1, and with compact support in a ball of radius 2. Set $\psi_R := \chi_R \psi$. It is clear from the assumption that ψ_R belongs to $L^2(\mathbb{R}^n, \mathbb{C})$, with

$$\|\psi_R\|_{L^2(\mathbb{R}^n)} \leq C\, R^{n/2}\, \|\psi\|_{L^\infty(\mathbb{R}^n)}. \tag{2.29}$$

We shall now show that if $\lambda \notin \sigma(\mathcal{P})$, then there exist $k > 0$ and $C_k > 0$ depending on λ such that, for all $R \geq 1$, we have

$$\|\psi_R\|_{L^2(\mathbb{R}^n)} \leq C\, R^{-k}\,. \tag{2.30}$$

Once the above inequality is proved, letting $R \to +\infty$ will lead to a contradiction.

Proof of (2.30). From (2.29) we see that there exist $k_0 \in \mathbb{R}$ and $C_0 > 0$ such that (2.30) holds for $k = k_0$ and $C = C_0$. We now show that there exists C_1 such that (2.30) holds for $k = k_0 + 1$ and $C = C_1$. To this end we first observe that $\chi_R \psi$ satisfies the following equation in the sense of distributions

$$(P_{\mathbf{A},V} - \lambda)(\chi_R \psi) = -2(\nabla \chi_R) \cdot (\nabla_{\mathbf{A}} \psi) - (\Delta \chi_R)\psi\,. \tag{2.31}$$

It is immediately seen (using the local regularity of $P_{\mathbf{A},V}$) that the right hand side is in $L^2(\mathbb{R}^n,\mathbb{C})$ and consequently that $\psi_R \in D(\mathcal{P})$. To obtain an effective estimate for the L^2-norm of this right hand side we take the scalar product of (2.31) with $\chi_R\psi$, integrating by parts, and then taking real part, to obtain

$$\operatorname{Re}\langle (P_{\mathbf{A},V}-\lambda)(\chi_R\psi),\chi_R\psi\rangle = \|(\nabla\chi_R)\psi\|^2_{L^2(\mathbb{R}^n)}. \tag{2.32}$$

Using the assumption that $\operatorname{Re}V \geq 0$, and the fact that $\nabla\chi_R$ is supported in a ball with radius $2R$, we have

$$\begin{aligned}&\int_{\mathbb{R}^n} |\nabla_{\mathbf{A}}\psi_R|^2 dx \leq \|(\nabla\chi_R)\psi\|^2_{L^2(\mathbb{R}^n)} + |\operatorname{Re}\lambda|\|\psi_R\|^2_{L^2(\mathbb{R}^n)}\\ \leq& C'R^{-2}\|\psi_{2R}\|^2_{L^2(\mathbb{R}^n)} + |\operatorname{Re}\lambda|\|\psi_R\|^2_{L^2(\mathbb{R}^n)} \leq C'R^{-2}C_0(2R)^{-2k_0} + |\operatorname{Re}\lambda|C_0^2R^{-2k_0}.\end{aligned}$$

Hence we can find positive constants C and $\hat{C}$ depending on λ, such that for any $R \geq 1$

$$\|\nabla_{\mathbf{A}}(\chi_R\psi)\|_{L^2(\mathbb{R}^n)} \leq C_0CR^{-(k_0+1)} + |\operatorname{Re}\lambda|C_0R^{-k_0} \leq \hat{C}\,R^{-k_0}\,. \tag{2.33}$$

Then, we have from (2.31) and (2.33) (with R replaced by $3R$)

$$\|(\mathcal{P}-\lambda)(\chi_R\psi)\|_{L^2(\mathbb{R}^n)} \leq CR^{-(k_0+1)}\,. \tag{2.34}$$

By the assumption λ is not in the spectrum of $\mathcal{P}$, so there exists $C_1 > 0$ depending on λ such that (2.30) is satisfied for $k = k_0+1$ and $C = C_1$.

Thus, we can repeat the above argument to show that there exists a constant C_2 depending on λ such that (2.30) holds for $k = k_0+2$ and $C = C_2$. After a finite number of iterations we reach the conclusion that (2.30) holds form some $k > 0$. ∎

Corollary 2.8. *For any $c \neq 0$, there is no L^∞ spectral pair (ψ,λ) for $\mathcal{A}_{0,c}$.*

Proof. We have indeed $\sigma(\mathcal{A}) = \emptyset$ for $\mathcal{A} = \overline{A_{0,c}}$ as proven in Corollary 2.6. Combining this with Theorem 2.7 we see that no L^∞ spectral pair for $\mathcal{A}_{0,c}$ exists. ∎

3. The Anharmonic Oscillator

In this section we consider the operator

$$\mathcal{L}_\beta = -\frac{d^2}{dx^2} + \Big(\frac{1}{2}x^2+\beta\Big)^2, \tag{3.1}$$

which is the well-known anharmonic oscillator [22, 26]. We consider its selfadjoint realization on $L^2(\mathbb{R})$. It is well known that this operator has a compact resolvent, and we are particularly interested in the limiting behaviour as $\beta \to \pm\infty$ of its discrete spectrum.

3.1. **Large $|\beta|$ asymptotics of the first eigenvalues.** The next proposition lists a few properties of the eigenvalues and the eigenfunctions of $\mathcal{L}_\beta$ in $L^2(\mathbb{R})$ in the limit $\beta \to -\infty$.

Proposition 3.1. *Let $\{E_k^{(1)}(\beta)\}_{k=0}^\infty$ (respectively $\{E_k^{(2)}(\beta)\}_{k=0}^\infty$) denote the eigenvalues of $\mathcal{L}_\beta$ corresponding to the even (respectively odd) spectrum and $\{\phi_k^{(1)}(x,\beta)\}_{k=0}^\infty$ (respectively $\{\phi_k^{(2)}(x,\beta)\}_{k=0}^\infty$) denote their corresponding eigenmodes in $L^2(\mathbb{R})$, i.e.,*

$$\mathcal{L}_\beta \phi_k^{(\ell)} = E_k^{(\ell)} \phi_k^{(\ell)} \quad \text{in } \mathbb{R}\,, \tag{3.2}$$

with $\|\phi_k^{(\ell)}\|_{L^2(\mathbb{R})} = 1$, $\ell = 1,2$. Then we have the following conclusions.

(i) *For all $S \in (0, \frac{4\sqrt{2}}{3})$ and $k \geq 0$, there exist $C_k > 0$ and $\beta_k < 0$ such that, for $\beta \leq \beta_k$ it holds that*

$$0 \leq E_k^{(2)}(\beta) - E_k^{(1)}(\beta) \leq C_k e^{-S|\beta|^{3/2}}.$$

(ii) *For any $k \geq 0$, we have, for sufficiently large $-\beta$ and for $\ell = 1,2$,*

$$|E_k^{(\ell)}(\beta) - (2k+1)\sqrt{-2\beta}| \leq \frac{C_k}{|\beta|}\,, \tag{3.3}$$

and

$$\big|(E_k^{(\ell)})'(\beta)\big| \leq \frac{C_k}{\sqrt{-\beta}}. \tag{3.4}$$

(iii) *For sufficiently large $-\beta$ and for all $k \geq 0$, $\ell = 1,2$, we can choose the $\phi_k^{(\ell)}$ such that*

$$\big\|\phi_k^{(\ell)}(x,\beta) - \frac{1}{\sqrt{2}}\big[|2\beta|^{1/8} h_k\big(|2\beta|^{1/4}(x - \sqrt{-2\beta})\big)\big]\big\|_{L^2(\mathbb{R})} \leq \frac{\hat{C}_k}{|\beta|^{3/4}}, \tag{3.5}$$

where

$$h_k(x) = H_k(x) e^{-x^2/2},$$

and H_k is the k'th Hermite polynomial, normalized by the condition $\|h_k\|_{L^2(\mathbb{R})} = 1$, and $\hat{C}_k$ depends only on k.

Proof. We omit from now on the superscript (ℓ) to simplify the notation. The statements in the sequel (in this specific limit) are equally true for both the even and the odd cases.

Except for (3.4) all the statements of the proposition have been proved in [13, 14, 26] (sometimes in a refined way, see [16, Chapters 2-4]).

Note that after an appropriate dilation we arrive at a standard semi-classical problem with the semiclassical parameter

$$\hbar = |\beta|^{-3/2}.$$

More precisely, by introducing the new coordinate

$$\tilde{x} = \beta^{-1/2} x,$$

we obtain the operator

$$\beta^2\big(-\hbar^2 \frac{d^2}{d\tilde{x}^2} + W(\tilde{x})\big), \quad \text{with} \quad W(\tilde{x}) = (\frac{1}{2}\tilde{x}^2 - 1)^2.$$

To prove (3.4), we take the derivative of (3.2) with respect to β and obtain

$$(\mathcal{L}_\beta - E_k)\frac{\partial \phi_k}{\partial \beta} = \big(-(x^2+2\beta) + E_k'(\beta)\big)\phi_k.$$

Taking inner product with ϕ_k yields the following Feynman-Hellmann formula [11]:

$$E_k'(\beta) = \int_{\mathbb{R}} (x^2+2\beta)|\phi_k(x,\beta)|^2\,dx.$$

Then, (3.4) is readily verified with the aid of (3.5) and the decay properties of ϕ_k far from $x=\pm\sqrt{-2\beta}$. ∎

Similar approximations are valid in the limit $\beta \to +\infty$.

Proposition 3.2. *For the above-defined $\{E_k^{(\ell)}(\beta)\}_{k=0}^\infty$ and $\{\phi_k^{(\ell)}(x,\beta)\}_{k=0}^\infty$ we have, as $\beta \to +\infty$, the asymptotics*

$$\big|E_k^{(\ell)}(\beta) - \beta^2 - [2(2k+\ell)-1]\beta^{1/2}\big| \le \frac{C_k}{\beta}, \tag{3.6a}$$

and

$$\|\phi_k^{(\ell)}(x,\beta) - |\beta|^{1/8}h_{2k+\ell-1}(|\beta|^{1/4}x)\|_{L^2(\mathbb{R})} \le \frac{\hat{C}_k}{|\beta|^{3/2}}. \tag{3.6b}$$

In contrast with the case $\beta \to -\infty$, the Schrödinger operator (3.1) with $\beta > 0$ has a single well potential. The proof is again a standard application of semi-classical analysis (see [16, Section 3.4] for instance).

In the following we denote

$$E_0(\beta) = E_0^{(1)}(\beta),$$

which is the lowest eigenvalue of $\mathcal{L}_\beta$. By (3.3) and (3.6), and since $\mathcal{L}_\beta$ is positive-definite, it is clear that $E_0(\beta)$ has a strictly positive infimum E^*,

$$E^* = \inf_{\beta\in\mathbb{R}} E_0(\beta), \tag{3.7}$$

and that there exists at least one $\beta^* < 0$ such that

$$E^* = E_0(\beta^*)\,.$$

In [22] the values of E^* and β^* have been numerically computed. In [24] Pan and Kwek stated that β^* is unique. In a recent contribution Helffer [15] proved, using a different approach, that β^* is indeed unique and that in addition the minimum is non degenerate.

3.2. **Auxiliary functions of E_0.** For later use, we define some auxiliary quantities depending on the behavior of $E_0(\beta)$ as $|\beta| \to \infty$. For sufficiently large ρ the equation

$$E_0(\beta) = \rho, \tag{3.8}$$

has exactly two distinct solutions $\beta_-(\rho)$ and $\beta_+(\rho)$ satisfying

$$\beta_-(\rho) < 0 < \beta_+(\rho).$$

Moreover, we have the following asymptotics by (3.3) and (3.6)

$$\beta_-(\rho) \sim -\frac{\rho^2}{2} + \mathcal{O}(\rho^{-1}), \qquad \beta_+(\rho) \sim \sqrt{\rho} + \mathcal{O}(\rho^{-1/4}) \quad \text{as } \rho \to +\infty. \tag{3.9}$$

We then define the natural quantity

$$\Psi(\rho) = \int_{\beta_-(\rho)}^{\beta_+(\rho)} E_0(\beta)\, d\beta\,. \tag{3.10}$$

By (3.3), (3.6), and (3.9) we have (cf. [23, section 1.2] or [5, section 3.8]), as $\rho \to +\infty$

$$\Psi(\rho) = \frac{1}{3}\big(\rho^3 + \rho^{3/2}\big) + \mathcal{O}(\ln \rho)\,. \tag{3.11}$$

We further define the quantity $\rho(\tau)$ as the solution of

$$\beta_+(\rho(\tau)) - \beta_-(\rho(\tau)) = \tau\,, \tag{3.12}$$

which has the asymptotics (by (3.9))

$$\rho(\tau) \sim \sqrt{2\tau} + \mathcal{O}(\tau^{-1/4}) \quad \text{as } \tau \to +\infty. \tag{3.13}$$

Then we set

$$\widehat{\Psi}(\tau) = \Psi(\rho(\tau)). \tag{3.14}$$

We observe from (3.11) and (3.13) that

$$\widehat{\Psi}(\tau) = \frac{2\sqrt{2}}{3}\tau^{3/2} + \mathcal{O}(\tau^{3/4}) \quad \text{as } \tau \to +\infty. \tag{3.15}$$

Finally, we define the function

$$L(\widehat{\Psi}, c, \mu) = \sup_{t\in\mathbb{R}}\big(\mu t - \frac{1}{c}\widehat{\Psi}(ct)\big). \tag{3.16}$$

Observe first that

$$L(\widehat{\Psi}, c, \mu) = \frac{1}{c} L(\widehat{\Psi}, 1, \mu). \tag{3.17}$$

Hence, in order to compute (3.16) as $\mu \to \infty$, we need to approximate only $L(\widehat{\Psi}, 1, \mu)$, which is called the Legendre transform of $\widehat{\Psi}$ (see [25, chapter 11]).

Using (3.10) and (3.14), together with a short computation, gives that for sufficiently large μ, when $c = 1$, the supremum in (3.16) is attained at $t = t(\mu)$ which is defined by

$$t(\mu) = \beta_+(\mu) - \beta_-(\mu),$$

and that

$$L(\widehat{\Psi}, c, \mu) = \frac{1}{c}\int_{\beta_-(\mu)}^{\beta_+(\mu)} (\mu - E_0(\beta))\, d\beta\,. \tag{3.18}$$

The asymptotics of $L(\widehat{\Psi}, c, \mu)$ for large μ can be derived by using (3.9), (3.10) and (3.11):

$$L(\widehat{\Psi}, c, \mu) = \frac{\mu^3}{6c} + \mathcal{O}(\mu^{3/2}) \quad \text{as } \mu \to +\infty. \tag{3.19}$$

4. Analysis on Time-Dependent Problems

4.1. **Long time decay.** We start this section by considering the long time behaviour of the solutions of (1.9). Using the properties of the operator $\mathcal{A}_{0,c}$ that were derived in section 2, we can apply semigroup theory to prove the *global* existence of solutions [8]. Our goal here is to improve the general results of semi-group theory using the more particular structure of the operator $\mathcal{A}$ in (2.4).

Let $u(x,y,t)$ be a solution of (1.9) in $\mathbb{R}^2$. We denote by $u(t)$ the one-parameter family of functions $(x,y) \to u(x,y,t)$. Their norm is given by

$$\|u(t)\|_{L^2(\mathbb{R}^2)} = \left(\int_{\mathbb{R}^2} |u(x,y,t)|^2 dxdy \right)^{1/2} .$$

Proposition 4.1. *Let $u(x,y,t)$ be a solution of* (1.9) *defined for all $t \geq 0$ such that*

$$u(x,y,0) = u_0(x,y) \in L^2(\mathbb{R}^2, \mathbb{C}).$$

Then there exists $T_0 > 0$ such that, for any $t \geq T_0$ and any $u_0 \in L^2(\mathbb{R}^2)$,

$$\|u(t)\|_{L^2(\mathbb{R}^2)} \leq \exp\Big(-\frac{1}{c}\widehat{\Psi}(ct) + \lambda t\Big)\, \|u_0\|_{L^2(\mathbb{R}^2)}, \tag{4.1}$$

with $\widehat{\Psi}$ introduced in (3.10) *and* (3.14). *As a consequence there exists a constant $C > 0$ such that for $t \geq T_0$ and any $u_0 \in L^2(\mathbb{R}^2, \mathbb{C})$,*

$$\|u(t)\|_{L^2(\mathbb{R}^2)} \leq \exp(-\frac{2\sqrt{2c}}{3} t^{3/2} + \lambda t + C t^{3/4}) \|u_0\|_{L^2(\mathbb{R}^2)}. \tag{4.2}$$

Proof. We prove (4.1) first for $u_0 \in \mathcal{S}(\mathbb{R}^2)$ (where $\mathcal{S}(\mathbb{R}^2)$ denotes the Schwartz space of the rapidly decreasing functions in x, y). The extension to all $u_0 \in L^2(\mathbb{R}^2, \mathbb{C})$ then follows by density. Thus, it is sufficient to prove (4.1) for the partial Fourier transform of u with respect to y which is denoted by

$$(x,\omega,t) \mapsto \hat{u}(x,\omega,t).$$

For given ω, we multiply (1.11) by $\bar{\hat{u}}(x,\omega,t)$ (the complex conjugate of $\hat{u}$), and integrate the resulting equality over $\mathbb{R}$ with respect to x to obtain

$$\frac{1}{2}\frac{d}{dt}\|\hat{u}(\cdot,\omega,t)\|^2_{L^2(\mathbb{R}_x)} = -\langle \hat{u}(\cdot,\omega,t)\,,\, \mathcal{L}_{ct-\omega}\hat{u}(\cdot,\omega,t)\rangle + \lambda \|\hat{u}(\cdot,\omega,t)\|^2_{L^2(\mathbb{R}_x)},$$

where $\mathcal{L}_\beta$ is defined by (3.1), and $\langle\cdot,\cdot\rangle$ denotes the $L^2(\mathbb{R}_x)$ inner product. Clearly,

$$\langle \hat{u}(\cdot,\omega,t), \mathcal{L}_{ct-\omega}\hat{u}(\cdot,\omega,t)\rangle \geq E_0(ct-\omega)\|\hat{u}(\cdot,\omega,t)\|^2_{L^2(\mathbb{R}_x)},$$

where $E_0(\beta) = E_0^{(1)}(\beta)$ is defined in section 2. Consequently, we have

$$\frac{d}{dt}\|\hat{u}(\cdot,\omega,t)\|^2_{L^2(\mathbb{R}_x)} \leq 2(\lambda - E_0(ct-\omega))\|\hat{u}(\cdot,\omega,t)\|^2_{L^2(\mathbb{R}_x)}.$$

Hence, it readily follows that, for any $t > t_0 \geq 0$,

$$\|\hat{u}(\cdot,\omega,t)\|^2_{L^2(\mathbb{R}_x)} \leq \|\hat{u}(\cdot,\omega,t_0)\|^2_{L^2(\mathbb{R}_x)} \exp\Big(2\big[\lambda(t-t_0) - \int_{t_0}^t E_0(cs-\omega)ds\big]\Big). \tag{4.3}$$

Setting $t_0 = 0$, integrating over ω, and making use of the Plancherel formula, we obtain

$$\|u(t)\|_{L^2(\mathbb{R}^2)} \leq \exp\Big(\lambda t - \inf_{\omega \in \mathbb{R}} \Phi(\omega, t)\Big) \|u_0\|_{L^2(\mathbb{R}^2)}, \tag{4.4}$$

with

$$\Phi(\omega, t) := \int_{-\omega/c}^{t-\omega/c} E_0(cs)\, ds = \frac{1}{c} \int_{-\omega}^{ct-\omega} E_0(\beta)\, d\beta. \tag{4.5}$$

Hence, it remains necessary to estimate the quantity

$$\inf_{\omega \in \mathbb{R}} \Phi(\omega, t). \tag{4.6}$$

In view of the asymptotic behaviour, as $|\beta| \to \infty$, of $E_0(\beta)$ it follows that, for any given t, the infimum (4.6) exists and must be attained at a point $\omega = \omega_1(t) \in \mathbb{R}$ such that

$$\frac{\partial}{\partial \omega} \Phi(\omega, t)\Big|_{\omega=\omega_1(t)} = 0,$$

implying that

$$E_0(-\omega_1) = E_0(ct - \omega_1).$$

Then we may use (3.8) to obtain for sufficiently large t that, the number $\rho = E_0(-\omega_1)$ satisfies

$$\beta_-(\rho) = -\omega_1, \quad \beta_+(\rho) = ct - \omega_1\,.$$

By these equalities, (4.5) and (3.10) we then have

$$\Phi(\omega_1, t) = \frac{1}{c} \int_{\beta_-(\rho)}^{\beta_+(\rho)} E_0(\beta) d\beta = \frac{\Psi(\rho)}{c}, \qquad \rho = E_0(-\omega_1). \tag{4.7}$$

However, as

$$\beta_+(\rho) - \beta_-(\rho) = ct\,,$$

$\rho = E_0(-\omega_1)$ is the solution of (3.12) for $\tau = ct$, so

$$E_0(-\omega_1) = \rho = \rho(ct).$$

So we obtain from (4.7) and (3.14) that

$$\Phi(\omega_1, t) = \frac{\Psi(\rho(ct))}{c} = \frac{\widehat{\Psi}(ct)}{c}\,,$$

which, together with (4.4), establishes (4.1).

Using (3.15), we obtain (4.2) as well. ∎

Coming back to (4.3), one can define a one-parameter semigroup

$$\mathcal{S}_t = \exp(-t\mathcal{A}) \tag{4.8}$$

on $L^2(\mathbb{R}^2, \mathbb{C})$ associated to the operator $\mathcal{A} = \overline{\mathcal{A}_{0,c}}$ introduced in (2.4), such that the solution of (1.9) with initial data is given by

$$u(x, y, t) = \mathcal{S}_t u(x, y, 0). \tag{4.9}$$

As a direct consequence of Proposition 4.1 with $\lambda = 0$ we have:

Theorem 4.2. *Let $\mathcal{A}$ be the operator defined in* (2.4)*. Then there exists $T_0 > 0$ such that for any $t \geq T_0$ we have*

$$\| \exp(-t\mathcal{A})\| \leq \exp\Big(-\frac{\widehat{\Psi}(ct)}{c}\Big). \tag{4.10}$$

In particular, there exists a constant $C > 0$ such that, for $t \geq T_0$,

$$\| \exp(-t\mathcal{A})\| \leq \exp\Big(-\frac{2\sqrt{2c}}{3}t^{3/2} + Ct^{3/4}\Big). \tag{4.11}$$

Remark 4.3. *It follows from Theorem 4.2 and* [8, Theorem 8.2.1] *that $\mathcal{A}$ has an empty spectrum. We provide another proof of this fact in section 2.*

4.2. **Upper bounds of the resolvent with large $|\lambda|$.** We look for a bound for the norm of the resolvent $(\mathcal{A} - \lambda)^{-1}$ with large $|\lambda|$ for the operator $\mathcal{A}$ defined in (2.4), where λ is a complex number. Let

$$\lambda = \mu + i\nu, \tag{4.12}$$

with both μ and ν being real. Without loss of generality we can set $\nu = 0$, otherwise we translate y by ν/c (that is, mapping $y \to y + \nu/c$). For $\mathcal{A}_{0,c}$ with real constant c we have $E^* \geq 0$ (see the definition of E^* for $\mathcal{A}_{0,c}$ in (2.6)). So from (2.14) we see that for $\lambda < 0$,

$$\|(\mathcal{A} - \lambda)^{-1}\| \leq \frac{1}{|\lambda|}. \tag{4.13}$$

On the other hand, as $\lambda \to +\infty$ we have the following estimate.

Lemma 4.4. *For the operator $\mathcal{A}$ defined in* (2.4)*, there exist positive constants λ_0 and C such that, for all $\lambda > \lambda_0$,*

$$\|(\mathcal{A} - \lambda)^{-1}\| \leq \exp\Big(\frac{1}{6c}\lambda^3 + C\lambda^{3/2}\Big). \tag{4.14}$$

Proof. We use the formula relating the semi-group and the resolvent given in the book [8] :

$$(\mathcal{A} - \lambda)^{-1} = \int_0^{+\infty} \exp(\lambda t)\mathcal{S}_t \, dt, \tag{4.15}$$

with

$$\mathcal{S}_t = \exp(-t\mathcal{A}). \tag{4.16}$$

Using (4.10), we get the universal upper bound

$$\|(\mathcal{A} - \lambda)^{-1}\| \leq \int_0^{+\infty} \exp\big(\lambda t - \frac{\widehat{\Psi}(ct)}{c}\big) \, dt. \tag{4.17}$$

Using (4.11) and the substitution

$$t = \lambda^2 \tau$$

we obtain

$$\|(\mathcal{A} - \lambda)^{-1}\| \leq \lambda^2 \int_0^{+\infty} \exp\Big(\lambda^3\big(\tau - \frac{2\sqrt{2c}}{3}\tau^{3/2} + C\lambda^{-3/2}\tau^{3/4}\big)\Big) d\tau \,. \tag{4.18}$$

The integral on the right hand side can be estimated using Laplace method (see [23, chapter 2] for instance) to obtain (4.14). ∎

Remark 4.5. *One can improve the upper bound of the norm of the resolvent in* (4.14) *by using* (3.15)-(3.19) *and using an estimate with greater accuracy the Legendre transform of* $\widehat{\Psi}(ct)$*. The Laplace integral method can then provide better estimates for the right hand side of* (4.18).

5. Long Time Asymptotics for the Fourier Transform and Applications

5.1. **Technical preliminaries.** In this section we look for a finer estimate of the norm of the resolvent of $\mathcal{A}$. We shall work on the equation (1.15) which is obtained first making Fourier transform to the equation (1.9) and then making translation after fixing ω as a parameter. To obtain with greater accuracy the asymptotics of solutions for (1.15), we need to obtain some additional spectral properties of the anharmonic oscillator (1.13). We use semi-classical analysis to obtain these properties, which all involve asymptotics in the limit $|\beta| \to \infty$.

Let $E_k^{(\ell)}(\beta)$, $\ell = 1, 2$, $k = 0, 1, 2, \cdots$ be the eigenvalues of the operator $\mathcal{L}_\beta$ introduced in section 3, and $\phi_k^{(\ell)}$ be the associated eigenfunctions of unit L^2 norm. Since the following discussion is valid for both $\ell = 1$ and 2, and we omit the superscript ℓ in the sequel, namely, we replace $E_k^{(\ell)}(\beta)$ and $\phi_k^{(\ell)}$ by $E_k(\beta)$ and ϕ_k. Then, we set

$$\widehat{\Delta}_k(\beta) = E_k(\beta) - E_0(\beta)\,. \tag{5.1}$$

We have seen from Proposition 3.1 that

$$\widehat{\Delta}_1(\beta) = 2\sqrt{-2\beta} + \mathcal{O}(|\beta|^{-1}) \quad \text{as } \beta \to -\infty. \tag{5.2}$$

We further introduce

$$g_k(\beta) = \left[\sum_{m=2}^{\infty} \left| \frac{1}{E_m(\beta) - E_k(\beta)} \int_{\mathbb{R}} x^2 \phi_k(x,\beta)\phi_m(x,\beta)dx \right|^2\right]^{1/2}, \quad k = 0, 1\,, \tag{5.3}$$

and

$$f(\beta) = \frac{1}{\widehat{\Delta}_1(\beta)} \int_{\mathbb{R}} x^2 \phi_1(x,\beta)\phi_0(x,\beta)dx\,. \tag{5.4}$$

Finally, let

$$g(\beta) = [g_0^2(\beta) + f^2(\beta)]^{1/2}\,. \tag{5.5}$$

The following result will be useful in the next subsection.

Lemma 5.1. *There exist* $\alpha \neq 0$, $C > 0$ *and* $\beta_0 > 0$ *such that*

$$\left| f(\beta) - \alpha|\beta|^{-1/4} \right| \leq C|\beta|^{-1}, \quad \forall \beta < -\beta_0, \tag{5.6a}$$

$$|f'(\beta)| \leq C\,|\beta|^{-5/4}, \quad \forall \beta < -\beta_0, \tag{5.6b}$$

$$|g_0(\beta)| \leq C\,|\beta|^{-1}, \quad \forall \beta < -\beta_0, \tag{5.6c}$$

$$|g_1(\beta)| \leq C\,|\beta|^{-1/4}, \quad \forall \beta < -\beta_0. \tag{5.6d}$$

$$g(\beta) < C\beta^{-3/2} \quad \forall \beta > \beta_0, \tag{5.6e}$$

Note that (5.6e) is related to the asymptotic behaviour of the anharmonic oscillator in the limit $\beta \to +\infty$, in contrast with the rest of the statements that deal with the limit $\beta \to -\infty$.

Proof. Proof of (5.6a). We observe that, since $\{\phi_k(\cdot,\beta)\}_{k=0}^\infty$ are orthogonal to each other,

$$\int_{\mathbb{R}} x^2\phi_j(x,\beta)\phi_k(x,\beta)dx = \int_{\mathbb{R}} (x^2+2\beta)\phi_j(x,\beta)\phi_k(x,\beta)dx, \quad j \neq k. \tag{5.7}$$

By (3.5), and the exponential rate of decay of ϕ_0 and ϕ_1 away from $x = \pm\sqrt{-2\beta}$ (see for instance in [16, chapter 3]) we have

$$\|(x^2+2\beta)\phi_k\|_{L^2(\mathbb{R})} = \|(x+\sqrt{-2\beta})(x-\sqrt{-2\beta})\phi_k\|_{L^2(\mathbb{R})} \leq C\beta^{1/4}, \quad \text{for } k=0,1. \tag{5.8}$$

(5.6a) is easily verified using (3.5) and (5.2).

Proof of (5.6b). We first write, for $i = 0, 1$,

$$\Big\langle x^2\phi_i, \frac{\partial \phi_{1-i}}{\partial\beta}\Big\rangle = \sum_{k=0}^\infty \langle x^2\phi_i,\phi_k\rangle\Big\langle \frac{\partial\phi_{1-i}}{\partial\beta},\phi_k\Big\rangle. \tag{5.9}$$

Here $\langle\cdot,\cdot\rangle$ denotes the inner product in $L^2(\mathbb{R})$. We then use the fact that $\|\phi_j\|_{L^2(\mathbb{R})} = 1$ and differentiate (3.2) with $k = j$ in β to obtain

$$\Big\langle \frac{\partial\phi_j}{\partial\beta},\phi_k\Big\rangle = \begin{cases} -\frac{1}{E_k-E_j}\langle x^2\phi_j,\phi_k\rangle & \text{if } k\neq j,\\ 0 & \text{if } k=j.\end{cases} \tag{5.10}$$

Substituting (5.10) with $j = 1 - i$ into (5.9) yields

$$f'(\beta) = -\frac{\widehat{\Delta}_1'(\beta)}{\widehat{\Delta}_1^2(\beta)}f(\beta) - \frac{1}{\widehat{\Delta}_1(\beta)}\sum_{k=2}^\infty\Big(\frac{1}{E_k-E_0}+\frac{1}{E_k-E_1}\Big)\langle x^2\phi_0,\phi_k\rangle\langle x^2\phi_1,\phi_k\rangle. \tag{5.11}$$

The first term in the right side of (5.11) can be easily estimated. To control the second term we recall that the sequence $\{E_k\}$ is monotonically increasing, and hence for any $k \geq 2$,

$$\frac{1}{\widehat{\Delta}_1(\beta)}\Big(\frac{1}{E_k-E_0}+\frac{1}{E_k-E_1}\Big) \leq \frac{1}{\widehat{\Delta}_1}\frac{2}{E_2-E_1}.$$

Using (3.3) we see that the right hand side of the above inequality is controlled by $C/|\beta|$ as $\beta \to -\infty$. Hence the second term in the right side of (5.11) is bounded by

$$\frac{C}{|\beta|}\Big[\sum_{k=2}^\infty |\langle x^2\phi_0,\phi_k\rangle|^2\Big]^{1/2}\Big[\sum_{k=2}^\infty |\langle x^2\phi_1,\phi_k\rangle|^2\Big]^{1/2}.$$

Using (5.7) with β replaced by 2β and then using Parseval equality we have, for $k \geq 2$,

$$\begin{aligned}&\sum_{k=2}^\infty |\langle x^2\phi_0,\phi_k\rangle|^2 = \sum_{k=2}^\infty |\langle (x^2+2\beta)\phi_0,\phi_k\rangle|^2\\ =&\sum_{k=0}^\infty |\langle x^2\phi_0,\phi_k\rangle|^2 - |\langle (x^2+2\beta)\phi_0,\phi_0\rangle|^2 - |\langle (x^2+2\beta)\phi_0,\phi_1\rangle|^2\\ =&\|(x^2+2\beta)\phi_0\|_{L^2(\mathbb{R})}^2 - |\langle (x^2+2\beta)\phi_0,\phi_0\rangle|^2 - |\langle (x^2+2\beta)\phi_0,\phi_1\rangle|^2.\end{aligned}$$

Now we claim that,

$$\|(x^2+2\beta)\phi_0\|^2_{L^2(\mathbb{R})} - |\langle (x^2+2\beta)\phi_0,\phi_0\rangle|^2 - |\langle (x^2+2\beta)\phi_1,\phi_0\rangle|^2 \leq \frac{C}{|\beta|}. \tag{5.12}$$

This is true due to the orthogonality of all Hermite functions h_k for $k \geq 2$ with xh_0. The complete proof of (5.12) is given in appendix A. Using (5.12) we get

$$\sum_{k=2}^{\infty} |\langle x^2\phi_0,\phi_k\rangle|^2 \leq \frac{C}{|\beta|}. \tag{5.13}$$

Similarly we have

$$\sum_{k=2}^{\infty} |\langle x^2\phi_1,\phi_k\rangle|^2 = \sum_{k=2}^{\infty} |\langle (x^2+2\beta)\phi_1,\phi_k\rangle|^2 \leq \|(x^2+2\beta)\phi_1\|^2_{L^2(\mathbb{R})} \leq C|\beta|^{1/2}. \tag{5.14}$$

Here we have used the inequality (5.8). Combining (5.11), (5.13) and (5.14) yields (5.6b).

The proof of (5.6c) and (5.6d) follows in exactly the same manner from (5.8) and (5.12).

Finally we prove (5.6e) by first observing that

$$g(\beta) \leq \frac{1}{\hat{\Delta}_1}\|x^2\phi_0\|^2_{L^2(\mathbb{R})}\,.$$

From (3.6b) we now obtain (5.6e). ∎

5.2. **An evolution problem.** More detailed information on the solutions u of (1.7) can be obtained by deriving the large t asymptotic behavior of their (partial) Fourier transform which satisfies (1.11) or the large τ asymptotic behavior of the solutions of (1.15), with the relationship between t and τ given by (1.14). Note that in (1.15) the initial value corresponding to $t=0$ is $\tau = -\omega/c$, and that $-\tau$ might be very large. Hence we need to separately consider two different regions of the variable τ:

Case 1. $1 \ll -\tau \leq \frac{\omega}{c}$;

Case 2. $1 \ll \tau$.

The next proposition deals with the first case. We assume the initial data to be either even or odd, thus saving the need for marking each eigenvalue and eigenfunction by an appropriate superscript as in section 3. For a given

$$T = \frac{\omega}{c} > 0 \tag{5.15}$$

and some even L^2-normalized function $v_0 \in L^2(\mathbb{R})$, we analyze the properties of the unique solution v of (1.15) in the region $-T < \tau < +\infty$ such that

$$v(x,-T) = v_0(x), \qquad \|v_0\|_{L^2(\mathbb{R})} = 1\,. \tag{5.16}$$

We denote by $\mathcal{C}(T)$ the union set of all such solutions with initial data v_0 satisfying (5.16).

Remark 5.2. *We note that this problem is an evolution problem attached to a τ-dependent problem. We can no more use the semi-group theory but fortunately our case enters in the theory developed by Kato for extending Hille-Yosida theorem to time-dependent problems. We refer to* [30, chapter XIV] *for a presentation of the theory. We observe that the* *domain of $\mathcal{L}_{c\tau}$ is independent of τ and contains $C_0^\infty(\mathbb{R})$ as a dense subspace. Moreover if the initial condition at $\tau = -T$ is in $\mathcal{S}(\mathbb{R})$ then the solution is in $C^\infty([-T, +\infty), \mathcal{S}(\mathbb{R}))$.*

Proposition 5.3. *For every $0 < \delta < 1/2$, there exist $T_0 \geq 1$, $T_1 \geq 1$, $C_\delta > 0$, $\hat{C}_\delta > 0$ and*

$$T(\delta) = T - \frac{C_\delta}{2} T^{-\delta}, \tag{5.17}$$

such that, if

$$T > T_0, \qquad -T(\delta) \leq \tau \leq -T_1, \tag{5.18}$$

and if $v \in \mathcal{C}(T)$, then there exists C_1 such that

$$\Big\| v(\cdot, \tau) \exp\big(-\lambda(\tau + T) + \int_{-T}^{\tau} E_0(cs)\, ds\big) - C_1 \Big|\frac{\tau}{T}\Big|^\gamma \phi_0(\cdot, c\tau) \Big\|_{L^2(\mathbb{R})} \leq \hat{C}_\delta\, |\tau|^{-3/4}, \tag{5.19}$$

and

$$\big| C_1 - \langle v_0, \phi_0(\cdot, -cT) \rangle \big| \leq \hat{C}_\delta\, T^{-1/2}, \tag{5.20}$$

where

$$\gamma = \frac{|\alpha|^2 c}{2\sqrt{2}}, \tag{5.21}$$

and α is the positive number given in Lemma 5.1.

Proof. We shall use (4.3) to derive the estimates for $v(x, \tau)$.

Step 1. We first transform (1.15) into an equivalent equation whose solution is both bounded and independent of λ. From (1.14), (5.15), (5.16), and using (4.3) (with $t_0 = 0$) we find that, for $-T < \tau < -T_1$,

$$\|v(\cdot, \tau)\|_{L^2(\mathbb{R})} \leq \exp\big(\lambda(\tau + T) - \int_{-T}^{\tau} E_0(cs)\, ds\big). \tag{5.22}$$

Set then

$$w(x, \tau) = v(x, \tau) \exp\big(-\lambda(\tau + T) + \int_{-T}^{\tau} E_0(cs)\, ds\big). \tag{5.23}$$

Clearly, by (5.22) we have

$$\|w(\cdot, \tau)\|_{L^2(\mathbb{R})} \leq 1\,. \tag{5.24}$$

Substituting (5.23) into (1.15), we obtain

$$\begin{cases} \partial_\tau w(x, \tau) = \partial_{xx} w(x, \tau) - \Big[\big(\frac{1}{2}x^2 + c\tau\big)^2 - E_0(c\tau)\Big] w(x, \tau), \\ w(x, -T) = v_0(x). \end{cases} \tag{5.25}$$

Step 2. We now define an expansion using the eigenfunctions $\{\phi_k(x, c\tau)\}$ of $\mathcal{L}_{c\tau}$, and derive the equations for the first two coefficients and the remainder.

Set then

$$a_k(\tau) = \langle w(\cdot, \tau), \phi_k(\cdot, c\tau) \rangle, \quad k \in \mathbb{N}, \tag{5.26}$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $L^2(\mathbb{R}, \mathbb{C})$. Taking the inner product of (5.25) with $\phi_k(x, c\tau)$ and integrating over $\mathbb{R}_x$, we obtain

$$\frac{da_k}{d\tau}(\tau) + \Delta_k(\tau) a_k(\tau) = \Big\langle w(\cdot, \tau), \frac{\partial \phi_k}{\partial \tau}(\cdot, c\tau) \Big\rangle, \tag{5.27}$$

in which

$$\Delta_k(\tau) \equiv \widehat{\Delta}_k(c\tau) = E_k(c\tau) - E_0(c\tau)\,. \tag{5.28}$$

Since

$$w(\cdot,\tau) = \sum_{m=0}^{\infty} a_m(\tau)\phi_m(\cdot,c\tau),$$

we have

$$\Big\langle w(\cdot,\tau), \frac{\partial\phi_k}{\partial\tau}(\cdot,c\tau)\Big\rangle = \sum_{m=0}^{\infty} a_m \int_{\mathbb{R}} \frac{\partial\phi_k}{\partial\tau}(x,c\tau)\phi_m(x,c\tau)\,dx.$$

From (5.10) it now follows that

$$\Big\langle w(\cdot,\tau), \frac{\partial\phi_k}{\partial\tau}(\cdot,c\tau)\Big\rangle = -c\sum_{\substack{m=0\\ m\neq k}}^{\infty} \frac{a_m}{E_m(c\tau)-E_k(c\tau)} \int_{\mathbb{R}} x^2\phi_k(x,c\tau)\phi_m(x,c\tau)\,dx\,. \tag{5.29}$$

Next we set

$$\check{w}(x,\tau) = w(x,\tau) - a_0(\tau)\phi_0(x,c\tau) - a_1(\tau)\phi_1(x,c\tau)\,. \tag{5.30}$$

From (5.26) and (5.30) we have

$$\|\check{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R})} = \sum_{m=2}^{\infty} |a_m|^2\,. \tag{5.31}$$

By (5.27) and (5.29) we have

$$\begin{aligned}\frac{da_0(\tau)}{d\tau} &= -c\sum_{m=1}^{\infty} \frac{a_m(\tau)}{E_m(c\tau)-E_0(c\tau)} \int_{\mathbb{R}} x^2\phi_0(x,c\tau)\phi_m(x,c\tau)\,dx \\ &= -ca_1(\tau)f(c\tau) - c\sum_{m=2}^{\infty} \frac{a_m(\tau)}{E_m(c\tau)-E_0(c\tau)} \int_{\mathbb{R}} x^2\phi_0(x,c\tau)\phi_m(x,c\tau)\,dx.\end{aligned}$$

The second term can be controlled by

$$\begin{aligned}&c\Big[\sum_{m=2}^{\infty} a_m^2\Big]^{1/2}\Big[\sum_{m=2}^{\infty}\Big|\frac{1}{E_m(c\tau)-E_0(c\tau)}\int_{\mathbb{R}} x^2\phi_0(x,c\tau)\phi_m(x,c\tau)\,dx\Big|^2\Big]^{1/2} \\ &= cg_0(c\tau)\|\check{w}(\cdot,c\tau)\|_{L^2(\mathbb{R})}\,,\end{aligned}$$

where $g_0(c\tau)$ and $f(c\tau)$ are given in (5.3) and (5.4). Hence

$$\Big|\frac{da_0}{d\tau}(\tau) + cf(c\tau)a_1(\tau)\Big| \leq Cg_0(c\tau)\|\check{w}(\cdot,\tau)\|_{L^2(\mathbb{R})}. \tag{5.32}$$

Similarly we have

$$\Big|\frac{da_1}{d\tau}(\tau) + \Delta_1(\tau)a_1(\tau) + cf(c\tau)a_0(\tau)\Big| \leq Cg_1(c\tau)\|\check{w}(\cdot,\tau)\|_{L^2(\mathbb{R})}, \tag{5.33}$$

where $g_1(c\tau)$ is given by (5.3).

Note for later use that, since $a_k(\tau)$ is bounded in $\tau \ll -1$ for each k, (5.32) together with (5.6a) and (5.6c) imply that

$$\Big|\frac{da_0}{d\tau}(\tau)\Big| \leq C|\tau|^{-1/4}. \tag{5.34}$$

From the definition of $\breve{w}$ it follows that

$$\frac{\partial \breve{w}(\cdot,\tau)}{\partial \tau} + \big(\mathcal{L}_{c\tau} - E_0(c\tau)\big)\breve{w}(\cdot,\tau) = -\Delta_1(\tau)a_1(\tau)\phi_1(\cdot,c\tau) - \partial_\tau \sum_{k=0}^{1} a_k(\tau)\phi_k(\cdot,c\tau). \tag{5.35}$$

Recall that $\breve{w}(\cdot,\tau)$ is orthogonal to $\phi_i(\cdot,c\tau)$ for $i=0,1$. Taking inner product of (5.35) with $\breve{w}(\cdot,\tau)$ and using this orthogonality yield that

$$\begin{aligned} &\frac{1}{2}\frac{d}{d\tau}\|\breve{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R})} + \big\langle \breve{w}(\cdot,\tau), \big(\mathcal{L}_{c\tau} - E_0(c\tau)\big)\breve{w}(\cdot,\tau)\big\rangle \\ &= -\, c\sum_{k=0}^{1} a_k(\tau)\Big\langle \breve{w}(\cdot,\tau), \frac{\partial \phi_k}{\partial \tau}(\cdot,c\tau)\Big\rangle. \end{aligned} \tag{5.36}$$

Using (5.36) it is easy to show that

$$\frac{d}{d\tau}\|\breve{w}(\cdot,\tau)\|_{L^2(\mathbb{R})} + \Delta_2(\tau)\|\breve{w}(\cdot,\tau)\|_{L^2(\mathbb{R})} \le c\sum_{k=0}^{1} |a_k(\tau)|\, g_k(c\tau)\,. \tag{5.37}$$

Step 3. Now we establish a preliminary upper bound estimate of $a_1(\tau)$. Set

$$\widehat{r}_1(\tau) = \frac{da_1}{d\tau}(\tau) + \Delta_1(\tau)a_1(\tau). \tag{5.38}$$

By (5.33) we have

$$|\widehat{r}_1(\tau)| \le Cg_1(c\tau)\|\breve{w}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)} + c|f(c\tau)|\,|a_0(\tau)|\,. \tag{5.39}$$

Since

$$\frac{d}{d\tau}\Big[\exp\big(\int_{-T}^{\tau}\Delta_1(s)ds\big)\; a_1(\tau)\Big] = \exp\big(\int_{-T}^{\tau}\Delta_1(s)ds\big)\widehat{r}_1(\tau)\;,$$

we have

$$\exp\big(\int_{-T}^{\tau}\Delta_1(s)ds\big)\; a_1(\tau) = a_1(-T) + \int_{-T}^{\tau}\exp\big(\int_{-T}^{s}\Delta_1(\eta)d\eta\big)\widehat{r}_1(s)\,ds\,.$$

Hence, by (5.39)

$$\begin{aligned} &\Big|a_1(\tau) - a_1(-T)\exp\big(-\int_{-T}^{\tau}\Delta_1(s)ds\big)\Big| \\ &\le \int_{-T}^{\tau}\exp\big(-\int_{s}^{\tau}\Delta_1(\eta)\,d\eta\big)\Big[Cg_1(cs)\|\breve{w}(\cdot,s)\|_{L^2(\mathbb{R})} + c|f(cs)||a_0(s)|\Big]\,ds\,. \end{aligned} \tag{5.40}$$

In view of (5.24) we have that

$$a_0^2(\tau) + a_1^2(\tau) + \|\breve{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R})} \le 1\,. \tag{5.41}$$

Hence, by (5.6) and (5.40),

$$|a_1(\tau)| \le |a_1(-T)|\exp\big(-\int_{-T}^{\tau}\Delta_1(s)ds\big) + C\int_{-T}^{\tau}\exp\big(-\int_{s}^{\tau}\Delta_1(\eta)d\eta\big)|s|^{-1/4}ds. \tag{5.42}$$

By (3.3) we have that

$$\Delta_1(s) > C\Delta_1(\tau) \quad \text{for } s<\tau<0.$$

Therefore,

$$\begin{aligned}|a_1(\tau)| \leq & |a_1(-T)| \exp\big(-\int_{-T}^{\tau}\Delta_1(s)ds\big)\\ & + \frac{C}{|\tau|^{1/4}\Delta_1(\tau)}\int_{-T}^{\tau}\exp\big(-\int_s^{\tau}\Delta_1(\eta)d\eta\big)\Delta_1(s)\,ds\,.\end{aligned} \tag{5.43}$$

We now estimate the right hand side of (5.43). From (5.2) and (5.28), there exists $C>0$ such that for all $\tau<0$

$$\big|\Delta_1(\tau) - 2\sqrt{2c|\tau|}\big| \leq \frac{C}{|\tau|}\,. \tag{5.44}$$

Consequently, we obtain, for $-T<\tau\leq -1$,

$$\exp\big(-\int_{-T}^{\tau}\Delta_1(s)ds\big) \leq \Big|\frac{T}{\tau}\Big|^C \exp\big(-2\sqrt{2c|\tau|}(\tau+T)\big)\,. \tag{5.45}$$

From the above inequality and the fact that $|a_1(-T)|\leq 1$ it is readily verified that the first term on the right hand side of (5.43) is bounded from above by the right side of (5.45). The second term on the right-hand-side of (5.43) can be immediately integrated. So we obtain, for $-T<\tau\leq -1$,

$$|a_1(\tau)| \leq C\max\big\{|\tau|^{-3/4}, \Big|\frac{T}{\tau}\Big|^C \exp\big(-2\sqrt{2c|\tau|}(\tau+T)\big)\big\}. \tag{5.46}$$

Let now $0<\delta<1/2$. Then for sufficiently large T_0, there exists $C_\delta>0$ such that

$$\begin{aligned} & T-T_1 > \tau+T > \frac{C_\delta}{2}T^{-\delta} \quad \text{and} \quad T\geq T_0\\ \Longrightarrow \quad & \Big|\frac{T}{\tau}\Big|^C \exp\big(-2\sqrt{2c|\tau|}(\tau+T)\big) \leq |\tau|^{-5/2}\,.\end{aligned} \tag{5.47}$$

Let $T(\delta)$ be given by (5.17). For fixed δ, if T is large then $T(\delta)>T/2>0$, and

$$\tau\in[-T(\delta),-T_1] \quad \Longrightarrow \quad |a_1(\tau)| \leq C\,|\tau|^{-3/4}\,. \tag{5.48}$$

Step 4. Now we establish some differential inequalities for $\|\breve{w}(\cdot,\tau)\|_{L^2(\mathbb{R})}$, $a_0(\tau)$ and $a_1(\tau)$.

We deduce from (5.37), (5.41), (5.46), (5.6c) and (5.6d) that there exists $C>0$ such that

$$\|\breve{w}(\cdot,\tau)\|_{L^2(\mathbb{R})} \leq C\max\big\{|\tau|^{-3/2}, |\tau|^{-3/4}\Big|\frac{T}{\tau}\Big|^C \exp\big(-2\sqrt{2c|\tau|}(\tau+T)\big)\big\}. \tag{5.49}$$

From the above and (5.47), for $\tau\in[-T(\delta),-T_1]$ we have

$$\|\breve{w}(\cdot,\tau)\|_{L^2(\mathbb{R})} \leq C|\tau|^{-3/2}\,. \tag{5.50}$$

By(5.32), (5.6a), (5.6c), (5.46) and (5.49) we now have for $-T\leq\tau\leq -T_1$

$$\Big|\frac{da_0}{d\tau}(\tau)\Big| \leq C\,\max\big\{|\tau|^{-1}, |\tau|^{-1/4}\Big|\frac{T}{\tau}\Big|^C \exp\big(-2\sqrt{2c|\tau|}(\tau+T)\big)\big\}. \tag{5.51}$$

We now derive a differential inequality for $a_1(\tau)$. To this end we use (5.33), (5.49) and (5.6d) to obtain for $\tau \in [-T(\delta), -T_1]$ that

$$\Big|\frac{da_1}{d\tau}(\tau) + \Delta_1(\tau)a_1(\tau) + cf(c\tau)a_0(\tau)\Big| \leq C|\tau|^{-7/4}. \tag{5.52}$$

Step 5. Now we establish an asymptotic expansion for $a_1(\tau)$ for $\tau \in [-T(\delta), -T_1]$.
Integrating (5.52) from $-T(\delta)$ to τ we obtain

$$\begin{aligned}\Big|a_1(\tau) - a_1(-T(\delta))\exp\big(-\int_{-T(\delta)}^{\tau}\Delta_1(s)ds\big)\\ + c\int_{-T(\delta)}^{\tau}\exp\big(-\int_s^{\tau}\Delta_1(\eta)d\eta\big)f(cs)a_0(s)\,ds\Big| \leq C|\tau|^{-9/4}.\end{aligned} \tag{5.53}$$

In the following we obtain the asymptotic expansion, in the large T_1 limit, for the second and the third terms on the left-hand-side of (5.53). The second term can be easily estimated. In fact, from (5.41) we see that $|a_1(-T(\delta))| \leq 1$, and using (5.47) we obtain that, for $\tau \in [-T(\delta), -T_1]$,

$$\Big|a_1(-T(\delta))\exp\big(-\int_{-T(\delta)}^{\tau}\Delta_1(s)ds\big)\Big| \leq C|\tau|^{-5/2}. \tag{5.54}$$

To approximate the third term on the left-hand-side of (5.53) we represent it in the form

$$\begin{aligned}&\frac{ca_0(\tau)f(c\tau)}{\Delta_1(\tau)}\int_{-T(\delta)}^{\tau}\exp\big(-\int_s^{\tau}\Delta_1(\eta)\,d\eta\big)\Delta_1(s)ds\\ &+ c\int_{-T(\delta)}^{\tau}\exp\big(-\int_s^{\tau}\Delta_1(\eta)d\eta\big)\Big[\frac{a_0(s)f(cs)}{\Delta_1(s)} - \frac{a_0(\tau)f(c\tau)}{\Delta_1(\tau)}\Big]\Delta_1(s)\,ds.\end{aligned} \tag{5.55}$$

The first term in (5.55) can be readily integrated, to obtain for large $T(\delta)+\tau$:

$$\begin{aligned}&\frac{ca_0(\tau)f(c\tau)}{\Delta_1(\tau)}\int_{-T(\delta)}^{\tau}\exp\big(-\int_s^{\tau}\Delta_1(\eta)\,d\eta\big)\Delta_1(s)ds\\ =&\frac{ca_0(\tau)f(c\tau)}{\Delta_1(\tau)} + \mathcal{O}\Big(\exp\big\{-\hat\beta\sqrt{|\tau|}(T(\delta)+\tau)\big\}\Big),\end{aligned} \tag{5.56}$$

for some $0 < \hat\beta \leq \sqrt{2c}$.

To estimate the second term in (5.55), we observe that, for $-T(\delta) \leq s \leq \tau$

$$\Big|\frac{a_0(s)f(cs)}{\Delta_1(s)} - \frac{a_0(\tau)f(c\tau)}{\Delta_1(\tau)}\Big| \leq \sup_{\xi\in(s,\tau)}\Big|\frac{d}{d\xi}\Big[\frac{a_0(\xi)f(c\xi)}{\Delta_1(\xi)}\Big]\Big|(\tau - s).$$

In view of (5.6a), (5.6b), (3.4), (5.41) and (5.51), for

$$-T(\delta) \leq s \leq \xi < \tau \leq -T_1,$$

we also have

$$\Big|\frac{d}{d\xi}\Big[\frac{a_0(\xi)f(c\xi)}{\Delta_1(\xi)}\Big]\Big| \leq C|\xi|^{-7/4} \leq C|\tau|^{-7/4}.$$

Hence,

$$
\begin{aligned}
&\Big|\int_{-T(\delta)}^{\tau} \exp\big(-\int_s^\tau \Delta_1(\eta)d\eta\big)\Big[\frac{a_0(s)f(cs)}{\Delta_1(s)}-\frac{a_0(\tau)f(c\tau)}{\Delta_1(\tau)}\Big]\,\Delta_1(s)\,ds\Big| \\
\le& C|\tau|^{-7/4}\int_{-T(\delta)}^{\tau} \exp\big(-\int_s^\tau \Delta_1(\eta)\,d\eta\big)(\tau-s)\Delta_1(s)ds.
\end{aligned}
\tag{5.57}
$$

Integration by parts then yields for $\tau\in[-T(\delta),-T_1]$,

$$
\begin{aligned}
0\le& \int_{-T(\delta)}^{\tau} \exp\big(-\int_s^\tau \Delta_1(\eta)\,d\eta\big)(\tau-s)\Delta_1(s)ds \\
=& \int_{-T(\delta)}^{\tau} \exp\big(-\int_s^\tau \Delta_1(\eta)\,d\eta\big)ds-(\tau+T(\delta))\exp\big(-\int_{-T(\delta)}^{\tau}\Delta_1(\eta)\,d\eta\big) \\
\le& \frac{C}{\Delta_1(\tau)}\int_{-T(\delta)}^{\tau} \exp\big(-\int_s^\tau \Delta_1(\eta)\,d\eta\big)\Delta_1(s)ds\le C|\tau|^{-1/2}\,,
\end{aligned}
$$

which together with (5.57) yields

$$
\begin{aligned}
&\Big|\int_{-T(\delta)}^{\tau} \exp\big(-\int_s^\tau \Delta_1(\eta)d\eta\big)\Big[\frac{a_0(s)f(cs)}{\Delta_1(s)}-\frac{a_0(\tau)f(c\tau)}{\Delta_1(\tau)}\Big]\,\Delta_1(s)\,ds\Big| \\
\le& C|\tau|^{-7/4}|\tau|^{-1/2}=C|\tau|^{-9/4}.
\end{aligned}
\tag{5.58}
$$

Substituting (5.56) and (5.58) into (5.55) we obtain that the third term on the left-hand-side of (5.53) admits the expansion

$$
\begin{aligned}
&\frac{ca_0(\tau)f(c\tau)}{\Delta_1(\tau)}+\mathcal{O}(1)\exp\big(-\hat\beta\sqrt{|\tau|}(T(\delta)+\tau)\big)+\mathcal{O}(|\tau|^{-9/4}) \\
=&\frac{ca_0(\tau)f(c\tau)}{\Delta_1(\tau)}+\mathcal{O}(|\tau|^{-9/4}).
\end{aligned}
$$

Inserting the above and (5.54) into (5.53) yields, for $\tau\in[-T(\delta),-T_1]$ with T_1 large,

$$
a_1(\tau)=\frac{ca_0(\tau)f(c\tau)}{\Delta_1(\tau)}+\mathcal{O}(|\tau|^{-9/4}).
\tag{5.59}
$$

Step 6. We now estimate $a_0(\tau)$.

We substitute (5.59) into (5.32), and use (5.6c), (5.47) and (5.49) to obtain that for all $\tau\in[-T(\delta),-T_1]$ we have

$$
\Big|\frac{da_0}{d\tau}(\tau)+\frac{c^2|f(c\tau)|^2}{\Delta_1(\tau)}a_0(\tau)\Big|\le C|\tau|^{-9/4}
$$

From (5.6a) and (5.44) it follows that for $\tau\in[-T(\delta),-T_1]$ we have

$$
\Big|\frac{da_0}{d\tau}(\tau)+\frac{\gamma}{|\tau|}a_0(\tau)\Big|\le C|\tau|^{-7/4}\,,
$$

where γ is given in (5.21). Consequently,

$$
a_0(\tau)=a_0(-T(\delta))\Big|\frac{\tau}{T(\delta)}\Big|^{\gamma}+\mathcal{O}(|\tau|^{-3/4})\,.
\tag{5.60}
$$

Step 7. Denote $C_1 = a_0(-T(\delta))$. From (5.30), (5.48), (5.50) and (5.60) we have, for $\tau \in [-T(\delta), -T_1]$,

$$\begin{aligned}
&\Big\| w(\cdot,\tau) - C_1 \Big|\frac{\tau}{T(\delta)}\Big|^{\gamma}\phi_0(\cdot,c\tau)\Big\|_{L^2(\mathbb{R})} \\
\le& \|\check{w}(\cdot,\tau)\|_{L^2(\mathbb{R})} + \|a_1\phi_1(\cdot,c\tau)\|_{L^2(\mathbb{R})} + \|(a_0(\tau)-a_0(-T(\delta)))\phi_0(\cdot,c\tau)\|_{L^2(\mathbb{R})} \\
\le& C|\tau|^{-3/2} + C|\tau|^{-3/4} + C|\tau|^{-3/4} \le C|\tau|^{-3/4}.
\end{aligned} \tag{5.61}$$

Thus (5.19) is proved.

To obtain (5.20) we use (5.34), which is valid also for $-T \le \tau \le -T(\delta) = -T + C_\delta T^{-\delta}$, to obtain

$$\big|a_0(-T + C_\delta T^{-\delta}) - a_0(-T)\big| \le C\, T^{-\delta-1/4}\,.$$

Hence, (5.20) follows for $\delta \ge 1/4$. Note that the theorem follows for all $\delta < 1/2$, since by decreasing δ we impose a weaker constraint on the τ domain where (5.19) is valid. ∎

We conclude this subsection by stating the asymptotic behaviour of u in the limit $\tau \to +\infty$.

Proposition 5.4. *Given $T > 0$ there exist positive numbers T_2 and C such that, for any $v \in \mathcal{C}(T)$ and $\tau > T_2$, there exists C_0 such that*

$$\Big\| v\, \exp\big(-\lambda(\tau+T) + \int_{-T}^{\tau} E_0(cs)\, ds\big) - C_0\phi_0(\cdot,c\tau)\Big\|_{L^2(\mathbb{R})} \le C|T_2|^{-5/2}\,. \tag{5.62}$$

Proof. We define w once again according to (5.23) to obtain (5.25). Set again

$$a_0(\tau) = \langle \phi_0(\cdot,c\tau), w(\cdot,\tau)\rangle\,.$$

Then, following the steps of the proof of Proposition 5.3 we obtain the inequality

$$\Big|\frac{da_0}{d\tau}(\tau)\Big| \le c\|\tilde{w}(\cdot,\tau)\|_{L^2(\mathbb{R})}\, g(c\tau) \quad i = 1,2\,, \tag{5.63}$$

where

$$\tilde{w}(x,\tau) = w(x,\tau) - a_0(\tau)\phi_0(x,c\tau)\,,$$

and g is given by (5.5). Furthermore, we obtain that

$$\frac{d}{d\tau}\|\tilde{w}(\cdot,\tau)\|_{L^2(\mathbb{R})} + \Delta_1\|\tilde{w}(\cdot,\tau)\|_{L^2(\mathbb{R})} \le g(c\tau)\,. \tag{5.64}$$

By (5.64) and (5.6e) we obtain that

$$\|\tilde{w}(\cdot,\tau)\|_{L^2(\mathbb{R})} \le \frac{C}{\tau^2}\,.$$

Hence, by (5.63) we obtain that

$$|a_0(\tau) - a_0(T_2)| \le C\, T_2^{-5/2}\,.$$

Let $C_0 = a_0(T_2)$ and write

$$w(x,\tau) = \tilde{w}(x,\tau) + C_0\phi_0(x,c\tau) + (a_0(\tau) - a_0(T_2))\phi_0(x,c\tau).$$

An integration yields (5.62). ∎

5.3. **Lower bounds for the resolvent.** We now provide a lower bound for the norm of the resolvent of the operator $\mathcal{A}$ defined in (2.4).

Proposition 5.5. *Let*

$$\theta = \frac{4\gamma}{4\gamma+3}\,, \tag{5.65}$$

where γ is given by (5.21). *Then there exists a positive constant μ_1 such that, for all $\mu > \mu_1$ and $\tilde{\theta} > \theta$*

$$\|(\mathcal{A}-\mu)^{-1}\| \geq \exp\Big(\frac{\mu^3}{3c} - 2\mu^{1+2\tilde{\theta}}\Big), \tag{5.66}$$

where c is the constant appearing in the definition of $\mathcal{A}_{0,c}$.

Proof. To prove (5.66) we will construct a pair (u,f) with $u \in D(\mathcal{A})$ such that

$$(\mathcal{A}-\mu)u = f, \tag{5.67}$$

where

$$\|f\|_{L^2(\mathbb{R}^2)} = 1, \tag{5.68}$$

and the norm of u will be estimated from below. To simplify notation we set

$$c = 1.$$

For general values of c the proof is almost identical.

Step 1. We continue by applying a partial Fourier transform

$$u(x,y) \mapsto \hat{u}(x,\omega)$$

to (5.67) to obtain,

$$(\widehat{\mathcal{A}}-\mu)\hat{u} = \hat{f}\,, \tag{5.69}$$

in which $\hat{u}$ and $\hat{f}$ are the respective partial Fourier transforms of u and f and

$$\widehat{\mathcal{A}} = \partial_\omega + \mathcal{L}_\omega, \tag{5.70}$$

where $\mathcal{L}_\omega$ is the operator in (3.1) with β replaced by ω.

Choose f such that

$$\hat{f}(x,\omega) = \mu^{1/4} 1_I(\omega)\phi_0^{(1)}(x,\omega), \tag{5.71}$$

where I is the interval

$$I = \Big[-\frac{\mu^2}{2} - \frac{1}{2\sqrt{\mu}}, -\frac{\mu^2}{2} + \frac{1}{2\sqrt{\mu}}\Big]\,, \tag{5.72}$$

$1_I(\omega)$ is the characteristic function of the interval I, and $\phi_0^{(1)}(\cdot,\omega)$ is the eigenfunction of the operator $\mathcal{L}_\omega$ associated to $E_0^{(1)}(\omega)$. It is immediately seen, using the Fubini Theorem, that the condition (5.68) is satisfied. Since $\mathcal{A}$ has empty spectrum in $L^2(\mathbb{R}^2,\mathbb{C})$ (see Corollary 2.6), $\widehat{\mathcal{A}}$ cannot have a non-empty spectrum in $L^2(\mathbb{R}^2,\mathbb{C})$, and hence $(\widehat{\mathcal{A}}-\mu)$ is invertible. Therefore (5.69) has a unique solution $\hat{u}$.

Step 2. We next consider (5.69) locally as an evolution equation associated to the ω-dependent unbounded operator $(\mathcal{L}_\omega - \mu)$, where the time is the parameter ω. To find the solution we follow the Duhamel's principle for solving non-homogeneous equations. We thus introduce an additional parameter $\xi \in I$ and consider the following problem:

$$\begin{cases} \partial_\omega v_\xi(\cdot,\omega) = -\Big[(\mathcal{L}_\omega - \mu)v_\xi\Big](\cdot,\omega) & \text{for } \omega \geq \xi\,, \\ v_\xi(\cdot,\xi) = \phi_0^{(1)}(\cdot,\xi)\,. \end{cases} \tag{5.73}$$

This problem has a unique solution v_ξ which is well defined for $\omega > \xi$ using Kato's theorem (as recalled in Remark 5.2). We extend v_ξ into the region $\omega < \xi$ by letting

$$v_\xi(\cdot,\omega) = 0 \quad \text{for } \omega < \xi\,.$$

Once v_ξ is obtained, we claim that the following equality holds:

$$\hat{u}(x,\omega) = \mu^{1/4}\int_I v_\xi(x,\omega)\,d\xi\,. \tag{5.74}$$

We can indeed verify that the function in the right side of (5.74) belongs to $L^2(\mathbb{R}^2)$, and is a distribution solution of (5.69).

Step 3. We now estimate the $L^2(\mathbb{R}^2,\mathbb{C})$-norm of $\hat{u}$. For simplicity we denote the $L^2(\mathbb{R},\mathbb{C})$ norm by $\|\cdot\|_2$. We set μ to be so large that we can make use of Proposition 5.3 for any $\xi \in I$. We first state the obvious identity (recall that we have taken $c = 1$)

$$\begin{aligned} v_\xi(x,\omega) =& \mu^{1/4}\Big|\frac{\omega}{\xi}\Big|^\gamma \phi_0^{(1)}(x,\xi)e^{\mu(\omega-\xi)}\,e^{-\int_\xi^\omega E_0^{(1)}(\tau)\,d\tau} \\ &+\Big[v_\xi(x,\omega)e^{-\mu(\omega-\xi)}\,e^{\int_\xi^\omega E_0^{(1)}(\tau)\,d\tau} - \mu^{1/4}\Big|\frac{\omega}{\xi}\Big|^\gamma\phi_0^{(1)}(x,\xi)\Big]e^{\mu(\omega-\xi)}\,e^{-\int_\xi^\omega E_0^{(1)}(\tau)\,d\tau}. \end{aligned} \tag{5.75}$$

Hence

$$\begin{aligned} &\|\int_I v_\xi(\cdot,\omega)\,d\xi\|_2 \\ \geq& \mu^{1/4}\Big\|\int_I\Big|\frac{\omega}{\xi}\Big|^\gamma\phi_0^{(1)}(\cdot,\xi)e^{\mu(\omega-\xi)}\,e^{-\int_\xi^\omega E_0^{(1)}(\tau)\,d\tau}\,d\xi\Big\|_2 \\ -&\|\int_I\Big[v_\xi(\cdot,\omega)e^{-\mu(\omega-\xi)}\,e^{\int_\xi^\omega E_0^{(1)}(\tau)\,d\tau} - \mu^{1/4}\Big|\frac{\omega}{\xi}\Big|^\gamma\phi_0^{(1)}(\cdot,\xi)\Big]e^{\mu(\omega-\xi)}\,e^{-\int_\xi^\omega E_0^{(1)}(\tau)\,d\tau}\,d\xi\|_2. \end{aligned} \tag{5.76}$$

Using (5.76) and (5.19) we obtain that, if $\mu \geq 1$, then for all $\omega > -\mu^2/2+1$ (hence $\omega \geq \xi$ for all $\xi \in I$), we have

$$\begin{aligned} \Big\|\int_I v_\xi(\cdot,\omega)\,d\xi\Big\|_2 \;\geq\; &\Big\|\int_I\Big|\frac{\omega}{\xi}\Big|^\gamma\phi_0^{(1)}(\cdot,\xi)e^{\mu(\omega-\xi)}\,e^{-\int_\xi^\omega E_0^{(1)}(\tau)\,d\tau}\,d\xi\Big\|_2 \\ &-\frac{C_2}{|\omega|^{3/4}}\int_I e^{\mu(\omega-\xi)}\,e^{-\int_\xi^\omega E_0^{(1)}(\tau)\,d\tau}\,d\xi. \end{aligned}$$

We now observe that by (3.5) and the expression of $h_k(x)$ that

$$\inf_{(\xi_1,\xi_2)\in I\times I}\int_{\mathbb{R}}\phi_0^{(1)}(x,\xi_1)\phi_0^{(1)}(x,\xi_2)\,dx \geq C_3 > 0\,, \tag{5.77}$$

where C_3 is independent of μ. Consequently, for all $\omega > -\mu^2/2+1$ we have

$$\Big\|\int_I v_\xi(\cdot,\omega)\,d\xi\Big\|_2 \geq C_4\int_I\Big[\Big|\frac{\omega}{\xi}\Big|^\gamma - \frac{1}{|\omega|^{3/4}}\Big]e^{\mu(\omega-\xi)}\,e^{-\int_\xi^\omega E_0^{(1)}(\tau)\,d\tau}\,d\xi\,. \tag{5.78}$$

Step 4. This step involves an estimate of the right hand side in (5.78). Let θ be the number defined in (5.65), and define

$$\tilde{I} = \Big[-\frac{\mu^2}{2} + 1, -\mu^{2\theta}\Big].$$

If $\mu > 2$ and $\xi \in I$, then

$$|\xi| \leq \frac{\mu^2}{2} + \frac{1}{2\sqrt{\mu}} \leq \frac{2\mu^2}{3}.$$

Hence for all $\omega \in \tilde{I}$ we have

$$\Big|\frac{\omega}{\xi}\Big|^\gamma \geq \Big(\frac{3}{2}\Big)^\gamma \mu^{-2\gamma(1-\theta)}, \qquad \frac{1}{|\omega|^{3/4}} \leq \mu^{-3\theta/2}.$$

By our choice of θ we have

$$2\gamma(1-\theta) = \frac{3\theta}{2} = \frac{6\gamma}{4\gamma+3} \leq \frac{3}{2}\,.$$

Hence for $\mu > 2$, $\xi \in I$ and $\omega \in \tilde{I}$ we have

$$\Big|\frac{\omega}{\xi}\Big|^\gamma - \frac{1}{|\omega|^{3/4}} \geq \frac{d}{\mu^{3/2}}\,, \tag{5.79}$$

where

$$d = \Big(\frac{3}{2}\Big)^\gamma - 1 > 0.$$

Consequently for $\mu > 2$ and $\omega \in \tilde{I}$ we have by (5.78) and (5.79) that

$$\begin{aligned}\Big\|\int_I v_\xi(\cdot,\omega)\,d\xi\Big\|_2 \geq & \frac{C_5}{\mu^{3/2}}\int_I e^{\mu(\omega-\xi)}\, e^{-\int_\xi^\omega E_0^{(1)}(\tau)\,d\tau}\,d\xi \\ = & \frac{C_5}{\mu^{3/2}}\int_I \exp\Big(\int_\xi^\omega [\mu - E_0^{(1)}(\tau)]\,d\tau\Big)\,d\xi\,,\end{aligned} \tag{5.80}$$

where $C_5 = C_4 d$.

Note that by (3.3) and (3.4)

$$\sup_{\xi\in I}\big|E_0^{(1)}(\xi) - \mu\big| \leq \frac{C_6}{\mu^{3/2}}\,.$$

Set

$$z(\mu) = -\frac{\mu^2}{2} + \frac{1}{2\sqrt{\mu}}.$$

Then

$$\int_\xi^\omega [\mu - E_0^{(1)}(\tau)]\,d\tau \geq -C_6\mu^{-2} + \mu\omega + \frac{\mu^3}{2} - \frac{\sqrt{\mu}}{2} - \int_{z(\mu)}^\omega E_0^{(1)}(\tau)\,d\tau.$$

In view of (3.3) we have

$$\begin{aligned}\int_{z(\mu)}^\omega E_0^{(1)}(\tau)\,d\tau = & \int_{z(\mu)}^\omega \Big[\sqrt{-2\tau} + O(\frac{1}{|\tau|})\Big]d\tau \\ = & \frac{2\sqrt{2}}{3}(|z(\mu)|^{3/2} - |\omega|^{3/2}) + \mathcal{O}\big(\log\big|\frac{z(\mu)}{\omega}\big|\big).\end{aligned}$$

Hence

$$
\begin{aligned}
&\int_I \exp\Big\{\int_\xi^\omega [\mu - E_0^{(1)}(\tau)]\,d\tau\Big\}\,d\xi \\
\geq &\frac{1}{\mu^{1/2}}\exp\Big\{-C_6\mu^{-2} + \mu\omega + \frac{\mu^3}{2} - \frac{\sqrt{\mu}}{2} - \frac{2\sqrt{2}}{3}(|z(\mu)|^{3/2} - |\omega|^{3/2}) - \mathcal{O}\big(\log\big|\frac{z(\mu)}{\omega}\big|\big)\Big\}.
\end{aligned}
\tag{5.81}
$$

Note that for $\omega \in \tilde{I}$ we have

$$
\Big|\frac{z(\mu)}{\omega}\Big| \leq \mu^{2(1-\theta)}.
$$

Therefore we plug (5.81) into (5.80) and find that there exists a positive constant C_7 such that, for sufficiently large μ and for all $\omega \in \tilde{I}$

$$
\Big\|\int_I v_\xi(\cdot,\omega)\,d\xi\Big\|_2 \geq \frac{C_7}{\mu^2}\exp\Big\{\frac{\mu^3}{6} + \frac{2\sqrt{2}}{3}|\omega|^{3/2} + \mu\omega\Big\}. \tag{5.82}
$$

Step 5. Now we recall (see (5.74)) that

$$
\begin{aligned}
\|u\|^2_{L^2(\mathbb{R}^2)} =&\|\hat{u}\|^2_{L^2(\mathbb{R}^2_{x,\omega})} = \mu^{1/2}\int_{\mathbb{R}}\Big\|\int_I v_\xi(\cdot,\omega)\,d\xi\Big\|_2^2\,d\omega \\
\geq&\mu^{1/2}\int_{\omega\in\tilde{I}}\Big\|\int_I v_\xi(\cdot,\omega)\,d\xi\Big\|_2^2\,d\omega.
\end{aligned}
\tag{5.83}
$$

It follows from (5.82) and (5.83) that

$$
\begin{aligned}
\|u\|^2_{L^2(\mathbb{R}^2,\mathbb{C})} \geq&\frac{C_7^2}{\mu^{7/2}}\int_{-\frac{\mu^2}{2}+1}^{-\mu^{2\theta}}\exp\Big\{\frac{\mu^3}{3} + \frac{4\sqrt{2}}{3}|\omega|^{3/2} + 2\mu\omega\Big\}d\omega \\
=&\frac{C_7^2}{\mu^{7/2}}\int_{\mu^{2\theta}}^{\frac{\mu^2}{2}-1}\exp\Big\{\frac{\mu^3}{3} + \frac{4\sqrt{2}}{3}s^{3/2} - 2\mu s\Big\}ds.
\end{aligned}
$$

If μ is sufficiently large (for instance $\mu \geq 2^{3/(2(1-\theta))}$), then for any $s \in [\mu^{2\theta}, \frac{\mu^2}{2} - 1]$, we have $\mu - \sqrt{2s} > 0$, and

$$
1 = \frac{\mu - \sqrt{2s}}{\mu - \sqrt{2s}} \geq \frac{\mu - \sqrt{2s}}{\mu - \sqrt{2\mu^{2\theta}}} \geq \frac{2\mu - 2\sqrt{2s}}{2\mu}.
$$

Hence for large μ we have

$$
\begin{aligned}
\|u\|^2_{L^2(\mathbb{R}^2,\mathbb{C})} \geq&\frac{C_7^2}{\mu^{9/2}}\int_{\mu^{2\theta}}^{\frac{\mu^2}{2}-1}\exp\Big\{\frac{\mu^3}{3} + \frac{4\sqrt{2}}{3}s^{3/2} - 2\mu s\Big\}(2\mu - 2\sqrt{2s})ds \\
\geq&\exp\Big(\frac{\mu^3}{3} - 2\mu^{1+2\tilde{\theta}}\Big),
\end{aligned}
$$

for all $\tilde{\theta} > \theta$. Hence (5.66) is true for sufficiently large μ. ∎

5.4. **Estimates for dense subspaces.** Let $\mathcal{A}$ be the operator defined in (2.4). In view of the different asymptotic behaviors (3.3) for $\beta \to -\infty$ and (3.6) for $\beta \to +\infty$, one may expect the asymptotic dependence on μ of the norm

$$\|(\mathcal{A}-\lambda)^{-1}f\|_{L^2(\mathbb{R}^2)}$$

for fixed f to be different with that of the norm of the resolvent

$$\|(\mathcal{A}-\lambda)^{-1}\|.$$

This is indeed true. We now derive this different dependence if some additional conditions on the support of $\hat{f}$ are assumed. As an example consider, for $a \geq 0$, the space

$$L^2_a(\mathbb{R}^2,\mathbb{C}) = \left\{u \in L^2(\mathbb{R}^2,\mathbb{C}) \, : \, \mathrm{Supp}\,\hat{\mathrm{u}} \subset \{(\mathrm{x},\omega) : \omega \leq \mathrm{a}\}\right\}, \tag{5.84}$$

where, as above, we denote by $\hat{u}$ the partial Fourier with respect to the y variable of u. Denote by Π_a the orthogonal projector

$$\Pi_a : L^2(\mathbb{R}^2,\mathbb{C}) \to L^2_a(\mathbb{R}^2,\mathbb{C}).$$

The next theorem shows that $\exp(-t\mathcal{A})\circ\Pi_a$ and $(\mathcal{A}-\lambda)^{-1}\circ\Pi_a$ have a different respective behavior as $t \to +\infty$ or $\lambda \to +\infty$, than that of $\exp(-t\mathcal{A})$ and $(\mathcal{A}-\lambda)^{-1}$.

Theorem 5.6. *For any $a \geq 0$, there exists a constant $T_0(a)$ such that, for any $t \geq T_0(a)$ we have*

$$\|\exp(-t\mathcal{A})\circ\Pi_a\| \leq \exp\Big(-\frac{\Psi_a(ct)}{c}\Big), \tag{5.85}$$

where

$$\Psi_a(t) = \inf_{\omega \leq a} \Phi(\omega,t), \tag{5.86}$$

where Φ is given by (4.5). *In particular, for any $a \geq 0$, there exists $C(a) > 0$, such that for $t \geq T_0(a)$ we have*

$$\|\exp(-t\mathcal{A})\circ\Pi_a\| \leq C(a)\exp\left(-\frac{c^2}{3}t^3 + C(a)t^{3/2}\right), \tag{5.87}$$

The proof is the same as that for Theorem 4.2 and Lemma 4.4 except that now we take in consideration the information on $\mathrm{Supp}\,\hat{\mathrm{u}}$. Clearly, (4.3) is still valid, but as $\hat{u} \equiv 0$ for $\omega \geq a$ we obtain an estimate similar to (4.10) but with $\Psi_a(ct)$ instead of $\widehat{\Psi}(ct)$. To prove (5.87) we note that, for sufficiently large t we have

$$\Psi_a(t) = \Phi(a,t),$$

and hence (5.87) easily follows from (3.3).

As in Subsection 4.1, Theorem 5.6 implies :

Corollary 5.7. *For any $a \geq 0$, there exists $\lambda_0(a)$ and $C(a)$, such that, for $\lambda \geq \lambda_0(a)$,*

$$\|(\mathcal{A}-\lambda)^{-1}\circ\Pi_a\| \leq \exp\Big(\frac{2}{3c}\lambda^{3/2} + C(a)\lambda^{3/4}\Big). \tag{5.88}$$

Observing, that the subspace

$$V = \bigcup_{a \geq 0} L^2_a(\mathbb{R}^2,\mathbb{C})$$

is dense in $L^2(\mathbb{R}^2,\mathbb{C})$, we also obtain

Corollary 5.8. *There exists a dense set V in $L^2(\mathbb{R}^2, \mathbb{C})$, such that for each $f \in V$ there exist positive constants $\alpha = \alpha(f)$ and $\mu_0 = \mu_0(f)$ such that for all λ with $\operatorname{Re} \lambda = \mu > \mu_0(f)$ it holds that*

$$\|(\mathcal{A} - \lambda)^{-1} f\|_{L^2(\mathbb{R}^2)} \leq \exp(\alpha \mu^{3/2}) \|f\|_{L^2(\mathbb{R}^2)}. \tag{5.89}$$

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Appendix A. Proof of (5.12)

By (3.5) we may write

$$\phi_k = \phi_{k,0} + |\beta|^{-3/4}\phi_{k,1} \quad k = 1, 2\,,$$

in which $\|\phi_{k,1}\|_2 \leq C$ where C is independent of β. Note that since $\|\phi_k\|_2 = 1$ and $\|\phi_{k,0}\|_2 = 1 + \mathcal{O}(e^{-S\beta})$ for some $S > 0$, we have

$$1 = \|\phi_{k,0} + |\beta|^{-3/4}\phi_{k,1}\|_2^2 = 1 + 2|\beta|^{-3/4}\langle\phi_{k,0}, \phi_{k,1}\rangle + |\beta|^{-3/2}\|\phi_{k,1}\|_2^2 + \mathcal{O}(e^{-S\beta})\,,$$

which leads to

$$|\langle\phi_{k,0}, \phi_{k,1}\rangle| \leq \frac{C}{|\beta|^{3/4}} \quad k = 1, 2\,. \tag{A.1}$$

Furthermore, by (3.5) and (5.8) we have that

$$\|(x^2 + 2\beta)\phi_k\|_2 \leq C|\beta|^{1/4} \quad k = 1, 2\,. \tag{A.2}$$

We can then write

$$\begin{aligned}&\|(x^2 + 2\beta)\phi_0\|_2^2 \\ =&\|(x^2 + 2\beta)\phi_{0,0}\|_2^2 + 2|\beta|^{-3/4}\langle(x^2 + 2\beta)\phi_{0,0}, (x^2 + 2\beta)\phi_{0,1}\rangle + \mathcal{O}(|\beta|^{-1})\,,\end{aligned} \tag{A.3a}$$

$$\begin{aligned}&|\langle(x^2 + 2\beta)\phi_0, \phi_0\rangle|^2 \\ =&|\langle(x^2 + 2\beta)\phi_{0,0}, \phi_{0,0}\rangle|^2 + 4|\beta|^{-3/4}\langle(x^2 + 2\beta)\phi_{0,0}, \phi_{0,0}\rangle\langle(x^2 + 2\beta)\phi_{0,0}, \phi_{0,1}\rangle + \mathcal{O}(|\beta|^{-1})\,,\end{aligned} \tag{A.3b}$$

and

$$\begin{aligned}&|\langle(x^2 + 2\beta)\phi_0, \phi_1\rangle|^2 = |\langle(x^2 + 2\beta)\phi_{0,0}, \phi_{1,0}\rangle|^2 \\ &+2|\beta|^{-3/4}\big[\langle(x^2 + 2\beta)\phi_{0,0}, \phi_{1,1}\rangle + \langle(x^2 + 2\beta)\phi_{1,0}, \phi_{0,1}\rangle\big]\langle(x^2 + 2\beta)\phi_{0,0}, \phi_{1,0}\rangle + \mathcal{O}(|\beta|^{-1})\,.\end{aligned} \tag{A.3c}$$

Furthermore, orthogonality properties of Hermite functions yield

$$\Big|\|(x^2 + 2\beta)\phi_{0,0}\|_2^2 - |\langle(x^2 + 2\beta)\phi_{0,0}, \phi_{0,0}\rangle|^2 - |\langle(x^2 + 2\beta)\phi_{1,0}, \phi_{0,0}\rangle|^2\Big| \leq \frac{C}{\beta}\,.$$

We thus obtain from (A.3) that

$$\begin{aligned}&\|(x^2 + 2\beta)\phi_0\|_2^2 - |\langle(x^2 + 2\beta)\phi_0, \phi_0\rangle|^2 - |\langle(x^2 + 2\beta)\phi_1, \phi_0\rangle|^2 \\ =&|\beta|^{-3/4}\big\{2\langle(x^2 + 2\beta)\phi_{0,0}, (x^2 + 2\beta)\phi_{0,1}\rangle - 4\langle(x^2 + 2\beta)\phi_{0,0}, \phi_{0,0}\rangle\langle(x^2 + 2\beta)\phi_{0,0}, \phi_{0,1}\rangle \\ &- 2\big[\langle(x^2 + 2\beta)\phi_{0,0}, \phi_{1,1}\rangle + \langle(x^2 + 2\beta)\phi_{1,0}, \phi_{0,1}\rangle\big]\big\}\langle(x^2 + 2\beta)\phi_{0,0}, \phi_{1,0}\rangle + \mathcal{O}(|\beta|^{-1})\,.\end{aligned} \tag{A.4}$$

By the standard theory of orthogonal polynomials and Hermite functions in particular (see [1] for instance) we have that

$$\big|\langle(x^2 + 2\beta)\phi_{0,0}, \phi_{0,0}\rangle\big| \leq C|\beta|^{-1/2}\,. \tag{A.5}$$

Furthermore,

$$\big|\langle (x^2+2\beta)\phi_{0,0},\phi_{1,1}\rangle\big| \leq |2\beta|^{1/4}\big|\langle \phi_{1,0},\phi_{1,1}\rangle\big| + \|(x^2+2\beta)\phi_{0,0} - |2\beta|^{1/4}\phi_{1,0}\|_2\|\phi_{1,1}\|_2\,.$$

By (A.1) and by evaluating the integral of Hermite functions on the right-hand-side of the above inequality we obtain

$$\big|\langle (x^2+2\beta)\phi_{0,0},\phi_{1,1}\rangle\big| \leq C|\beta|^{-1/2}\,.$$

Substituting the above together with (A.5) into (A.4) yields

$$\begin{aligned}
&\|(x^2+2\beta)\phi_0\|_2^2 - |\langle (x^2+2\beta)\phi_0,\phi_0\rangle|^2 - |\langle (x^2+2\beta)\phi_1,\phi_0\rangle|^2 \\
=&|\beta|^{-3/4}\big\{2\langle (x^2+2\beta)\phi_{0,0},(x^2+2\beta)\phi_{0,1}\rangle - 2\langle (x^2+2\beta)\phi_{1,0},\phi_{0,1}\rangle\langle (x^2+2\beta)\phi_{0,0},\phi_{1,0}\rangle\big\} \\
&+\mathcal{O}(|\beta|^{-1})\,.
\end{aligned} \tag{A.6}$$

To complete the proof we expand $(x^2+2\beta)\phi_{0,0}$ into a series of Hermite functions to obtain

$$\begin{aligned}
&\langle (x^2+2\beta)\phi_{0,0},(x^2+2\beta)\phi_{0,1}\rangle \\
=&\langle (x^2+2\beta)\phi_{0,0},\phi_{0,0}\rangle\langle (x^2+2\beta)\phi_{0,0},(x^2+2\beta)\phi_{0,1}\rangle \\
&+\langle (x^2+2\beta)\phi_{0,0},\phi_{1,0}\rangle\langle (x^2+2\beta)\phi_{0,0},(x^2+2\beta)\phi_{0,1}\rangle + \mathcal{O}(\beta^{-1/4})\,.
\end{aligned}$$

Substituting into (A.6) we obtain (5.12) with the aid of (A.5).

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