Ensemble-average versus suspension-scale Cauchy continuum-mechanical definitions of stress in polarized suspensions: Global homogenization of a dilute suspension of dipolar spherical particles

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(Received 13 August 1997; accepted 30 October 1998)

The macroscale rheological properties of a dilute suspension exposed to a uniform external field and composed of identical, rigid, inhomogeneous, dipolar, spherical particles dispersed in an incompressible Newtonian fluid and possessing the same mean density as the latter fluid are derived from knowledge of its microscale properties by applying a global ensemble-averaging technique. Each dipole, which is permanently embedded in the particle, is assumed to be generated by the presence of an inhomogeneous external body-force field in the particle interior resulting from the action of the uniform external field on an inhomogeneous distribution of interior matter. It is shown that although the ensemble-average stress tensor is symmetric, the suspension nevertheless behaves macroscopically as if it possessed an asymmetric stress tensor. This seeming contradiction can be traced to the fact that the average body force acting on the contents of any arbitrarily drawn volume lying in the interior of the suspension does not vanish despite the fact that each particle is ''neutrally buoyant.'' That this force is not zero stems from the fact that some particles necessarily straddle the closed surface bounding that volume, and that the distribution of external body forces over the interiors of these particles is nonuniform. As such, that portion of the spherical particle lying outside of the surface enclosing the domain exerts a force on the remaining portion of the sphere lying within that domain. We then demonstrate that the natural macroscopic model, which is derived by equating the divergence of the suspension-scale stress appearing in that model to the ensemble-average external body-force field, and which predicts a *symmetric* stress tensor, is macroscopically deficient with respect to the more intuitive *asymmetric* stress model usually proposed by continuum mechanicians for such a suspension. It is shown that the latter, continuum-mechanical model recovers all the physically interesting properties of the suspension.
 [S1070-6631(99)02602-1]

I. INTRODUCTION

While the present contribution nominally addresses itself to the somewhat arcane subject of the rheological properties of polar and dipolar suspensions, it actually goes beyond that topic in raising fundamental questions about the definition of the suspension-scale stress tensor. In particular, it questions whether the macroscopic stress tensor proposed for such suspensions by adopting a continuum-mechanical point of view should be identical to the ensemble-average stress tensor—the former specifying the contact force acting across an arbitrarily drawn surface in the ''continuous suspension.'' This issue arises in the context of asymmetric states of stress in heterogeneous continua, such as are encountered in ferrofluid suspensions.[1]

It is well known in continuum mechanics[2,3] that in absence of an external body-couple density field, angular momentum is conserved if and only if the stress tensor is symmetric (provided that rotational inertia and couple-stress fields are also absent). Such materials are termed nonpolar. Conversely, in the presence of such external body-couple fields (and under the same parenthetic circumstances described in the preceding sentence), continuum-mechanical models require the existence of an asymmetric stress tensor. Such continua are termed polar. The necessity for an antisymmetric stress arises from the fact that the couple generated by a symmetric stress tensor acting over a (net force-free) surface bounding an arbitrary volume vanishes identically. Consequently, in the absence of an inhomogeneous external body-force density field, any body-couple field present must be balanced by an antisymmetric stress tensor contribution.

One of the rheologically interesting systems for which body couples and asymmetric stresses are assumed to exist macroscopically is posed by a dilute suspension composed of identical, density-matched, rigid, dipolar particles dispersed in a homogeneous incompressible Newtonian fluid in the presence of a uniform external field. In continuum mechanics[1] it is usually assumed *a priori* that since the particle size and average distance between proximate particles are each small relative to the macroscopic length scale on which the mean velocity field varies, that the suspension can be viewed macroscopically as consisting of polar matter, thereby enabling the discrete system of particles to be replaced by a continuous distribution of infinitesimal dipoles—manifesting itself at the suspension scale as a body-couple field when subjected to an external field.

Batchelor[4] and Brenner[5,6] present formal rheological calculations justifying the above assumption. Both works demonstrate (Batchelor[4] by ensemble averaging the stress over a suspension-scale continuum element, and Brenner[5,6] by averaging over a suspension cell) the existence of an asymmetric bulk stress tensor in cases where the microscale stress inside the particles is itself asymmetric due to the presence there of an external body-couple field (and the external body-force field, if any, is homogeneous throughout both the fluid and particles). While the couple on the particle in that case is generated by the existence of a body-couple field, a couple can also be generated even in the absence of such body-couple fields by the existence of an *inhomogeneous* external body-force distribution within the particle, such as arises from an inhomogeneous mass distribution in the presence of gravity—the so-called ''loaded-sphere'' case.[5,7–9] (A loaded sphere is distinguished by the fact that its center of mass does not coincide with its center of volume, through which the buoyant force exerted by the fluid acts, resulting in the creation of a mass dipole.) In the loaded-sphere case the stress inside the particles is symmetric, whence the symmetry of the stress tensor both inside and outside of the particles necessitates that the ensemble-average stress tensor be symmetric as well! Another case in which a couple on a suspended particle can be sustained arises when the couple is equilibrated by couple stresses existing in the particle interior.[10] In that case the source of the resulting ensemble-average asymmetric stress is different from that of Batchelor[4] and Brenner,[5,6] although the effect is the same. We will not further discuss that case.

Despite the above arguments regarding the symmetry of the ensemble-average stress tensor in the loaded-sphere case, we nevertheless expect the suspension-scale stress tensor to be asymmetric (see the first paragraph of Sec. IV). Unlike the ensemble-average stress, which depends upon the detailed manner in which the stresses within the particles are distributed, the ensemble-average velocity field as well as the ensemble-average surface traction on the external boundaries of the suspension are independent of this internal stress distribution. Rather, they are completely determined, both inside and outside of the particles by the resultant net couple exerted on each particle by the external field, and not by the explicit manner via which this couple arises from the internal constitution of the particle. The formal derivation of the various suspension-scale mean properties proposed by Batchelor[4] and Brenner[5,6] is therefore in need of generalization since it is based upon the local stress field existing inside of the particles.

In the present work we formally demonstrate that the dipolar suspension may indeed be regarded as macroscopically composed of polar matter, and that the asymmetric suspension-scale stress model[4–6] indeed recovers the physically interesting properties of the suspension. (No consensus exists in the literature as to the choice of terminology used to distinguish between ''dipolar'' and ''polar'' suspensions; in the present contribution we adopt the historical, chronological definitions of these terms,[2,7] which differ from those used later by Batchelor[4] and Rosensweig.[1]) We calculate, using formal ensemble averaging, the average stress and body-force fields. We then derive a macroscopic model which recovers these average fields, and subsequently compare it with the more traditional approach, in which the macroscopic asymmetric stress tensor is *a priori* assumed asymmetric.

Explicitly, in Sec. II which follows, we derive both the ensemble-average stress, and body-force fields, up to $O(c)$, c being the particle volume fraction. Next, in Sec. III we quantify the macroscopic model satisfied by the hypothetical homogeneous medium which can reproduce the gross, ensemble-average properties of the suspension, subsequently comparing it with the more ''traditional'' approach,[4–6] in which an asymmetric macroscopic stress tensor is introduced. Concluding remarks in Sec. IV provide some further insights into the macroscopic model presented in Sec. III.

II. THE ENSEMBLE-AVERAGE FIELDS

Consider N identical rigid dipolar spherical particles of radii a, each containing an embedded dipole, dispersed in a homogeneous Newtonian fluid of viscosity μ. Each such permanently embedded dipole is regarded as arising from an inhomogeneous external body-force field within the particle. The centers of spheres $1,2,\ldots,N$ are, respectively, situated at the points $(\mathbf{x}_1,\ldots,\mathbf{x}_N)$ and the dipole moments of the spheres are, respectively, oriented in the directions $(\mathbf{e}_1,\ldots,\mathbf{e}_N)$, the latter set representing unit vectors embedded along the dipole axes of the spheres. Such a set of locations and orientations will be termed a ''configuration.''

Denote by Ω the domain of the suspension, and by $(\overline{\mathbf{u}},\overline{p})$ the ambient flow which satisfies the prescribed microscale boundary conditions on the external boundaries $\partial\Omega$ of the suspension. The domain, ambient flow, and particle number density $n(\mathbf{x})$ are all characterized by a macroscopic length scale R', say, which is much larger than both the particle size and average distance l between the particles. It is convenient to non-dimensionalize the spatial coordinate with l. Denote the dimensionless radius of the suspended spheres by $\epsilon\,(\epsilon\ll 1)$, supposed small compared with unity since the suspension is dilute. The dimensionless macroscopic length scale R'/l will be denoted by $R\,(R\gg 1)$.

When inertial effects are negligible it can be shown[11] that the ensemble-average stress is given to $O(c)$ by

$$\langle\boldsymbol{\sigma}\rangle(\mathbf{x})\cong\Sigma(\langle\mathbf{u}\rangle)+c\overline{p}\mathbf{I}+\int_{\xi\leqslant\epsilon}\langle\boldsymbol{\sigma}^{(p)}(\mathbf{x},\mathbf{x}+\boldsymbol{\xi})\rangle^{(1)}n(\mathbf{x}+\boldsymbol{\xi})d\boldsymbol{\xi}+O(c^2). \tag{2.1}$$

In the above the angular brackets $\langle\cdot\rangle$ denote averaging over all possible N-sphere locations and orientations; that is, for any tensor field $\mathbf{A}(\mathbf{x},\mathbf{x}_1,\ldots,\mathbf{x}_N,\mathbf{e}_1,\ldots,\mathbf{e}_N)$,

$$\langle \mathbf{a}\rangle(\mathbf{x})=\int \mathbf{A}q_N(\mathbf{x}_1,\ldots,\mathbf{x}_N,\mathbf{e}_1,\ldots,\mathbf{e}_N)d\mathbf{x}_1\cdots\mathbf{x}_N d\mathbf{e}_1\cdots\mathbf{e}_N,$$

where q_N denotes the multiparticle probability density. Similarly, the bracketed superscript operator $\langle\cdot\rangle^{(1)}$ denotes averaging over all possible N-sphere locations and orientations while excluding the location (but including the orientation) of one of the particles. (Clearly, since all the particles are identical, the explicit choice of which particle is to be excluded is arbitrary.) Thus, $\langle\boldsymbol{\sigma}\rangle$ denotes the average stress field, whereas $\Sigma(\langle\mathbf{u}\rangle)$ denotes the stress due to the average velocity field:

$$\Sigma_{ij}(\langle\mathbf{u}\rangle)=\mu\left(\frac{\partial\langle u_i\rangle}{\partial x_j}+\frac{\partial\langle u_j\rangle}{\partial x_i}\right)-\langle p\rangle\delta_{ij}, \tag{2.2}$$

where $\langle p\rangle$ denotes the average pressure field. Finally, $\boldsymbol{\sigma}^{(p)}(\mathbf{x},\mathbf{x}_1)$ is the stress at the point $\mathbf{x}$ which lies inside of the particle whose center is situated at $\mathbf{x}_1$.

Assuming that $\boldsymbol{\sigma}^{(p)}(\mathbf{x},\mathbf{x}+\boldsymbol{\xi})\cong\boldsymbol{\sigma}^{(p)}(\mathbf{x}-\boldsymbol{\xi},\mathbf{x})+O(\epsilon/R)$,[11] we obtain the following relation upon utilizing the slow variation of n, i.e., $n(\mathbf{x}+\boldsymbol{\xi})\cong n(\mathbf{x})+O(\epsilon/R)$ together with the divergence theorem:

$$\langle\sigma_{ij}\rangle=\Sigma_{ij}(\langle\mathbf{u}\rangle)+c\bar{p}\delta_{ij}-\left[\int_{\xi=\epsilon}\langle f_i\rangle^{(1)}\xi_j ds_\xi-\int_{\xi\le\epsilon}\langle b_i\rangle^{(1)}\xi_j d\boldsymbol{\xi}\right]n(\mathbf{x}). \tag{2.3}$$

In the latter, $\mathbf{f}$ denotes the surface traction exerted by the body on the fluid ($\mathbf{f}=\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}$, with $\hat{\mathbf{n}}$ the inward unit normal), and $\mathbf{b}$ denotes the external body force density field, which is generated by external means. For the polar case, discussed by Batchelor[4] and Brenner,[5] $\mathbf{b}\equiv$const. Consequently, since the antisymmetric portion of the first integral on the right-hand side, representing the torque exerted by the fluid on the particle, does not vanish, the average stress must be asymmetric, reflecting the asymmetry of $\boldsymbol{\sigma}^{(p)}$. For the dipolar case, however, the torques exerted by the body force and the fluid on the particle must balance each other. We now approximate[11] $\langle\mathbf{f}\rangle^{(1)}$ by the surface traction on a neutrally buoyant spherical particle on which a couple $\langle\mathbf{L}\rangle^{(1)}$ is exerted by external means, and which is subjected to a homogeneous shear flow. Explicitly, in the absence of boundaries,

$$\langle f_i\rangle^{(1)}\cong\frac{3}{8\pi\epsilon^3}\epsilon_{ijk}\hat{n}_j\langle L_k\rangle^{(1)}+(5\mu\bar{E}_{ij}-\bar{p}\delta_{ij})\hat{n}_j, \tag{2.4}$$

in which $\bar{\mathbf{E}}$ denotes the local ambient rate of strain. Substituting in (2.3) we obtain

$$\langle\sigma_{ij}\rangle=\mu\left(1+\frac{5}{2}c\right)\left(\frac{\partial\langle u_i\rangle}{\partial x_j}+\frac{\partial\langle u_j\rangle}{\partial x_i}\right)-\langle p\rangle\delta_{ij}+\frac{1}{2}\int_{\xi\le\epsilon}[\langle b_i\rangle^{(1)}\xi_j+\langle b_j\rangle^{(1)}\xi_i]d\boldsymbol{\xi}n(\mathbf{x}). \tag{2.5}$$

Examination of (2.5) leads immediately to two interesting observations: The first is that the ensemble-average stress is symmetric! Trivially, the average of a symmetric field (the stress field is symmetric both inside and outside of the particles) over the configuration space must be symmetric too. However, as stated in Sec. I, intuition rather suggests that the suspension-scale stress tensor should be asymmetric as a consequence of the couples experienced by the particles, similar to the asymmetry observed by Batchelor[4] and Brenner[5,6] (see the arguments in the first paragraph of Sec. IV). Of course, their calculations assume a *homogeneous* external body-force field inside the particles, considering instead the polar case in which the couple is exerted by an external body-couple field inside the particles. In that case the angular momentum equation requires the stress field within the particles to be asymmetric (at least in the absence of couple stresses and rotary inertia effects), whence the ensemble-average stress tensor too must be asymmetric.

The second interesting consequence of (2.5) is the existence of an additional symmetric contribution to the ensemble-average stress arising from the external body forces. This term is, in general, absent when the body forces are either homogeneously distributed or else entirely absent.[4–6] However, a similar term appears in the expression for the average stress tensor in a polymeric liquid composed of a Newtonian fluid in which Hookian dumbbell-like bead-spring particles are suspended.[12] Its presence there arises from the recognition that internal forces acting within their particles (namely the tensile force acting along the spring connecting a pair of beads) must have an effect on the average stress.

We now calculate the ensemble-average external force exerted on a macroscopic volume situated within the interior of Ω, away from the boundaries. To this end we first calculate the external force acting on such a macroscopic domain [denoted by B with boundaries ∂B, and of $O(R)$] of the suspension for a given N-particle configuration. Since each particle experiences no *net* force owing to its "neutral buoyancy," such a force may be exerted on B only by those particles intersected by (i.e., straddling) the surface ∂B. Consequently,

$$\mathbf{F}=\sum_{n=1}^{k}\mathbf{F}_n(\mathbf{x}_n,\mathbf{e}_n), \tag{2.6}$$

wherein $\mathbf{F}$ is the resultant dimensionless external force acting on B, k is the number of particles intersecting its boundaries, and $\mathbf{F}_n$ is the dimensionless external force acting on that portion of the nth particle which lies inside of B. The latter is expressible in the form

$$\mathbf{F}_n=\int_{\substack{\xi\le\epsilon\\ \boldsymbol{\xi}+x_n\in B}}\mathbf{b}(\boldsymbol{\xi})d\boldsymbol{\xi}. \tag{2.7}$$

Averaging (2.6) over configuration space yields

$$\langle\mathbf{F}\rangle=\int_{\partial B_\epsilon}\int_{\substack{\xi\le\epsilon\\ \boldsymbol{\xi}+\mathbf{x}_n\in B}}\langle\mathbf{b}\rangle(\boldsymbol{\xi})d\boldsymbol{\xi}n(\mathbf{x}_1)d\mathbf{x}_1, \tag{2.8}$$

in which

$$\partial B_\epsilon=\{\mathbf{x}_1:d(\mathbf{x}_1,\partial B)\le\epsilon\}$$

is a layer of width 2ϵ surrounding the boundaries ∂B of B. If the radius of curvature of ∂B is of $O(R)$ at all points we may approximate the integral over ∂B_ϵ, thereby generating an

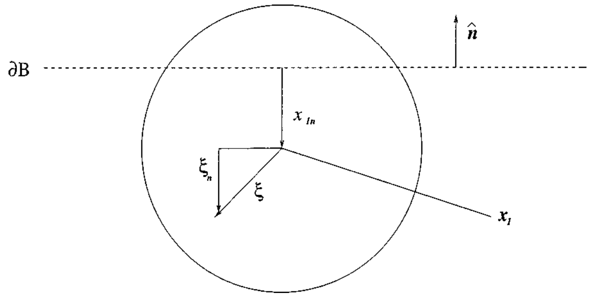

FIG. 1. Description of the various coordinates appearing in (2.9).

error of $O(\epsilon/R)$, by integrating first with respect to the coordinate normal to the boundaries, and then over ∂B to obtain

$$\langle \mathbf{F}\rangle=\int_{\partial B} ds_{x_1}\int_{-\epsilon}^{\epsilon} n(\mathbf{x}_1)\,d\mathbf{x}_{1n}\int_{-\epsilon}^{-x_{1n}} d\boldsymbol{\xi}_n\int_{\substack{\xi\leqslant\epsilon\\ \boldsymbol{\xi}\cdot\hat{\mathbf{n}}=\xi_n}}\langle \mathbf{b}\rangle ds_{\xi}. \tag{2.9}$$

Here, $\hat{\mathbf{n}}$ is the outward normal to ∂B, whereas $x_{1n}=\mathbf{x}_1\cdot\hat{\mathbf{n}}$ and $\xi_n=\boldsymbol{\xi}\cdot\hat{\mathbf{n}}$. (Figure 1 defines the various coordinates.)

It is easy to show by changing the order of integration that the above ensemble-average external force possesses the alternative form

$$\langle \mathbf{F}\rangle=-\int_{\partial B}\left[\int_{\xi\leqslant\epsilon}\langle \mathbf{b}\rangle\boldsymbol{\xi}\,d\boldsymbol{\xi}\right]\cdot\hat{\mathbf{n}}\,n(\mathbf{x})\,ds_x. \tag{2.10}$$

Suppose now that a portion of ∂B coincides with $\partial\Omega$, the solid boundaries of the suspension as a whole. Designate by $\partial B_{\mathrm{int}}$ the remainder of ∂B. In this case the average external force on the contents of B is given by

$$\langle \mathbf{F}\rangle=-\int_{\partial B_{\mathrm{int}}}\left[\int_{\xi\leqslant\epsilon}\langle \mathbf{b}\rangle\boldsymbol{\xi}\,d\boldsymbol{\xi}\right]\cdot\hat{\mathbf{n}}\,n(\mathbf{x})\,ds_x, \tag{2.11}$$

since particles cannot intersect the solid boundaries of the suspension. Hence, a discontinuity is predicted to exist in the average external force near the boundaries. This discontinuity may be represented by the additional surface traction $\mathbf{f}_e=\boldsymbol{\tau}_e\cdot\hat{\mathbf{n}}$, where $\boldsymbol{\tau}_e$ is given by

$$\boldsymbol{\tau}_e=-\int_{\xi\leqslant\epsilon}\langle \mathbf{b}\rangle\boldsymbol{\xi}\,d\boldsymbol{\xi}\,n\mathbf{x}) \tag{2.12}$$

on the boundaries. The need for introducing such an additional force distribution clearly arises as a consequence of the fact that suspended particles cannot penetrate the solid boundaries surrounding the suspension.

III. THE HYPOTHETICAL HOMOGENEOUS MEDIUM

To complete the process of homogenization it is necessary to formulate a boundary-value problem for a hypothetical homogeneous medium which is kinematically and dynamically ''equivalent'' in its macroscopic, suspension-scale consequences to that predicted by its ensemble-average counterpart. In particular, the velocity and stress fields obtained from the solution of this continuum-mechanical boundary-value problem must be asymptotically identical to the respective ensemble-average velocity and stress fields.

The natural way to derive this macroscopic continuum-mechanical model is to require that the sum of the divergence of the suspension-scale stress field, $\boldsymbol{\tau}$, say, and the ensemble-average external body-force field, given by the divergence of (2.12), would vanish. Then, with $\mathbf{v}$ the suspension-scale velocity field, the pertinent continuum-mechanical equations governing the unknown fields $\boldsymbol{\tau}$ and $\mathbf{v}$ to be determined are as follows:

$$\boldsymbol{\nabla}\cdot\mathbf{v}=0, \tag{3.1a}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{\tau}+\boldsymbol{\nabla}\cdot\boldsymbol{\tau}^e=0, \tag{3.1b}$$

$$\mathbf{v}|_{\partial\Omega_u}=\overline{\mathbf{u}}|_{\partial\Omega_u}, \tag{3.1c}$$

$$\boldsymbol{\tau}\cdot\hat{\mathbf{n}}|_{\partial\Omega_f}=\overline{\mathbf{f}}+\boldsymbol{\tau}^e\cdot\hat{\mathbf{n}}|_{\partial\Omega_f}, \tag{3.1d}$$

in which $\boldsymbol{\tau}$ is related to $\mathbf{v}$ by the constitutive relation

$$\boldsymbol{\tau}\cong\mu\left(1+\frac{5}{2}c\right)(\boldsymbol{\nabla}\mathbf{v}+\boldsymbol{\nabla}\mathbf{v}^{\dagger})-p\mathbf{I}+\frac{1}{2}\int_{\xi\leqslant\epsilon}[\langle \mathbf{b}\rangle^{(1)}(\mathbf{x},\boldsymbol{\xi})\boldsymbol{\xi}+\boldsymbol{\xi}\langle \mathbf{b}\rangle^{(1)}(\mathbf{x},\boldsymbol{\xi})]\,d\boldsymbol{\xi}\,n(\mathbf{x})+O(c^2), \tag{3.1e}$$

and wherein $\boldsymbol{\tau}^e$ is given by (2.12).

The velocity field must agree (and can be shown to agree) with that obtained for the case in which the particles are composed of polar matter,[4–6] for which circumstances an *asymmetric*, suspension-scale, macroscopic stress tensor is obtained. In that case, to $O(c)$ the macroscopic velocity and stress fields satisfy the following equations:

$$\boldsymbol{\nabla}\cdot\mathbf{v}=0, \tag{3.2a}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{\tau}=0, \tag{3.2b}$$

accompanied by the constitutive relation

$$\boldsymbol{\tau}=\mu(1+\tfrac{5}{2}c)(\boldsymbol{\nabla}\mathbf{v}+\boldsymbol{\nabla}\mathbf{v}^{\dagger})+\boldsymbol{\tau}^a-p\mathbf{I}, \tag{3.2c}$$

$$\mathbf{v}|_{\partial\Omega_u}=\overline{\mathbf{u}}|_{\partial\Omega_u}, \tag{3.2d}$$

$$\boldsymbol{\tau}\cdot\hat{\mathbf{n}}|_{\partial\Omega_f}=\overline{\mathbf{f}}|_{\partial\Omega_f}, \tag{3.2e}$$

in which the antisymmetric stress is given by the expression

$$\tau^a=\boldsymbol{\epsilon}\cdot\tfrac{1}{2}n\langle \mathbf{L}\rangle^{(1)}. \tag{3.2f}$$

Note that both (3.1) and (3.2) lead to the same boundary-value problem for $\mathbf{v}$.

For this polar particle case, no force distribution needs to be introduced on the boundaries beyond the fluid contact forces, as is clearly seen in (3.2e). Thus, in that case, to calculate the average force exerted by the fluid on a surface $S\subset\partial\Omega$, we may write

$$\mathbf{F}=\int_S\boldsymbol{\tau}\cdot\hat{\mathbf{n}}\,ds.$$

In contrast, for the dipolar case treated in this paper, where the couple arises from an inhomogeneous external body-force distribution within the particles, the force on S is of the form

$$\mathbf{F}=\int_S(\boldsymbol{\tau}+\boldsymbol{\tau}^e)\cdot\hat{\mathbf{n}}\,ds.$$

Since $\boldsymbol{\tau}+\boldsymbol{\tau}^e$ is identical to the polar matter stress tensor, the force distribution on the boundaries is identical in the two cases.

Despite this equality, the ensemble-average force and couple exerted by the external field on a macroscopic domain *which lies entirely in the interior of* Ω depends upon the internal constitution of the suspended particles. For dipolar matter the force is given by (2.10), whereas for polar matter it must vanish. Moreover, the external couple exerted on the material contained in a domain B which lies entirely in the interior of Ω is

$$\int_B \mathbf{r}\times\boldsymbol{\nabla}\cdot\boldsymbol{\tau}^e dv=\int_{\partial B}\mathbf{r}\times\boldsymbol{\tau}^e\cdot\hat{\mathbf{n}}\,ds-\int_B \boldsymbol{\epsilon}:\boldsymbol{\tau}^e\,dv,$$

in which we have employed the (macroscale) divergence theorem to obtain the right-hand side of the latter equation. In this equation $\mathbf{r}$ denotes the position vector. While the second term on the right represents the macroscopic external body-couple density in the polar matter case (again assuming the absence of couple stresses and rotary inertia effects), the first term is nonexistent for that case. It results from the forces exerted by the external field on those particles which straddle the boundary ∂B, and no such forces exist in the polar matter case. Though many more particles lie in the interior of B than on its boundary ∂B, the couple which is exerted on each interior particle is much smaller than the couple exerted by the forces on the boundaries. As such, both terms are generally of comparable order. Of course, both models predict the same overall external couple on the solid surface Ω bounding the suspension, since both predict the same force distribution on Ω.

IV. CONCLUDING REMARKS

From a continuum-mechanical, Cauchy stress point of view the source of the asymmetric stress in *both* polar and dipolar suspensions is patently obvious. Inasmuch as each of the suspended particles experiences an external couple—but no net force, so that the torque exerted on each particle is represented by an origin-independent couple—a macroscale domain (of volume V, say, centered at a point $\mathbf{x}$ of the suspension) and containing N particles will experience an external couple $\Sigma_{n=1}^{N}\mathbf{L}^{(n)}$, where $\mathbf{L}^{(n)}$ is the external couple on the nth particle in V (all variables in this paragraph being dimensional). The quantity $\overline{\mathbf{L}}$ defined as

$$\overline{\mathbf{L}}=\lim_{V\to 0}\frac{1}{V}\sum_{n=1}^{N}\mathbf{L}^{(n)}$$

thus represents the volumetric suspension-scale external body-couple density field at the point $\mathbf{x}$. According to angular momentum principles applied to the suspension-scale continuum, as a consequence of the existence of this body-couple field the suspension must experience an antisymmetric stress, $\overline{\boldsymbol{\tau}}^a$, say, whose vector invariant $\overline{\boldsymbol{\tau}}_\times\equiv-\boldsymbol{\epsilon}:\overline{\boldsymbol{\tau}}^a$ satisfies the Cauchy moment-of-momentum equation, $\overline{\boldsymbol{\tau}}_\times+\overline{\mathbf{L}}=0$ (in which intrinsic rotary inertia and couple stresses are both assumed absent).[2] Consequently, the Cauchy principle requires that antisymmetric stresses exist in all polarized suspensions. And this is true whether the external couples acting on the suspended particles arise from body-couple fields existing within the (''polar'') particles or from inhomogeneous body-force fields present within the otherwise neutrally buoyant (''dipolar'') particles. That the ensemble-average stress distinguishes sharply between these two cases is a manifestation of the fundamental conflict as to which of the two stress fields, namely the ensemble-average stress or the Cauchy continuum-mechanical stress, properly represents the appropriate suspension-scale stress!

The fundamental issue in the dipolar case is, however, to some extent semantic rather than substantiative. The *physical* definition of the suspension-scale stress tensor is that its dot product with a directed surface element gives the contact force exerted by the material lying on one side of the surface upon the material lying on the other side. Choosing this surface to lie in the interior of the suspension allows particles to straddle that surface, whence the appropriate stress tensor consistent with this definition is the (symmetric) ensemble-average stress tensor, given by Eq. (3.1e). On the other hand, if the surface is chosen to lie on the rigid boundaries of the apparatus housing the suspension, then no particles can straddle that surface. In such circumstances the appropriate stress tensor consistent with the continuum-mechanical definition of stress given above is the asymmetric stress tensor given by Eq. (3.2c). The discontinuous transition from one form to the other at the boundaries thus mirrors the corresponding discontinuity in the spatial distribution of particles there. These two disparate views are physically reconciled by the ''additional surface traction'' defined following Eq. (2.11), or equivalently by the ''external stress tensor'' defined in Eq. (2.12). Since it is unlikely that one would be physically interested in knowing the force on any surface other than that bounding the suspension, the most natural choice of the stress tensor is likely to be (3.2c). And that stress is asymmetric!

From a microscale point of view the hypothetical homogeneous medium, continuum-mechanical model (3.1) (which predicts a *symmetric* state of stress) is more accurate than the polar, asymmetric stress model (3.2), since it correctly reproduces the ensemble-average stress and body-force fields. Nevertheless, the model is macroscopically deficient in failing to faithfully mirror all the key attributes that one would expect of an appropriate physical model of the phenomena. Despite the greater accuracy of Eq. (3.1), in the sense of the (asymptotic) agreement of its predictions with formal ensemble-average results, Eq. (3.2) nevertheless possesses important conceptual advantages in applications. Explicitly:

(1) It provides a macroscopic description which can be intuitively realized from a continuum-mechanical point of view.
(2) No additional force distribution on the boundaries needs to be introduced.

ACKNOWLEDGMENTS

H.B. was supported by a ''Grand Challenge'' award sponsored by the Mathematical, Information, and Computational Sciences Division, Office of Computational and Technology Research, Office of Energy Research, U.S. Department of Energy.

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