# Apparent slip at the surface of a small rotating sphere in a dilute quiescent suspension

Y. ALMOG * H. BRENNER †

**Abstract**

We consider the case of a test sphere (ball) of radius $a_1$ rotating at constant angular velocity $\omega$ in an otherwise quiescent unbounded suspension of uniformly sized spheres of radii $a_2$ dispersed in a Newtonian fluid of viscosity $\mu$. To the first order in the volume fraction $c$ of suspended spheres it is shown that when the ball is small compared with the suspended spheres the suspension does not behave as regards the hydrodynamic torque $L$ exerted on the ball like a homogeneous Newtonian fluid characterized by the usual Einstein viscosity coefficient $\mu_s = \mu(1 + 5/2c)$. Explicitly, the torque on the rotating sphere does not obey Kirchoff's law, $L = 8\pi\mu_s a_1^3\omega$ for no slip. Rather, a modified form of Kirchoff's law is obtained in which the Einstein coefficient of 5/2 is multiplied by a coefficient which is less than unity in magnitude and is functionally dependent only upon the suspended-sphere/test-sphere size ratio, $\lambda = a_2/a_1$. In the 'continuum limit,' where $\lambda$ tends to zero, one recovers Kirchoff's law. Accordingly, the deviation from Kirchoff's law is interpreted in terms of an apparent Knudsen-like 'slip' at the rotating ball surface since this slip vanishes in the continuum limit. The existence of an apparent slip is consistent with recent experiments performed on small rotating spheres, albeit in concentrated suspensions, in which the 'viscosity' of the suspension – defined via Kirchoff's law in terms of the experimentally measured torque $L$ as $L/8\pi a_1^3\omega$ – was observed to be less than the viscosity of the suspension as measured by standard viscometric methods. Similar, although quantitatively different $O(c)$ theoretical Knudsen-like slip results were also obtained for the 'inverse' case, where the torque $L$ on the rotating ball is held constant for all time and its mean angular velocity calculated.

*Department of Applied Mathematics, The Weizmann Institute of Science, Rehovot 76100, ISRAEL.
†Department of Chemical Engineering, M.I.T, Cambridge, MA 02139-4307, USA.

It is well known ever since the work of Einstein [1] that when the length scale on which the ambient velocity field varies is much larger than the size of the suspended particles, a suspension of neutrally buoyant spherical particles may be replaced by a homogeneous medium with an increased viscosity .[2,3] Such is not the case when the ambient velocity field varies on a length scale which is comparable in magnitude to the size of the suspended particles – as would occur, for example, during the motion through the suspension of a body of size comparable to that of the suspended particles. Two examples of such flows are:

I. A non-neutrally buoyant sphere settling in an unbounded suspension;

II. A sphere rotating at a constant rate in an unbounded suspension.

Case I has already been extensively studied: Batchelor [4] and Batchelor & Wen [5] calculated the average velocity of a sphere settling through a quiescent suspension under the influence of gravity; subsequently, Davis & Hill [6] further calculated the mean-squared fluctuation of the sedimenting sphere about its average path. Almog & Brenner [7] compared the force/velocity relation for case of a test sphere sedimenting at a given velocity through a quiescent suspension of spheres of comparable size with the more usual case of a test sphere on which the force is prescribed.

In the present work we calculate the torque on a test sphere which is rotating at a given rate and find that the torque on the test sphere and its angular velocity are not related by the Kirchoff's law linear factor of $8\pi\mu_s a_1^3$, with $\mu_s$ the Einstein viscosity of the suspension. Rather, the effective viscosity $\mu_s$ calculated by assuming the applicability of Kirchoff's law, albeit with a viscosity other than Einstein's, is found to be less than the Einstein value by an amount that depends only upon the ratio of suspended to test sphere radii. This deviation from the Einstein relation is interpreted in terms of slip occurring at the ball surface. This torque calculation is motivated in part by an attempt to confirm the experimentally observed (cf. Mondy *et al.* [8]) presence of 'Kirchoff-law slip' in concentrated, monodisperse, sphere suspensions animated by the rotation of a ball comparable in size to that of the suspended spheres. A comparable calculation for circumstances in which the torque on the test sphere is held fixed and its mean angular velocity calculated also reveals the presence of slip, although to a slightly different extent than that for the preceding, fixed rotation rate case!

*Calculation of the 'apparent viscosity'*

Consider a dilute suspension of identical, freely-suspended spherical particles of radii $a_2$ dispersed in a homogeneous Newtonian fluid of viscosity $\mu$ in which a test sphere of radius $a_1$ rotates. If the size of the container is much larger than the sizes of both the suspended and test spheres, it is reasonable to approximate results by those obtained for an unbounded domain.

Suppose that the test sphere is rotating at a given, fixed rate $\omega$. Due to hydrodynamic interactions with the freely suspended particles, the torque, exerted on the rotating sphere by the suspension will be greater than if the freely suspended particles were absent. We define the extra-torque on the test sphere by the difference between the actual torque $L$ and $8\pi\mu a_1^3\omega$, the latter being the torque on the test sphere in the absence of the freely suspended particles.

Obviously, the extra-torque depends upon the manner in which the suspended particles are distributed in space. We shall be interested, therefore, in its average over all possible multisphere locations. Since the suspension is supposed dilute, this multisphere average can be approximated by a two-sphere average (Batchelor ,[4] Almog & Brenner [7]), given by

$$\Delta L_i = \omega_j \int_{|\boldsymbol{x}_1| \geq a_1 + a_2} B_{ij}(\boldsymbol{x}_1) P(\boldsymbol{x}_1/\boldsymbol{x}_0) dx_1. \tag{1}$$

The function $P(\boldsymbol{x}_1/\boldsymbol{x}_0)$ denotes the long-time conditional probability density for finding a freely-suspended particle at $\boldsymbol{x}_1$ when the test sphere center is located at $\boldsymbol{x}_0$; (notice that the coordinate system chosen is such that the origin coincides with $\boldsymbol{x}_0$.) The tensor $\boldsymbol{B}$ may be obtained using the two-sphere numerics of Jeffrey & Onishi [9] as

$$B_{ij} = 8\pi\mu a_1^3 \left[ b_{\parallel} e_i e_j + b_{\perp}(\delta_{ij} - e_i e_j) - \delta_{ij} \right], \tag{2a}$$

$$b_{\parallel} = \frac{1}{x_c^{11}}, \tag{2b}$$

$$b_{\perp} = \left[ y_c^{11} - 3\frac{(y_b^{11})^2}{y_a^{11}} \right]^{-1}, \tag{2c}$$

in which $\boldsymbol{e}$ is a unit vector parallel to $\boldsymbol{x}_1 - \boldsymbol{x}_0$, and the various mobility functions $x_c^{11}$, etc. are defined in the work of Jeffrey & Onishi .[9]

If, instead, the angular velocity $\boldsymbol{\omega}$ is unknown and the torque $\boldsymbol{L}$ exerted on the sphere is prescribed, one may derive the reduction in angular velocity due to hydrodynamic interactions in a similar manner to (1) as

$$\Delta\omega_i = L_j \int_{|\boldsymbol{x}_1| \geq a_1 + a_2} C_{ij}(\boldsymbol{x}_1) P(\boldsymbol{x}_1/\boldsymbol{x}_0) dx_1, \tag{3}$$

wherein

$$C_{ij} = \frac{1}{8\pi\mu a_1^3} \left[ x_{11}^c e_i e_j + y_{11}^c (\delta_{ij} - e_i e_j) - \delta_{ij} \right]. \tag{4}$$

The long-time probability density $P(\boldsymbol{x}_1/\boldsymbol{x}_0)$ cannot be determined solely by considering convection alone. If we neglect diffusive effects (due to weak Brownian motion or hydrodynamic interactions), the time-dependent probability density $P(\boldsymbol{x}_1, t/\boldsymbol{x}_0)$ satisfies the following conservation problem:

$$\frac{\partial P(\boldsymbol{x}_1, t/\boldsymbol{x}_0)}{\partial t} + \nabla_{x_1} \cdot [\boldsymbol{U}(\boldsymbol{x}_1) P(\boldsymbol{x}_1, t/\boldsymbol{x}_0)] = 0, \tag{5a}$$

$$\boldsymbol{U}(\boldsymbol{x}_1) = D(|\boldsymbol{x}_1|)\boldsymbol{\omega} \times \boldsymbol{x}_1, \tag{5b}$$

$$P(\boldsymbol{x}_1, t/\boldsymbol{x}_0) \sim n(\boldsymbol{x}_1) \qquad \text{for } |\boldsymbol{x}_1| \gg a_1. \tag{5c}$$

The characteristic length scale quantifying the number density $n(\boldsymbol{x}_1)$ is much larger than $a_1$. Consequently, in order to obtain the behavior of $P(\boldsymbol{x}_1, t/\boldsymbol{x}_0)$ for $|\boldsymbol{x}_1| \sim O(a)$ it suffices to assume that $n(\boldsymbol{x}_1) = n(0)$, where $n(0)$ is the mean number density of suspended particles, namely $n(0) = c/v_p$, with $v_p = 4\pi a_2^3/3$ the volume of a suspended sphere. The function $D(|\boldsymbol{x}_1|)$ can be obtained using Jeffrey & Onishi's [9] results, but is not of explicit interest here.

The solution of (2.5) is of a time-periodic nature:

$$P(\boldsymbol{x}_1, t/\boldsymbol{x}_0) = P(\boldsymbol{x}_1, t + \frac{2\pi}{D(|\boldsymbol{x}_1|)\omega}/\boldsymbol{x}_0).$$

Thus, $P(\boldsymbol{x}_1, t/\boldsymbol{x}_0)$ depends upon the initial condition for all $t$; hence, no long-time asymptotic behavior is expected, in contrast to the sedimenting sphere case.[7] However, if we add the effect of weak Brownian motion to (2.5) it is easy to show that $P(\boldsymbol{x}_1, t/\boldsymbol{x}_0) \to n(0)$ in the long-time limit. Furthermore, were we to add any other diffusive effect which vanishes for constant $P(\boldsymbol{x}_1, t/\boldsymbol{x}_0)$, the quantity $n(0)$ would still be a solution since $\nabla_{x_1} \cdot \boldsymbol{U}(\boldsymbol{x}_1) = 0$.

In view of the above discussion it seems reasonable to assume that $P(\boldsymbol{x}_1/\boldsymbol{x}_0) = n(0)$ in the equilibrium state. Such an assumption, usually referred to as the 'Eisenschitz hypothesis' (cf. Batchelor & Green [10]), leads generally to incorrect results (cf. Leal & Hinch [11]) since for non-solenoidal velocity fields a non-uniform distribution is expected in the long-time limit.

Figure 1 displays the dependence on $\lambda = a_2/a_1$, the ratio between the respective radii of the freely-suspended spheres and the rotating sphere, of: (i) the dimensionless extra-torque, defined as $\Delta L' = |\Delta L|/8\pi\mu a_1^3 c\omega$; and (ii) the dimensionless reduction in angular velocity, defined as $\Delta\omega' = 8\pi\mu a_1^3 |\Delta\omega| c/L$. Both represent the additional 'apparent viscosity' experienced by the test sphere normalized by the volume fraction $c$. The various mobility functions were calculated up to $O(|\boldsymbol{x}_1|^{-100})$. Owing to near-field inaccuracies, the error is expected to be approximately 0.1 percent for $\lambda \sim O(1)$ .[12] The solid and dashed curves respectively denote the dimensionless extra-torque $\Delta L'$ and the dimensionless angular velocity reduction $\Delta\omega'$.

For $\lambda \ll 1$ both functions attain the asymptotic value of 5/2, in agreement with theory. For $\lambda \gg 1$ both $\Delta L'$ and $\Delta\omega'$ diminish monotonically with $\lambda$. The decrease of these quantities may be intuitively understood from the fact that the rotating sphere remains almost unaffected when the freely-suspended sphere is located at $|\boldsymbol{x}_1| \gg a_1$ (but not necessarily at $|\boldsymbol{x}_1| \gg a_1 + a_2$). It is expected therefore that $\Delta L'$ and $\Delta\omega'$ will decay at least like $O(1/\lambda)$.

This behavior contrasts with the case of a non-neutrally buoyant particle settling in an unbounded suspension, in which the apparent viscosity experienced by a small (relative to the suspended particles) falling ball is very large .[6,7] The difference arises from the very different modes of behavior displayed by the respective probability density functions for the cases of sedimenting vs rotating spheres when the test and suspended spheres nearly touch. While for the case of a rotating sphere $P(\boldsymbol{x}_1/\boldsymbol{x}_0)$ is constant, independent of $\lambda$, in the case of a sedimenting sphere it exhibits large gradients near $\boldsymbol{x}_0$ for $\lambda \gg 1$, rendering the near-field contribution dominant (cf. Davis & Hill ,[6] Batchelor & Wen [5]). Since the settling velocity decreases significantly when the settling and freely-suspended spheres nearly touch,

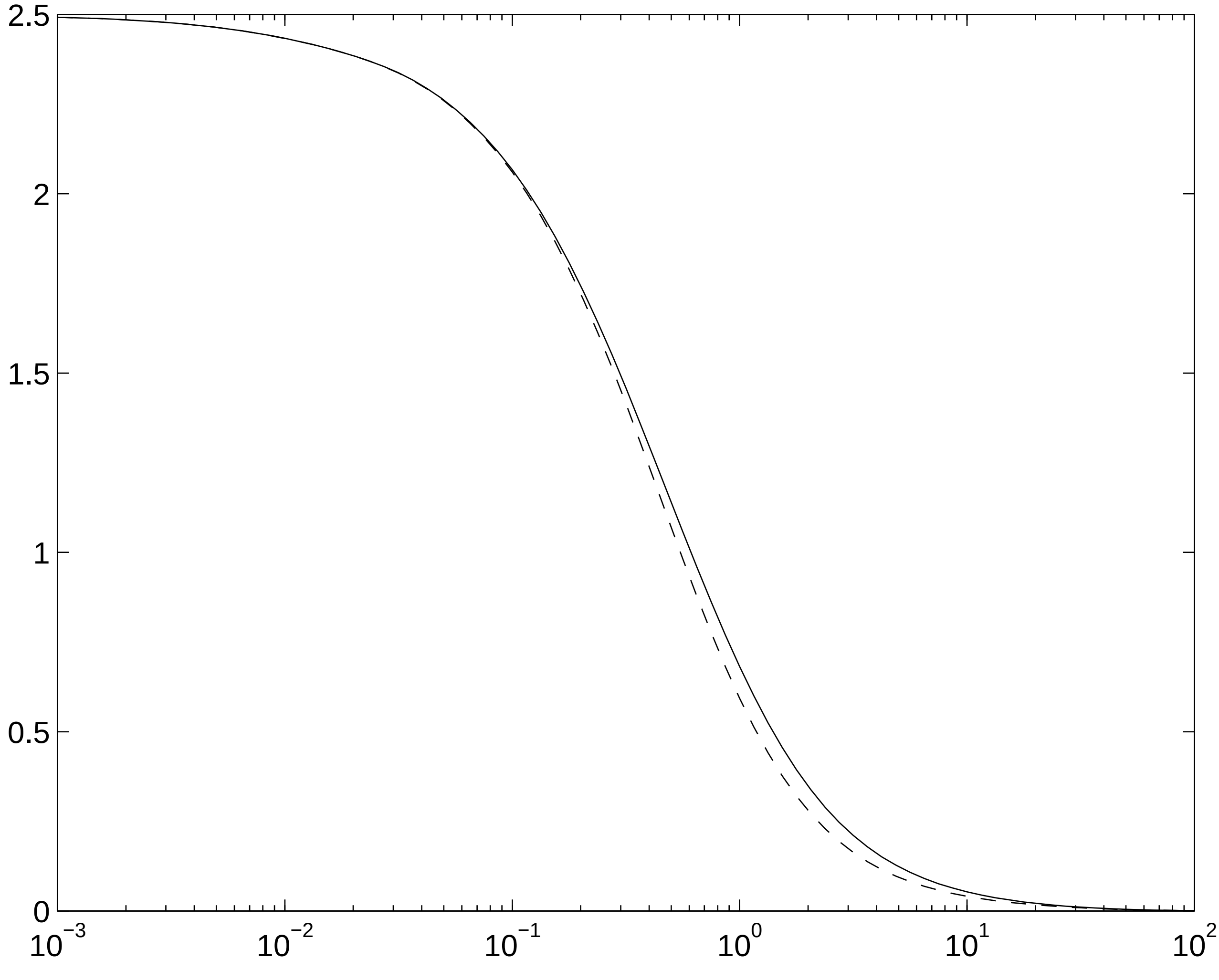


Figure 1: Variation with the suspended/test sphere radius ratio, $\lambda$ of: (i) the dimensionless extra-torque $\Delta L'$, denoted by the full curve; and (ii) the dimensionless angular velocity reduction $\Delta\omega'$, denoted by the dashed curve.

the apparent viscosity in the sedimentation case increases proportionally.

Experimental results [8] reveal the existence of apparent suspension-scale slip at the surface of a small test sphere rotating in a highly concentrated suspension. Explicitly, the suspension viscosity $\mu_s$ calculated on the basis of the supposed applicability of Kirchoff's law in the absence of slip, namely $L = 8\pi\mu_s a_1^3\omega$, in conjunction with the experimentally measured torque $L$ at the given rotation rate $\omega$ is observed to be less than the $\mu_s$ value measured experimentally for the given suspension by standard Couette viscometric methods .[13] It seems reasonable to assume that the origin of this apparent slip lies is the decay of the hydrodynamic interactions, since both theory and experiment predict maximal slip for large values of $\lambda$.

The nonequality of the solid and the dashed curves of Fig.1 points up the existence of yet another anomalous non-continuum phenomenon arising from the discrete nature of suspensions, a phenomenon which has also been demonstrated for the falling-ball case.[7] In particular, for $\lambda \ll 1$ both curves coincide, as expected, due to the weakness of the singularity. Moreover, for $\lambda \to \infty$, both the extra-torque and angular velocity reduction vanish due to the decay of the overall effect of hydrodynamic interactions. However, for $\lambda \sim O(1)$ a small but significant difference is observed between the solid and dashed curves. (Figure 2 displays, on a highly magnified scale, the variation with $\lambda$ of the difference $\Delta L' - \Delta\omega'$ between the two curves.) For some values of $\lambda$ the dimensionless extra-torque is seen to be almost 25 per cent larger than the comparable angular velocity reduction. This phenomenon cannot occur in a homogeneous medium, for which the constitutive stress/rate-of-strain relationship is an intrinsic material property of the system.

## Acknowledgments

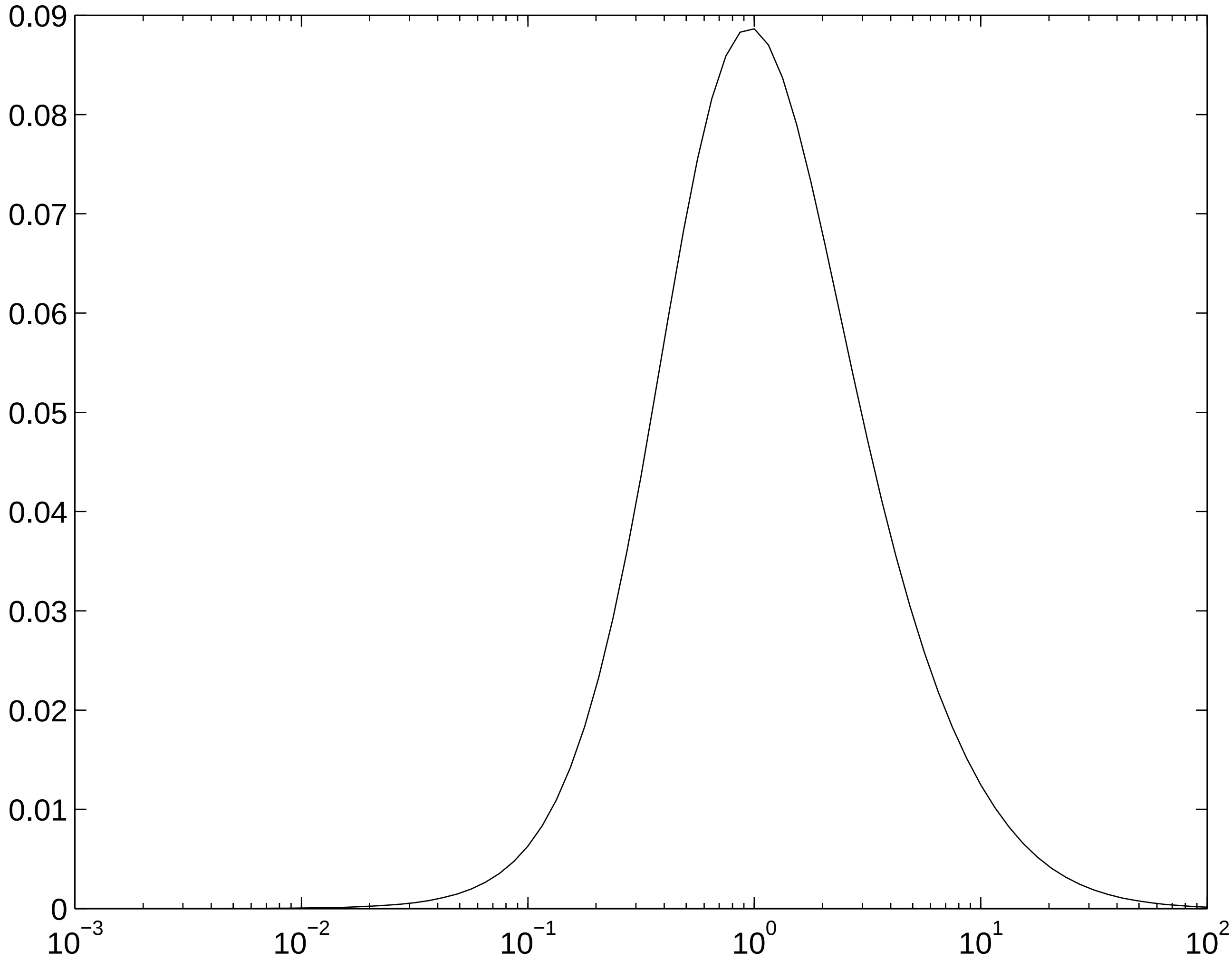


Figure 2: The dependence on the size ratio $\lambda$ of the difference $\Delta L' - \Delta\omega'$ in the dimensionless extra-torque/angular velocity reduction.

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