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Non-linear surface superconductivity for type II superconductors in the large domain limit

Y. ALMOG

Abstract

The Ginzburg-Landau model for superconductivity is considered in two dimensions. We show, for smooth bounded domains, that superconductivity remains concentrated near the surface when the applied magnetic field is decreased below H_{C_3} as long as it is greater than H_{C_2}. We demonstrate this result in the large domain limit, i.e, when the domain's size tends to infinity. Additionally, we prove that for applied fields greater than H_{C_2}, the only solution in $\mathbb{R}^2$ satisfying normal state condition at infinity is the normal state. The above results have been proved in the past for the linear case. Here we prove them for non-linear problems.

1. Introduction

Consider a planar superconducting body which is placed in a sufficiently low temperature (below the critical one) under the action of an external magnetic field . It is known both from experiments [20] and rigorous analysis [15] that for a sufficiently strong magnetic field the normal state would prevail. If the field is then decreased, there is a critical field, depending on the sample's geometry, where the material would enter the superconducting state. For samples with boundaries, this field is known as the onset field and has been termed H_{C_3}.

The simplest case in which the bifurcation from the normal state to the superconducting one was calculated is a half-plane [24]. The analysis in this case is one dimensional: the linearized Ginzburg-Landau equations, which are the most common model for macroscopic superconductivity, were solved on $\mathbb{R}_+$. Even in this simple case the onset field is substantially larger than the bifurcation field on $\mathbb{R}$ [14]. The situation is no different in two dimensions: it was proved in [19] and [11] that the bifurcating mode in $\mathbb{R}^2_+$ is one-dimensional and that the value of H_{C_3} is exactly the same as in the one-dimensional case. Similarly, the bifurcation from the normal state in $\mathbb{R}^2$ takes place when the applied magnetic field is identical with the bifurcation field for $\mathbb{R}$, which has been termed H_{C_2}.

In addition to the difference in the values of the applied field, it was found by Saint-James and de-Gennes [24] that superconductivity is concentrated at the onset near the boundary for a half-plane. This phenomenon, which appears only in the presence of boundaries have been termed, therefore, surface superconductivity. The significance of Saint-James and de Gennes' solution [24] extends far beyond the simple, one-dimensional example of a half-plane. It was proved, first for films [6], then for discs [4], and finally for general two-dimensional domains with smooth boundaries [18, 11], that as the domain's scale tends to infinity the onset field tends to de-Gennes' value. If the boundaries include wedges the onset field will be larger than de-Gennes' value [5, 16, 25, 17].

Surface superconductivity reflects another difference between the problems in $\mathbb{R}^2_+$ and $\mathbb{R}^2$, where the bifurcation takes place in the form of periodic solutions [1, 7, 3] known as Abrikosov's lattices. The transition, as the applied magnetic field decreases, from surface superconductivity to the experimentally observed [12] Abrikosov's lattices is not yet well understood. Rubinstein [23] conjectured that superconductivity remains limited to a neighborhood of the boundary until about H_{C_2} when a new solution which is similar in the bulk to Abrikosov lattice appears.

In the present contribution we focus on the first part of the conjecture in [23]. We show that superconductivity remains concentrated near the surface when the magnetic field is decreased below H_{C_3} as long as it is greater than H_{C_2}. We demonstrate this result in the large domain limit, i.e, when the domain's size tends to infinity. We consider here only smooth domains, domains with corners are left to future research.

Additionally, we prove that for applied fields greater than H_{C_2} only the normal state can exist in $\mathbb{R}^2$. This result has been proved in the linear case, near the bifurcation from the normal state [19,11], and for one-dimensional problems [2]. Here we prove it for the non-linear problem in $\mathbb{R}^2$.

In a recent contribution Pan [22] studies the same problem in the limit $\kappa \to \infty$. Pan's results are in some senses weaker than the results presented here. Nevertheless, he provides estimates for the energy of the solution near the boundary, a problem which is not addressed at all in the present contribution. We extend the discussion a little further at this point in the end of § 3.

The Ginzburg-Landau energy functional may be represented in the following dimensionless form [9]

$$E = \int_{\Omega} \left(-|\Psi|^2 + \frac{|\Psi|^4}{2} + |h - h_{ex}|^2 + \left| \frac{i}{\kappa} \nabla \Psi + A\Psi \right|^2 \right) dxdy \tag{1.1}$$

in which Ψ is the (complex) superconducting order parameter, such that $|\Psi|$ varies from $|\Psi| = 0$ (when the material is at a normal state) to $|\Psi| = 1$ (for the purely superconducting state). The magnetic vector potential is denoted by A (the magnetic field is, then, given by $h = \nabla \times A$), h_{ex} is the constant applied magnetic field, and κ is the Ginzburg-Landau parameter which is a material property. Superconductors for which $\kappa < 1/\sqrt{2}$ are termed type I superconductors, and those for which $\kappa > 1/\sqrt{2}$ have been termed type II. The domain Ω is the domain of superconductor whose Gibbs free energy is given by E. Note that E is invariant to the gauge transformation

$$\Psi \to e^{i\kappa\eta}\psi \qquad ; \qquad A \to A + \nabla\eta \; . \tag{1.2}$$

The Euler-Lagrange equations associated with E (the steady state Ginzburg-Landau equations) are given by

$$\left(\frac{i}{\kappa} \nabla + A \right)^2 \psi = \psi \left(1 - |\psi|^2 \right) \quad , \tag{1.3a}$$

$$-\nabla \times \nabla \times A = \frac{i}{2\kappa} \left(\psi^* \nabla \psi - \psi \nabla \psi^* \right) + |\psi|^2 A \quad , \tag{1.3b}$$

and the natural boundary conditions by

$$\left(\frac{i}{\kappa} \nabla + A \right) \psi \cdot \hat{n} = 0 \quad ; \quad h = h_{ex} \; . \tag{1.4a,b}$$

We consider two-dimensional settings where we can write $h = (0, 0, h(x, y))$ and $h_{ex} = (0, 0, h_{ex})$. In the next section we consider solutions of (1.3) in $\mathbb{R}^2$ satisfying the boundary conditions

$$|\psi| \xrightarrow[|x|\to\infty]{} 0 \tag{1.5a}$$

$$h = \nabla \times A \xrightarrow[|x|\to\infty]{} h_{ex} \tag{1.5b}$$

we prove that only the normal state, where $\psi \equiv 0$ and $h \equiv h_{ex}$ can satisfy (1.3) and (1.5) whenever $h_{ex} > \kappa = H_{C_2}$.

In § 3 we consider the global minimizer of (1.1) in smooth bounded domains. We consider the large domain limit, i.e., we stretch the domain with respect to a fixed point and let its scale tend to infinity. We show that for sufficiently large scales, the global minimizer of (1.1) which must solve (1.3) together with (1.4), tends exponentially fast away from the boundaries to a normal state as long as $h_{ex} > \kappa$.

2. Non-existence in $\mathbb{R}^2$

In this section we prove the following result:

Theorem 2.1. *Let $\psi : \mathbb{R}^2 \to \mathbb{C}$ and $A : \mathbb{R}^2 \to \mathbb{R}^2$ satisfy the equations*

$$\left(\frac{i}{\kappa}\nabla + A\right)^2 \Psi = \Psi\left(1 - |\Psi|^2\right) \quad x \in \mathbb{R}^2, \tag{2.1a}$$

$$-\nabla \times (\nabla \times A) = \frac{i}{2\kappa}\left(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*\right) + |\Psi|^2 A \quad x \in \mathbb{R}^2, \tag{2.1b}$$

together with the boundary conditions

$$|\psi| \xrightarrow[|x|\to\infty]{} 0, \tag{2.1c}$$

$$h = \nabla \times A \xrightarrow[|x|\to\infty]{} h_{ex}, \tag{2.1d}$$

where the applied magnetic field, h_{ex}, is a constant. (For convenience we pick the Coulomb gauge, i.e., $\nabla \cdot A = 0$.)

Then if either

1. $h_{ex} > \kappa$ *and* $\kappa > 1/\sqrt{2}$*,*

or

2. $h_{ex} < \frac{1}{2\kappa}$ *and* $\kappa < 1/\sqrt{2}$*,*

then, $\psi \equiv 0$*,* $h \equiv h_{ex}$ *is the unique solution of* (2.1)*.*

Proof: Let $\psi = \rho e^{i\phi}$. Multiplying (2.1a) by $e^{-i\phi}$ and taking the real part we obtain

$$\frac{1}{\kappa^2}\nabla^2\rho + \rho(1-\rho^2) = \rho\left|\frac{1}{\kappa}\nabla\phi - A\right|^2 = \frac{|\nabla h|^2}{\rho^3}, \tag{2.2a}$$

whenever $\rho \neq 0$. (To obtain the second equality in (2.2) we used (2.1b)). Dividing (2.1b) by ρ^2 and taking its curl (by standard elliptic estimates any solution of (2.1) must be C^∞ on any smooth compact subset of $\mathbb{R}^2$) yields

$$\nabla\cdot\left(\frac{\nabla h}{\rho^2}\right) - h = 0, \tag{2.2b}$$

whenever $\rho \neq 0$ [21]. Consider first the case $h_{ex} > \kappa$, $\kappa > 1/\sqrt{2}$. Define

$$u = h - \kappa + \frac{1}{2\kappa}\rho^2. \tag{2.3}$$

Utilizing (2.2) yields the following equation for u:

$$\nabla^2 u - \rho^2 u = \kappa\left|\frac{\nabla u}{\rho}\right|^2 + \left(\kappa - \frac{1}{2\kappa}\right)\rho^4, \tag{2.4}$$

whenever $\rho \neq 0$.

It is possible to derive a similar equation which is valid when $\rho = 0$ as well. To this end we define, following [26]

$$\frac{\partial}{\partial z} \overset{def}{=} \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad ; \quad \frac{\partial}{\partial z^*} \overset{def}{=} \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \quad ; \quad \hat{A} \overset{def}{=} A_x + iA_y$$

and

$$\hat{J} = \frac{2}{\kappa}\frac{\partial\psi}{\partial z^*} - i\hat{A}\psi \qquad J_3 = h - \kappa\left(1-\rho^2\right) \tag{2.5}$$

The Ginzburg-Landau equations (2.1) can now be written in the form

$$\frac{2}{\kappa}\frac{\partial\hat{J}}{\partial z} - i\hat{A}^*\hat{J} - \frac{1}{\kappa}J_3\psi = 0 \tag{2.6a}$$

$$2\frac{\partial J_3}{\partial z^*} - \psi^*\hat{J} = 2\left(\kappa - \frac{1}{2\kappa}\right)\frac{\partial\rho^2}{\partial z^*}\ . \tag{2.6b}$$

Differentiating (2.6b) with respect to z we obtain

$$\nabla^2 J_3 - \rho^2 J_3 = \kappa \left|\hat{J}\right|^2 + \left(\kappa - \frac{1}{2\kappa}\right) \nabla^2(\rho^2) \,,$$

and since

$$J_3 = u + \left(\kappa - \frac{1}{2\kappa}\right) \rho^4,$$

we obtain

$$\nabla^2 u - \rho^2 u = \kappa \left|\hat{J}\right|^2 + \left(\kappa - \frac{1}{2\kappa}\right) \rho^4. \tag{2.7}$$

Comparing with (2.4) we obtain

$$\left|\hat{J}\right|^2 = \left|\frac{\nabla u}{\rho}\right|^2 \tag{2.8}$$

whenever $\rho \neq 0$. While (2.7) is valid for all $x \in \mathbb{R}^2$ (even where $\rho = 0$), (2.8) would prove to be very useful in the next section.

By (2.7), u cannot have a positive maximum in $\mathbb{R}^2$ including points where ρ vanishes. Therefore,

$$u \leq h_{ex} - \kappa. \tag{2.9}$$

Let

$$u^+ = \begin{cases} u & u > 0 \\ 0 & u \leq 0 \end{cases} .$$

Multiplying (2.4) by u^+ and integrating over $B(0,r)$ we obtain

$$\int_{\partial B(0,r)} u^+ \frac{\partial u^+}{\partial r} \geq F(r) \tag{2.10a}$$

where

$$F(r) = \int_{B(0,r)} \kappa u^+ \left|\hat{J}\right|^2 + \rho^2 \left(u^+\right)^2 + \left|\nabla u^+\right|^2 \,. \tag{2.10b}$$

multiplying (2.10a) by $1/r$ and integrating between 1 and R yields

$$\frac{1}{2} \int_0^{2\pi} \left(u^+\right)^2 d\theta \Big|_1^R \geq \int_1^R \frac{F(r)}{r} dr$$

In view of (2.9) we have

$$\int_1^R \frac{F(r)}{r} dr \leq \pi (h_{ex} - \kappa)^2 \quad \forall R > 1. \tag{2.11}$$

Thus, since $F(r)$ is monotonically increasing it must identically vanish ($F(r) \equiv 0$). Consequently, to satisfy (2.1c,d), we must have

$$u \equiv h_{ex} - \kappa \quad ; \quad \rho \equiv 0.$$

To prove that the same result holds whenever $\kappa < 1/\sqrt{2}$ and $h_{ex} > \frac{1}{2\kappa}$ we define

$$w = h - \frac{1}{2\kappa}(1 - \rho^2) \tag{2.12}$$

to obtain

$$\nabla^2 w - \rho^2 w = \kappa \left| \hat{J} \right|^2 + \left(\frac{1}{2\kappa} - \kappa \right) (\rho^2 - \rho^4),$$

and proceed in the same manner as in the previous case.

□

We note that for type I superconductors, no solution can exist in the one-dimensional case whenever $h_{ex} \geq 1/\sqrt{2}$ [2]. Thus, it might be that $\frac{1}{2\kappa}$ is larger than the supremum of all the values of h_{ex} for which non-trivial solutions of (2.1) can exist. (It is expected that, for $\kappa > 1/\sqrt{2}$, any non-trivial solution would be unstable for $h_{ex} > \kappa$, where the normal state is known to be linearly stable [8], nevertheless unstable solution can exist [7,2].) For type II, however, it is well known from linear bifurcation theory (cf. [11,7] for instance) that at $h_{ex} = \kappa$ there is a supercritical bifurcation from the normal state. Finally, theorem 1 validity can trivially be extended to all cases where either u (for type II), or w (for type I) are known to be positive at some point in the plane.

3. Surface superconductivity for type II S.C.

In this section we consider the same equations but this time in a bounded domain. Denote by Ω_1 a planar domain whose boundaries are at least $C^{2,\alpha}$ for some $0 < \alpha < 1$. Denote by Ω_R the family of domains which are obtained from Ω_1 via the transformation

$$x \longrightarrow x_0 + R(x - x_0) \tag{3.1}$$

where x_0 is any point in $\mathbb{R}^2$. (If we change x_0 we merely translate the domains, and hence, the results in the sequel remain valid.) Let (ψ_R, A_R) denote the global minimizer of the energy functional (1.1) in $\mathcal{W}(\Omega_R)$, where

$$\mathcal{W}(\Omega_R) = \left\{ (\psi, A) \in H^1(\Omega_R) \times H^1(\Omega_R) | \left(\frac{i}{\kappa}\nabla + A \right) \Psi \in L^2(\Omega_R) \right\}.$$

Then (ψ_R, A_R) must be a solution of (2.1a,b) in Ω_R together with the boundary conditions

$$\left(\frac{i}{\kappa}\nabla + A_R \right) \psi_R \cdot \hat{n} = 0 \text{ on } \partial\Omega_R \tag{3.2a}$$

$$h_R = h_{ex} \text{ on } \partial\Omega_R. \tag{3.2b}$$

where $\hat{n}$ is the outward normal on $\partial\Omega_R$ and $h_R = \nabla \times A_R$.

We consider here the case $\kappa > 1/\sqrt{2}$, for which we prove the following theorem

Theorem 3.1. *Let $h_{ex} > \kappa$. Then, $\exists \beta > 0$, $R_0 > 0$ and a constant $\tilde{h}_R$ such that*

$$|D^\alpha \psi_R| \leq C_\alpha e^{-\beta d(x,\partial\Omega_R)} \qquad \textit{for all } \alpha \geq 0,\ R > R_0,\ \textit{and } x \in \Omega_R \tag{3.3a}$$

$$\left| D^\alpha (h_R - \tilde{h}_R) \right| \leq C_\alpha e^{-\beta d(x,\partial\Omega_R)} \qquad \textit{for all } \alpha \geq 0,\ R > R_0,\ \textit{and } x \in \Omega_R \tag{3.3b}$$

$$\left| \tilde{h}_R - h_{ex} \right| \leq \frac{C}{R^{1/2}} \tag{3.3c}$$

To prove the theorem we need first to derive several auxiliary results. The first of them is uniform boundedness of h_R and $\rho_R = |\psi_R|$ for all $R > 1$

Lemma 1. *Let $h_{ex} \geq 0$. Then,*

$$\|\rho_R\|_{L^\infty(\Omega_R)} < 1 \tag{3.4a}$$

$$\|h_R\|_{L^\infty(\Omega_R)} < \max\left(\kappa, h_{ex} + \frac{1}{2\kappa} \right) \tag{3.4b}$$

$$\left\| \hat{J}_R \right\|_{L^\infty(\Omega_R^s)} \leq C(s) \tag{3.4c}$$

where $\Omega_R^s = \{x \in \Omega_R \mid d(x, \partial\Omega_R) \geq s\}$ for some $s > 0$.

Proof: The proof of (3.4a) is well known and follows immediately from (2.2a) and the real part of the boundary condition (3.2a). To prove (3.4b) we define

$$u_R = h_R - \kappa + \frac{1}{2\kappa}\rho_R^2 .$$

Since u_R satisfies (2.7) it cannot have a positive maximum. Hence,

$$u_R \leq h_{ex} - \kappa + \frac{1}{2\kappa} \Rightarrow h_R \leq h_{ex} + \frac{1}{2\kappa} \tag{3.5}$$

In order to bound h_R from below we first note that $(\bar{\psi}_R, -A_R)$ is a solution of (2.1a,b) and (3.2a). It also satisfies (3.2b) with h_{ex} replaced by $-h_{ex}$. Proceeding in the same manner as before we obtain

$$-h_R - \kappa + \frac{1}{2\kappa}\rho_R^2 \leq 0 \Rightarrow h_R \geq -\kappa \tag{3.6}$$

We then prove (3.4b) by combining (3.5) and (3.6).

We prove (3.4c) using standard elliptic estimates. We write (2.1) in the form

$$\frac{1}{\kappa^2}\nabla^2\psi_R = 2A_R \cdot \left(\frac{i}{\kappa}\nabla + A_R\right)\psi_R + \psi_R\left(1 - |\psi_R|^2 + |A_R|^2\right)$$
$$\nabla^2 A = \Im\left[\bar{\psi}_R\left(\frac{i}{\kappa}\nabla + A_R\right)\psi_R\right]$$

Let $x_0 \in \Omega_R^s$. We first apply the transformation (1.2)

$$A_R \to A_R - A_R(x_0) \quad ; \quad \psi_R \to e^{i\kappa A_R(x_0)\cdot x}\psi_R , \tag{3.7}$$

which leaves (3.8) unaltered. In view of (3.4b) A_R must now be uniformly bounded in $B(x_0, s)$, i.e,

$$\|A_R\|_{L^\infty[B(x_0,s)]} \leq C . \tag{3.8}$$

By standard elliptic estimates [13] we have

$$\|\psi_R\|^2_{H^2[B(x_0,s/4)]} \leq C(s)\int_{B(x_0,s/2)}\left[\rho_R^2 + 4\left|A_R\cdot\left(\frac{i}{\kappa}\nabla + A_R\right)\psi_R\right|^2 + \rho_R^2\left(1 - |\psi_R|^2 + |A_R|^2\right)^2\right] \tag{3.9}$$

In view of (3.4a) we have

$$\int_{B(x_0,s/2)}\rho^2 \leq s/2$$

Define the cutoff function

$$\chi = \begin{cases} 1 & |x-x_0| \leq s/2 \\ 0 & |x-x_0| \geq s \end{cases}$$

where $|\nabla\chi| \leq C/s$. Then [19],

$$\int_{B(x_0,s)} \left|\left(\frac{i}{\kappa}\nabla + A_R\right)(\chi\psi_R)\right|^2 = \int_{B(x_0,s)} \chi^2\rho_R^2\left(1-\rho_R^2\right) + |\nabla\chi|^2\rho_R^2 \leq C\,.$$

Combining the above inequality (3.8) and (3.9) we obtain

$$\|\psi_R\|_{H^2[B(x_0,s/2)]} \leq C(s)\,.$$

In a similar manner we obtain the same inequality for the H^2 norm of A_R. Bootsraping and sobolev embedding then yiels (3.4c). Note that since $\hat{J}_R$ is gauge invariant (3.4c) would remain valid even if we did not apply (3.7).

□

We can now prove the following estimates on the L^2 norms of ρ_R and $h_R - h_{ex}$.

Lemma 2.

$$\int_{\Omega_R} \rho_R^2 \leq CR \tag{3.10a}$$

$$\int_{\Omega_R} |h_R - h_{ex}|^2 \leq CR \tag{3.10b}$$

where C is independent of R.

Proof: In a similar manner to [11] we define a local coordinate system near $\partial\Omega_R$. Let η denote the distance from the boundary and s the arclength along the boundary, with some point $x_0 \in \partial\Omega_R$ corresponding to $s=0$. This local coordinate system is well defined in the rectangle

$$S_R = \{(s,\eta)| - RL/2 < s < RL/2,\ 0 < \eta < \delta R\}$$

where L denotes the arclength of $\partial\Omega_1$. Denote by $\kappa_1(s)$ the curvature of the boundary of Ω_1, which must be uniformly bounded in $[-L/2, L/2]$. For sufficiently large R we have $\delta R > 1$ and then, in view of (3.4) we have

$$\int_0^1 \int_{-LR/2}^{LR/2} -\frac{\partial u_R}{\partial \eta}\left(1-\frac{\kappa_1}{R}\eta\right) ds d\eta = \int_{-LR/2}^{LR/2} -\left(1-\frac{\kappa_1}{R}\eta\right) u_R(s,\eta)\Big|_0^1 ds$$
$$-\int_0^1 \int_{-LR/2}^{LR/2} \frac{\kappa_1}{R} u_R(s,\eta) ds d\eta \leq CR$$

Hence, there exists $0<\alpha(R)<1$ such that

$$\int_{-LR/2}^{LR/2} -\frac{\partial u_R}{\partial \eta}(s,\alpha)\left(1-\frac{\kappa_1}{R}\alpha\right) ds \leq CR$$

or equivalently, $\exists 0<\alpha(R)<1$ such that

$$\int_{\eta=\alpha} \frac{\partial u_R}{\partial n} \leq CR$$

Denote by Ω'_α the domain enclosed in $\eta=\alpha$. We now integrate (2.7) on Ω'_α to obtain

$$\int_{\Omega'_\alpha} \rho_R^2 u_R + \left(\kappa - \frac{1}{2\kappa}\right)\int_{\Omega'_\alpha} \rho_R^4 \leq \int_{\partial\Omega'_\alpha} \frac{\partial u_R}{\partial n} \leq CR \tag{3.11}$$

Let

$$v_R = h_R - \kappa + \frac{1}{\sqrt{2}}\rho_R^2.$$

Upon substituting in (3.11) we utilize (3.4) to obtain

$$\int_{\Omega_R} \rho_R^2 v_R \leq CR \tag{3.12}$$

The energy functional (1.1) can be written, for (ψ_R, A_R) which solve (2.1a,b) and (3.2), in the form

$$E_R = \int_{\Omega_R} |h_{ex} - h_R|^2 - \frac{1}{2}\rho_R^4$$

Since $\rho_R \equiv 0$, $h_R \equiv h_{ex}$ is a solution of (2.1a,b) and (3.2) we obtain that at the global minimizer

$$\int_{\Omega_R} |h_{ex} - h_R|^2 - \frac{1}{2}\rho_R^4 \leq 0 \tag{3.13}$$

The above inequality may be written in the form

$$\int_{\Omega_R} |h_{ex} - \kappa - v_R|^2 + \sqrt{2}\int_{\Omega_R} \rho_R^2 (h_{ex} - \kappa - v_R) \leq 0 \tag{3.14}$$

Substituting (3.12) in the above relation, and then using (3.13) we prove both parts of (3.10).

The next lemma is perhaps the core of the analysis in this section.

Lemma 3. *Let $h_{ex} > \kappa$ and $\{x_R\}_{R\geq R_0}$ denote a family of points such that $x_R \in \Omega_R$. Let $s_R = d(x_R, \partial\Omega_R)$. Then,*

$$\forall n \in \mathbb{N},\ \exists \frac{1}{2} < \beta_n < 1 : \int_{B(x_R,\beta_n s_R)} \kappa u_R^+ \left|\hat{J}_R\right|^2 + \rho_R^2 \left(u_R^+\right)^2 + \left|\nabla u_R^+\right|^2 \leq \frac{C_n}{s_R^{\alpha_n}} \tag{3.15}$$

where $\alpha_n = 4\left(1 - \left(\frac{4}{5}\right)^n\right)$ and $\hat{J}_R$ is defined in (2.5) with (ψ_R, A_R) instead of (ψ, A).

Proof: We prove the lemma by invoking inductive arguments. First, we prove (3.15) for n=1. Let

$$F_R(x,r) = \int_{B(x,r)} \kappa u_R^+ \left|\hat{J}_R\right|^2 + \rho_R^2 \left(u_R^+\right)^2 + \left|\nabla u_R^+\right|^2 . \tag{3.16}$$

In a similar manner to the proof of (2.11) it is possible to show that

$$\int_1^{s_R} \frac{F_R(x_R,r)}{r} dr \leq \frac{1}{2} \int_0^{2\pi} \left[\left(u_R^+(s_R,\theta)\right)^2 - \left(u_R^+(1,\theta)\right)^2 \right] d\theta \leq C \tag{3.17}$$

In the following we use C to denote a constant which is independent of both R and x_R. As F_R is monotonically increasing in r,

$$\exists \frac{1}{2} < \beta_0 < 1 : \quad F_R(x_R, \beta_0 s_R) < C \tag{3.18}$$

and in particular

$$\int_{B(x_R,\beta_0 s_R)} \kappa u_R^+ \left|\hat{J}_R\right|^2 + \rho_R^2 \left(u_R^+\right)^2 \leq C\ . \tag{3.19}$$

It is easy to show that $1/2 < \beta < \beta_0$ exists such that

$$\int_{\partial B(x_R,\beta s_R)} \kappa u_R^+ \left|\hat{J}_R\right|^2 + \rho_R^2 \left(u_R^+\right)^2 \leq \frac{C}{s_R} \int_{B(x_R,\beta_0 s_R)} \kappa u_R^+ \left|\hat{J}_R\right|^2 + \rho_R^2 \left(u_R^+\right)^2 \tag{3.20}$$

Let $\xi_1, \xi_2 \in \partial B_\beta \stackrel{def}{=} \partial B(x_R, \beta s_R)$. Then,

$$\left| \left(u_R^+\right)^{5/2} (\xi_1) - \left(u_R^+\right)^{5/2} (\xi_2) \right| \leq C \int_{\partial B_\beta} \left(u_R^+\right)^{3/2} |\nabla u_R|$$

By (2.7) $|\nabla u_R| = \rho_R \left|\hat{J}_R\right|$ whenever $\rho_R \neq 0$. When ρ_R vanishes ∇u_R vanishes too in view of (2.1b) and (2.3). Hence,

$$\left| \left(u_R^+\right)^{5/2} (\xi_1) - \left(u_R^+\right)^{5/2} (\xi_2) \right| \leq C \left[\int_{\partial B_\beta} u_R^+ \left|\hat{J}_R\right|^2 \right]^{1/2} \left[\int_{\partial B_\beta} \rho_R^2 \left(u_R^+\right)^2 \right]^{1/2} \leq$$

$$\leq C \int_{\partial B_\beta} \kappa u_R^+ \left|\hat{J}_R\right|^2 + \rho_R^2 \left(u_R^+\right)^2 \leq \frac{C}{s_R}\ . \tag{3.21}$$

Let $0 < s < \beta s_R$, and let (r, θ) denote a polar coordinate system centered around x_R. Then,

$$\int_0^{2\pi}\int_s^{\beta s_R} \left(u_R^+\right)^{3/2} \frac{\partial u}{\partial r} dr d\theta \leq C\left[\int_A \kappa u_R^+ \left|\hat{J}_R\right|^2\right]^{1/2}\left[\int_A \rho_R^2 \left(u_R^+\right)^2 \frac{1}{r^2}\right]^{1/2} \tag{3.22}$$

where $A \stackrel{def}{=} B(x_R, \beta s_R) \setminus B(x_R, s)$. Hence, by (3.19),

$$\int_0^{2\pi} \left(u_R^+\right)^{5/2}\Big|_s^{\beta s_R} \leq \frac{C}{s}.$$

Since u_R cannot have a positive maximum, we obtain from (3.21)

$$\left(u_R^+\right)^{5/2}(s, \theta) \leq \left(u_R^+\right)^{5/2}(\beta s_R, \theta) + \frac{C}{s_R}$$

Utilizing the above inequality together with the inequality

$$\left|x^5 - y^5\right| \geq \left|x^4 - y^4\right|^{5/4} \tag{3.23}$$

and Hölder inequality we obtain

$$\begin{aligned}\int_0^{2\pi} \left(u_R^+\right)^{5/2}\Big|_s^{\beta s_R} d\theta + \frac{2\pi C}{s_R} &= \int_0^{2\pi}\left\{\left[\left(u_R^{+\,5/2}(\beta s_R) + \frac{C}{s_R}\right)^{1/5}\right]^5 - u_R^{+\,5/2}(s)\right\} d\theta \geq \\ &\geq \int_0^{2\pi}\left\{\left(u_R^{+\,5/2}(\beta s_R) + \frac{C}{s_R}\right)^{4/5} - u_R^{+\,2}(s)\right\}^{5/4} d\theta \geq \\ &\geq C\left\{\int_0^{2\pi}\left[\left(u_R^{+\,5/2}(\beta s_R) + \frac{C}{s_R}\right)^{4/5} - u_R^{+\,2}(s)\right] d\theta\right\}^{5/4}.\end{aligned}$$

In view of (3.17)

$$\int_0^{2\pi} \left(u_R^+\right)^2\Big|_s^{\beta s_R} d\theta \geq 0\,.$$

Consequently,

$$\left\{\int_0^{2\pi}\left[\left(u_R^{+\,5/2}(\beta s_R) + \frac{C}{s_R}\right)^{4/5} - u_R^{+\,2}(s)\right] d\theta\right\}^{5/4} \geq \left\{\int_0^{2\pi} \left(u_R^+\right)^2\Big|_s^{\beta s_R}\right\}^{5/4}$$

Combining the above inequalities yields

$$\int_0^{2\pi} \left(u_R^+\right)^2\Big|_s^{\beta s_R} d\theta \leq \frac{C}{s^{4/5}}$$

and by (3.17) we have

$$\int_s^{\beta s_R} \frac{F_R(x_R, r)}{r} dr \leq \frac{C}{s^{4/5}}$$

Thus, since F_R is monotone increasing

$$\exists \frac{1}{2} < \beta_1 < \beta : \; s < \beta s_R \Rightarrow F_R(x_R, s) \leq \frac{C_1}{s_R^{4/5}} \tag{3.24}$$

which is exactly (3.15) for $n = 1$.

We now make the inductive assumption

$$\exists \frac{1}{2} < \beta_n < 1 : \; s < \beta_n s_R \Rightarrow F_R(x_R, s) \leq \frac{C_n}{s_R^{\alpha_n}}.$$

By (3.20) and (3.21) we obtain that for some $1/2 < \beta < \beta_n$ we have for each $\xi_1, \xi_2 \in \partial B_\beta$

$$\left| \left(u_R^+\right)^{5/2} (\xi_1) - \left(u_R^+\right)^{5/2} (\xi_2) \right| \leq \frac{C}{s_R^{\alpha_n + 1}}.$$

By (3.22) we have

$$\int_0^{2\pi} \left(u_R^+\right)^{5/2} \Big|_s^{\beta s_R} \leq \frac{C}{s^{\alpha_n + 1}}.$$

Consequently,

$$\int_s^{\beta s_R} \frac{F_R(x_R, r)}{r} dr \leq \int_0^{2\pi} \left(u_R^+\right)^2 \Big|_s^{\beta s_R} d\theta \leq \frac{C}{s^{4(\alpha_n + 1)/5}}$$

Hence,

$$\alpha_{n+1} = \frac{4}{5} (\alpha_n + 1)$$

from which the lemma immediately follows.

□

Lemma 3 allows us to obtain uniform convergence in Ω of u_R to a constant, except for a boundary layer of $O(1)$ size (as $R \to \infty$).

Lemma 4. *For any family of points* $\{x_R\}_{R > R_0}$

$$\exists \tilde{u}_R : \; \left| u_R^+(x_R) - \tilde{u}_R \right| \leq \frac{C}{d(x_R, \partial\Omega_R)^{1/2}}$$

Proof: Let again $s_R = d(x_R, \partial\Omega_R)$. Let $x_R^1 \in B\left(x_R, \frac{1}{2} s_R\right)$. We first argue that, in view of (3.15), there must exist a contour $\mathcal{C}$ connecting x_R and x_R^1 which is embedded in $B\left(x_R, \frac{1}{2} s_R\right)$, such that

$$\int_{\mathcal{C}} \kappa u_R^+ \left| \hat{J}_R \right|^2 + \rho_R^2 \left(u_R^+\right)^2 \leq \frac{C}{s_R^{5/4}} \, . \tag{3.25}$$

To prove the latter statement consider the set of paths, enclosed in $B(x_R, \frac{1}{2}s_R)$ composed of two straight lines: the first of them connects x_R with the vertical median of the line connecting x_R and x_R^1, and the second line connect the end of the first line with x_R^1. Then, in view of (3.15) we must have along at least one of these paths

$$\int_0^{\bar{s}} \left[\kappa u_R^+ \left| \hat{J}_R \right|^2 + \rho_R^2 \left(u_R^+\right)^2 \right]_{x=x_R+r\hat{e}} rdr + \int_0^{\bar{s}} \left[\kappa u_R^+ \left| \hat{J}_R \right|^2 + \rho_R^2 \left(u_R^+\right)^2 \right]_{x=x_R^1+r\hat{e}_1} rdr \leq \frac{C}{s_R^{5/2}} \tag{3.26}$$

where $\hat{e}$ and $\hat{e}_1$ are unit vectors in the direction of the segments composing the path from x_R to x_R^1 and $\bar{s} < s_R/2$.

In view of (3.26) and (3.4) we have

$$\int_0^{\bar{s}} \left[\kappa u_R^+ \left| \hat{J}_R \right|^2 + \rho_R^2 \left(u_R^+\right)^2 \right]_{x=x_R+r\hat{e}} dr \leq \int_0^{t} \left[\kappa u_R^+ \left| \hat{J}_R \right|^2 + \rho_R^2 \left(u_R^+\right)^2 \right]_{x=x_R+r\hat{e}} dr +$$
$$+ \frac{1}{t} \int_t^{\bar{s}} \left[\kappa u_R^+ \left| \hat{J}_R \right|^2 + \rho_R^2 \left(u_R^+\right)^2 \right]_{x=x_R+r\hat{e}} rdr \leq C \left(t + \frac{1}{t s_R^{5/2}} \right)$$

Choosing $t = s_R^{-5/4}$ yields

$$\int_0^{\bar{s}} \left[\kappa u_R^+ \left| \hat{J}_R \right|^2 + \rho_R^2 \left(u_R^+\right)^2 \right]_{x=x_R+r\hat{e}} dr \leq \frac{C}{s_R^{5/4}} \,.$$

The integral along the second part of the path of can be esimated in the same manner, which completes the proof of (3.25).

In view of (3.25) we have,

$$\int_{\mathcal{C}} \left(u_R^+\right)^{3/2} |\nabla u_R| \leq C \left[\int_{\mathcal{C}} u_R^+ \left| \hat{J}_R \right|^2 \right]^{1/2} \left[\int_{\mathcal{C}} \rho_R^2 \left(u_R^+\right)^2 \right]^{1/2} \leq \frac{C}{s_R^{5/4}}$$

Consequently,

$$\left| \left(u_R^+\right)^{5/2}(x_R) - \left(u_R^+\right)^{5/2}(x_R^1) \right| \leq \frac{C}{s_R^{5/4}} \tag{3.27}$$

Using the inequality

$$\left| x^{5/2} - y^{5/2} \right|^{2/5} \geq |x - y|$$

yields

$$\left| \left(u_R^+\right)(x_R) - \left(u_R^+\right)(x_R^1) \right| \leq \frac{C}{s_R^{1/2}} \tag{3.28}$$

Let z_R be the closest point on $\partial\Omega_R$ to x_R. Let y_R denote the center of the maximal ball embedded in Ω_R which is tangent to $\partial\Omega_R$ at z_R. Let r_m denote the radius of the minimal ball which is both embedded in Ω_1 and tangent to $\partial\Omega_1$. Then clearly, $d(y_R,\partial\Omega_R) \geq r_m R$. We now define a finite sequence of points $\{x_R^n\}_{n=0}^{N_R}$ located on the line connecting x_R and y_R in the following way

$$x_R^0 = x_R, \quad x_R^{N_R} = y_R$$

$$d(x_R^n,\partial\Omega_R) = \frac{3}{2} d(x_R^{n-1}) \quad 1 \leq n \leq N_R - 1$$

$$\left(\frac{3}{2}\right)^{N_R-1} s_R < d(y_R,\partial\Omega_R) \leq \left(\frac{3}{2}\right)^{N_R} s_R$$

By (3.28) we have

$$\left|\left(u_R^+\right)(x_R^n) - \left(u_R^+\right)(x_R^{n-1})\right| \leq \frac{C}{s_R^{1/2}} \left(\frac{2}{3}\right)^{(n-1)/2} \quad \forall 1 \leq n \leq N_R$$

and thus,

$$\left|\left(u_R^+\right)(y_R) - \left(u_R^+\right)(x_R)\right| \leq \sum_{n=1}^{N_R} \left|\left(u_R^+\right)(x_R^n) - \left(u_R^+\right)(x_R^{n-1})\right| \leq$$

$$\leq \sum_{n=1}^{\infty} \frac{C}{s_R^{1/2}} \left(\frac{2}{3}\right)^{(n-1)/2} \leq \frac{C}{s_R^{1/2}}. \quad (3.29)$$

Let then $\tilde{u}_R \stackrel{def}{=} u_R^+(y_R)$. We need to show that

$$\left|\left(u_R^+\right)(\tilde{x}_R) - \tilde{u}_R\right| \leq \frac{C}{\tilde{s}_R^{1/2}}$$

for any $\tilde{x}_R \in \Omega_R$, where $\tilde{s}_R = d(\tilde{x}_R,\partial\Omega_R)$. Let $\tilde{z}_R$ be the closest point on $\partial\Omega_R$ to $\tilde{x}_R$. Let $\tilde{y}_R$ denote the center of the maximal ball embedded in Ω_R which is tangent to $\partial\Omega_R$ at $\tilde{z}_R$. Clearly, there is a curve $\tilde{\mathcal{C}}$ connecting y_R and $\tilde{y}_R$ whose distance from the boundary is strictly greater than $r_m R$, and whose length is bounded by LR where L is the arclength of $\partial\Omega_1$.

Let then $\{y_R^n\}_{n=0}^{\tilde{N}_R}$ denote a sequence of points along $\tilde{\mathcal{C}}$ such that

$$y_R^0 = y_R \quad ; \quad y_R^{\tilde{N}_R} = \tilde{y}_R$$

$$\left|y_R^n - y_R^{n-1}\right| = \frac{1}{2} r_m R \quad 1 \leq n \leq \tilde{N}_R - 1,$$

$$\left|y_R^{\tilde{N}_R} - y_R^{\tilde{N}_R - 1}\right| \leq \frac{1}{2} r_m R$$

Clearly,

$$\tilde{N}_R \leq \frac{2L}{r_m} \leq C \ .$$

Furthermore, for all $1 \leq n \leq \tilde{N}_R - 1$ we have

$$\left|\left(u_R^+\right)\left(y_R^n\right) - \left(u_R^+\right)\left(y_R^{n-1}\right)\right| \leq \frac{C}{R^{1/2}}.$$

Hence, following the same steps in (3.29), and taking advantage of the uniform boundedness of $\tilde{N}_R$ with respect to R, we obtain

$$\left|\left(u_R^+\right)\left(y_R\right) - \left(u_R^+\right)\left(\tilde{y}_R\right)\right| \leq \frac{C}{R^{1/2}}.$$

By (3.29) we have

$$\left|\left(u_R^+\right)\left(\tilde{x}_R\right) - \left(u_R^+\right)\left(\tilde{y}_R\right)\right| \leq \frac{C}{\tilde{s}_R^{1/2}},$$

and hence,

$$\left|\left(u_R^+\right)\left(\tilde{x}_R\right) - \tilde{u}_R\right| \leq \frac{C}{\tilde{s}_R^{1/2}} \quad \forall \tilde{x} \in \Omega_R.$$

□

We now find the value of the constant $\tilde{u}$ by using the energy estimates in lemma 2.

Lemma 5. *Let $h_{ex} > \kappa$. Then,*

$$|\tilde{u}_R - (h_{ex} - \kappa)| \leq \frac{C}{R^{1/2}} \tag{3.30}$$

Proof: Let $x_1 \in \Omega_1$ such that $\partial B(x_1, r) \subset int(\Omega_1)$. Denote by x_R the image of x_1 in Ω_R under the transformation (3.1). Clearly, $B(x_R, rR) \subset \Omega_R$. By lemma 2 we have

$$\|h_{ex} - \kappa - u_R\|_{L^2[B(x_R, rR)]} \leq \|h_{ex} - \kappa - u_R\|_{L^2[\Omega_R]} \leq \|h_{ex} - h_R\|_{L^2[\Omega_R]} + \frac{1}{2\kappa}\left\|\rho_R^2\right\|_{L^2[\Omega_R]} \leq CR^{1/2}.$$

However, by the previous lemma $|u_R^+ - \tilde{u}_R| \leq C/R^{1/2}$ in $B(x_R, rR)$, and hence

$$\|h_{ex} - \kappa - \tilde{u}_R\|_{L^2[B(x_R,rR)]} \leq \left\|h_{ex} - \kappa - u_R^+\right\|_{L^2[B(x_R,rR)]} + \left\|\tilde{u}_R - u_R^+\right\|_{L^2[B(x_R,rR)]} \leq CR^{1/2},$$

from which the lemma immediately follows.

□

We can now obtain better estimates for the rate of decay of $|u_R^+ - \tilde{u}_R|$ away from the boundaries as $R \to \infty$.

Lemma 6. *Let $h_{ex} > \kappa$ and $\{x_R\}_{R \geq R_0}$ denote a family of points such that $x_R \in \Omega_R$. Let $s_R = d(x_R, \partial\Omega_R) \xrightarrow[R\to\infty]{} \infty$. Then,*

$$\forall n \in \mathbb{N} \quad \exists \frac{1}{2} < \beta_n < 1,\ C_n > 0 : \ F_R(x_R, s) \leq \frac{C_n}{s_R^n} \tag{3.31a}$$

where F_R is defined in (3.16)

$$\exists \tilde{u}_R : \quad \left|u_R^+(x_R) - \tilde{u}_R\right| \leq \frac{C_n}{s_R^n} \tag{3.31b}$$

Proof: Clearly, by lemma 3

$$\exists \frac{1}{2} < \beta_1 < \beta,\ C_1 > 0 : \ s < \beta s_R \Rightarrow F_R(x_R, s) \leq \frac{C_1}{s_R}$$

Suppose, then, that

$$\exists \frac{1}{2} < \beta_n < 1\ C_n > 0 : \ s < \beta_n s_R \Rightarrow F_R(x_R, s) \leq \frac{C_n}{s_R^n}.$$

By (3.20) and (3.21) we obtain that for some $1/2 < \beta < \beta_n$ we have for each $\xi_1, \xi_2 \in \partial B_\beta$ where $B_\beta \overset{def}{=} B(x, \beta s_R)$

$$\left|\left(u_R^+\right)^{5/2}(\xi_1) - \left(u_R^+\right)^{5/2}(\xi_2)\right| \leq \frac{C}{s_R^{n+1}}.$$

By (3.22) we have

$$\int_0^{2\pi} \left(u_R^+\right)^{5/2}\Big|_s^{\beta s_R} \leq \frac{C}{s_R^{n+1}}.$$

Instead, however, of using (3.23) we write

$$\frac{x^5-y^5}{x^4-y^4}=\frac{x^4+x^3y+x^2y^2+xy^3+y^4}{x^3+x^2y+xy^2+y^3}\geq\frac{1}{6}(x+y)$$

which is correct when both x and y are positive. Consequently,

$$\int_0^{2\pi}\left(u_R^+\right)^{5/2}\Big|_s^{\beta s_R}d\theta+\frac{2\pi C}{s_R^{n+1}}=\int_0^{2\pi}\left\{\left[\left(u_R^{+\ 5/2}(\beta s_R)+\frac{C}{s_R^{n+1}}\right)^{1/5}\right]^5-u_R^{+\ 5/2}(s)\right\}d\theta\geq$$

$$\geq\frac{1}{6}\int_0^{2\pi}\left[\left(u_R^{+\ 5/2}(\beta s_R)+\frac{C}{s_R^{n+1}}\right)^{1/5}+u_R^{+\ 1/2}(s)\right]\left[\left(u_R^{+\ 5/2}(\beta s_R)+\frac{C}{s_R^{n+1}}\right)^{4/5}-u_R^{+\ 2}(s)\right]d\theta$$

However, from lemma 5 we can conclude that for sufficiently large R

$$u_R^+(\beta s_R)\geq\frac{1}{2}\left(h_{ex}-\kappa\right).$$

Hence,

$$\int_0^{2\pi}\left(u_R^+\right)^{5/2}\Big|_s^{\beta s_R}d\theta+\frac{2\pi C}{s_R^{n+1}}\geq C\int_0^{2\pi}\left[\left(u_R^{+\ 5/2}(\beta s_R)+\frac{C}{s_R^{n+1}}\right)^{4/5}-u_R^{+\ 2}(s)\right]d\theta\geq C\int_0^{2\pi}\left(u_R^+\right)^2\Big|_s^{\beta s_R}d\theta$$

from which we obtain

$$\exists\frac{1}{2}<\beta_{n+1}<\beta:\ s<\beta_{n+1}s_R\Rightarrow F_R(x_R,s)\leq\frac{C_{n+1}}{s_R^{n+1}}.$$

completing the proof of (3.31a).

The proof (3.31b) is almost identical with the proof of lemma 4.

□

The last auxiliary result we need is the uniform decay, away from the boundary, of ρ_R.

Lemma 7. *Let $h_{ex}>\kappa$. For any family of points $\{x_R\}_{R>R_0}$*

$$\forall n\in\mathbb{N}\quad\exists C_n>0:\rho_R(x_R)\leq\frac{C_n}{d(x_R,\partial\Omega_R)^n}.$$

Proof: Let $s_R=d(x_R,\partial\Omega_R)$. By (3.31a)

$$\exists\frac{1}{2}<\beta_n<1:\quad\int_{B(x_R,\beta s_R)}\rho_R^2\left(u_R^+\right)^2\leq\frac{C_n}{s_R^n}.$$

Writing

$$(h_{ex}-\kappa)\left\|\rho_R\right\|_{L^2[B(x_R,\beta_n s_R)]} \leq \left\|\rho_R u_R^+\right\|_{L^2[B(x_R,\beta_n s_R)]} + \\ + \left\|\rho_R\left(u_R^+-\tilde{u}_R\right)\right\|_{L^2[B(x_R,\beta_n s_R)]} + \left\|\rho_R\left(h_{ex}-\kappa-\tilde{u}_R\right)\right\|_{L^2[B(x_R,\beta_n s_R)]} \;,$$

we obtain, in view of (3.30) and (3.31b),

$$\left\|\rho_R\right\|_{L^2[B(x_R,\beta_n s_R)]} \leq \frac{C_n}{s_R^{n/2}} \;. \tag{3.32}$$

Consequently, for sufficiently large R, we may conclude the existence of $\frac{1}{2} < \bar{\beta}_n < \beta_n$ for which

$$\int_{\partial B(x_R,\bar{\beta}_n s_R)} \rho_R^2 \leq \frac{C_n}{s_R^n}. \tag{3.33}$$

We now multiply (2.1a) by ψ_R^* and integrate over $B(x_R,r)$ to obtain

$$\frac{1}{\kappa^2}\int_{\partial B(x_R,r)} \rho_R\frac{\partial\rho_R}{\partial r} + \int_{B(x_R,r)} \rho_R^2\left(1-\rho_R^2\right) = G_R(x_R,r)$$

where

$$G_R(x_R,r) = \int_{B(x_R,r)} \left|\left(\frac{i}{\kappa}\nabla + A\right)\psi\right|^2$$

Then, multiplying by $1/r$ and integrating between s and $\bar{\beta}_n s_R$ we obtain

$$\frac{1}{\kappa^2}\int_0^{2\pi} \rho_R^2\Big|_s^{\bar{\beta}_n s_R} d\theta + \int_s^{\bar{\beta}_n s_R}\frac{dr}{r}\int_{B(x_R,r)} \rho_R^2\left(1-\rho_R^2\right) = \int_s^{\bar{\beta}_n s_R}\frac{dr}{r}G_R(x_R,r)$$

Using (3.32) and (3.33) yields

$$\int_s^{\bar{\beta}_n s_R}\frac{dr}{r}G_R(x_R,r)dr \leq \frac{C_n}{s_R^n}$$

and since G_R is monontonically increasing we have

$$\forall n\in\mathbb{N}\;\exists\frac{1}{2} < \tilde{\beta}_n < \bar{\beta}_n < 1,\; C_n > 0:\; G_R(x_R,\beta_n s_R) \leq \frac{C_n}{s_R^n} \;.$$

Furthermore,

$$\left|\left(\frac{i}{\kappa}\nabla + A\right)\psi\right|^2 = \frac{1}{\kappa^2}\left|\nabla\rho_R\right|^2 + \rho_R^2\left|\frac{1}{\kappa}\nabla\phi - A\right|^2$$

when $\rho_R \neq 0$. When $\rho_R = 0$ we simply drop the second term on the right-hand-side, and hence,

$$\frac{1}{\kappa^2}\left|\nabla\rho_R\right|^2 \leq \left|\left(\frac{i}{\kappa}\nabla + A\right)\psi\right|^2 .$$

Consequently,

$$\frac{1}{\kappa^2}\int_{B(x_R,\tilde{\beta}_n s_R)}\left|\nabla\rho_R\right|^2 \leq \frac{C_n}{s_R^n}$$

Using the estimate

$$\int_{B(x_R,\tilde{\beta}_n s_R)}\left|\nabla\rho_R\right| \leq C s_R\left[\int_{B(x_R,\tilde{\beta}_n s_R)}\left|\nabla\rho_R\right|^2\right]^{1/2} \leq \frac{C_n}{s_R^{n/2-1}}$$

we can now complete the proof of the lemma by following the same steps as in the proof of lemma 4

□

Proof of Theorem 3.1: We use standard blow-up arguments in order to prove the theorem (cf. [11]). Let

$$\Omega(R,k,s) = \{x \in \Omega_R |\, d(x,\partial\Omega_R) \geq ks\}\,.$$

We prove exponential rate of decay by showing that

$$\exists R_0, s_0 :\ \|\psi_R\|_{L^\infty[\Omega(R,k+1,s)]} \leq \frac{1}{2}\|\psi_R\|_{L^\infty[\Omega(R,k,s)]} \quad \forall s > s_0,\ R > R_0,\ k \in \mathbb{N} \tag{3.34}$$

Suppose, for a contradiction, that (3.34) does not hold. Then, there exist sequences $\{R_j\}_{j=1}^\infty$, $\{s_j\}_{j=1}^\infty$, and $\{k_j\}_{j=1}^\infty$ satisfying $R_j \uparrow \infty$, $s_j \uparrow \infty$, $k_j \in \mathbb{N}$, and

$$\|\psi_R\|_{L^\infty[\Omega(R,k+1,s)]} \geq \frac{1}{2}\|\psi_R\|_{L^\infty[\Omega(R,k,s)]} \overset{def}{=} \frac{1}{2}m_j \tag{3.35}$$

Let

$$\tilde{\psi}_{R_j} \overset{def}{=} \frac{\psi_{R_j}}{m_j}.$$

By (3.34) there exists $x_j \in \Omega(R,k+1,s)$ such that $\tilde{\psi}_{R_j}(x_j) \geq \frac{1}{2}$. Furthermore, since $B(x_j,s_j) \in \Omega(R_j,k_j,s_j)$ we have

$$\frac{1}{2} \leq \left\|\tilde{\psi}_{R_j}\right\|_{L^\infty[B(x_j,s_j)]} \leq 1.$$

Define

$$f_j = \tilde{\psi}_{R_j}(x_j + x)e^{iA_{R_j}(x_j)\cdot x}.$$

It is easy to show that

$$\left(\frac{i}{\kappa}\nabla+\tilde{A}_j\right)^2 f_j = f_j\left(1-m_j^2\left|f_j\right|^2\right) \quad x\in B(0,s_j) \tag{3.36a}$$

wherein

$$\tilde{A}_j(x) = A_{R_j}(x_j+x) - A_{R_j}(x_j) \tag{3.36b}$$

We have seen in lemma 5 and 7 that

$$\left\|\nabla\times\tilde{A}_j - h_{ex}\hat{i}_z\right\|_{L^\infty[B(0,s_j)]}\to 0$$

and that

$$m_j\to 0.$$

In a similar manner to the one used in (3.4c) we now define a cut-off function

$$\eta_r = \begin{cases} 1 & \text{in } B(0,r) \\ 0 & \text{in } \mathbb{R}^2/B(0,r+1)\end{cases}$$

$$|\nabla\eta_r|\leq C \quad \text{in } \mathbb{R}^2$$

It is not difficult to show (cf. [19]) that

$$\int_{B(0,r+1)}\left|\left(\frac{i}{\kappa}\nabla+\tilde{A}_j\right)(\eta_r f_j)\right|^2 = \int_{B(0,r+1)}\eta_r^2 f_j^2\left(1-m_j^2\left|f_j\right|^2\right)+\frac{1}{\kappa^2}\left|\nabla\eta_r\right|^2 f_j^2$$

Consequently,

$$\int_{B(0,r)}\left|\left(\frac{i}{\kappa}\nabla+\tilde{A}_j\right)f_j\right|^2\leq C_r \tag{3.37}$$

We now rewrite (3.36) in the form

$$\frac{1}{\kappa^2}\nabla^2 f_j = 2\tilde{A}_j\cdot\left(\frac{i}{\kappa}\nabla+\tilde{A}_j\right)f_j - f_j\left(1-m_j^2\left|f_j\right|^2+\left|\tilde{A}_j\right|^2\right)$$

As the right-hand-side of the above equation is uniformly bounded in $L^2[B(0,r)]$ for all sufficiently large j (such that $r<s_j-1$), we can apply standard elliptic estimates to prove first that f_j is uniformly bounded

in $H^2_{loc}(\mathbb{R}^2)$, and then that a subsequence $\{f_{jk}\}_{k=1}^{\infty}$ which converges in $H^1_{loc}(\mathbb{R}^2)$ to a limit f_0 must exist. The limit is the (weak) solution of

$$\left(\frac{i}{\kappa}\nabla + h_{ex}\hat{A}\right)^2 f_0 = f_0 \quad x \in \mathbb{R}^2$$

wherein $\nabla \times \hat{A} = \hat{i}_z$. However, it has been proved in [19,11], that no bounded solution can exist to the above equation in $\mathbb{R}^2$ when $h_{ex} > \kappa$. Hence, (3.35) has been contradicted and we must have,

$$|\psi_R| \leq C_0 e^{-\beta d(x,\partial\Omega_R)} \tag{3.38}$$

where C_0 and β are independent of R.

To prove (3.3a) for $\alpha = 0$ we write (2.1b) in the form

$$\nabla^2 A_R = \nabla \times h_R = \Im\left[\bar{\psi}_R\left(\frac{i}{\kappa}\nabla + A_R\right)\psi_R\right] \tag{3.39}$$

Hence,

$$|\nabla h_R|^2 \leq |\psi_R|^2 \left|\left(\frac{i}{\kappa}\nabla + A_R\right)\psi_R\right|^2$$

Hence, by (3.37) and (3.38) we have

$$\int_{B\left(x,\frac{1}{2}d(x,\partial\Omega_R)\right)} |\nabla h_R|^2 \leq C_0 e^{-2\beta d(x,\partial\Omega_R)}.$$

Using the above inequality we can prove, following the same steps in the proof of lemma 4, the existence of a constant $\tilde{h}_R$, such that

$$\left|h_R - \tilde{h}_R\right| \leq C_0 e^{-\beta d(x,\partial\Omega_R)}$$

Using bootstraping and Sobolev embedding we can prove (3.3a,b) for $\alpha \geq 1$. (3.3c) follows immediately from lemma 5.

□

Pan [22] demonstrates that in the limit $\kappa \to \infty$ (where Ω is fixed) we have

$$\int_{\Omega}\left\{|\psi|^2 + \left|\frac{1}{\kappa}\nabla\psi - iA\psi\right|^2\right\} e^{\beta\sqrt{\kappa(h_{ex}-\kappa)}d(x,\partial\Omega)}dx \leq \frac{C}{\sqrt{\kappa(h_{ex}-\kappa)}}$$

whenever $h_{ex} - \kappa \gg 1$, for some $\beta > 0$ which is independent of κ. Pan's result is weaker than theorem 3.1 in the following senses:

1. The exponential rate of decay is in L^2 sense whereas here pointwise decay is proved.
2. Our results are valid for all $h_{ex} > \kappa$ and not only for $h_{ex} \gg \kappa$ (this advantage may be attributed to the difference between the limits).
3. We prove exponentially fast convergence of the magnetic field to a constant which is close to h_{ex}.

We mention again that Pan [22] approximates the behaviour of the solution near the boundary. This important problem has not been addressed in the present contribution.

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Faculty of Mathematics, Technion - Israel Institute ofTechnology, Haifa 32000, Israel.

e-mail: almog@math.technion.ac.il