The interface between the normal state and the fully superconducting state in the presence of an electric current

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Abstract

We consider the time-dependent Ginzburg-Landau equations, in the presence of electric current and the absence of magnetic field. We first study one-dimensional equilibrium solutions on a semi-infinite domain, describing a layer of transition from the normal state at one edge to the fully superconducting state at infinity. We find that the normal conductivity has a significant effect on the maximal current that can pass through such a transition layer. The global stability of the purely superconducting state in a finite domain is also considered, assuming zero potential drop between the conducting surfaces.

1 Introduction

When a superconductor is placed in a temperature lower than the critical one, it looses its electrical resistivity. This means that a current can flow through a superconducting sample and generate a vanishingly small voltage drop. If one raises the current above a certain critical level, superconductivity will be destroyed and the material will revert to the normal state, even if the temperature is kept fixed below the critical one.

The model we use in this work to study the above phenomena is the time-dependent Ginzburg-Landau model [10, 4], presented here in a dimensionless form

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$$\frac{\partial\psi}{\partial t} + i\phi\psi = (\nabla - iA)^2\,\psi + \psi\left(1 - |\psi|^2\right) \qquad \text{in } \Omega\times\mathbb{R}_+\,, \tag{1.1a}$$

$$-\kappa^2\nabla\times\nabla\times A + \sigma\left(\frac{\partial A}{\partial t} + \nabla\phi\right) = \Im\{\bar\psi\nabla\psi\} + |\psi|^2 A \qquad \text{in } \Omega\times\mathbb{R}_+\,, \tag{1.1b}$$

$$\psi = 0 \quad ; \quad -\sigma\left(\frac{\partial A}{\partial t} + \nabla\phi\right)\cdot\nu = J \qquad \text{on } \partial\Omega_c\times\mathbb{R}_+\,, \tag{1.1c}$$

$$(i\nabla + A)\psi\cdot\nu = 0 \quad ; \quad \left(\frac{\partial A}{\partial t} + \nabla\phi\right)\cdot\nu = 0 \qquad \text{on } \partial\Omega_i\times\mathbb{R}_+\,, \tag{1.1d}$$

$$\psi(x,0) = \psi_0 \qquad \text{in } \Omega\,, \tag{1.1e}$$

$$A(x,0) = A_0 \qquad \text{in } \Omega\,. \tag{1.1f}$$

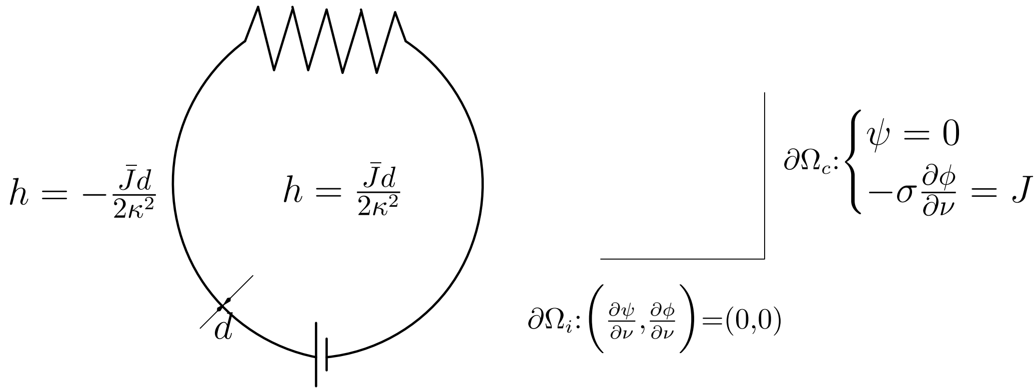
In the above ψ is the superconducting order parameter, so that $|\psi|$ represents the number density of superconducting electrons. Superconductors for which $|\psi| = 1$ are said to be purely superconducting, and those for which $\psi = 0$ are said to be at the normal state. We denote the magnetic vector potential by A, so that the magnetic field is given by $h = \nabla\times A$, and by ϕ the electric scalar potential. The constant σ represents the normal conductivity of the superconducting material so that $-\sigma(A_t + \nabla\phi)$ is the normal current, and κ is the Ginzburg-Landau parameter. Length has been scaled with respect to the coherence length ξ, which is the length-scale characterizing variations in ψ. The domain $\Omega\subset\mathbb{R}^n$ ($n\in\{1,2,3\}$) where the superconducting sample resides has interface, denoted by $\partial\Omega_c$, with a conducting metal which is at the normal state. The rest of the boundary, denoted by $\partial\Omega_i$ is adjacent to an insulator. The function $J : \partial\Omega_c\to\mathbb{R}$ represents the normal current entering the sample. The average current density flowing through the wire's cross-section will be denoted later by $\bar J$. It is possible to prescribe the electric potential on $\partial\Omega_c$ instead of the current.

Except for the initial conditions, (1.1) is invariant to the gauge transformation [4]

$$A\to A+\nabla\omega \quad ; \quad \psi\to\psi e^{i\omega} \quad ; \quad \phi\to\phi - \frac{\partial\omega}{\partial t}$$

for some smooth ω,. Thus, we fix the Coloumb gauge ($\nabla\cdot A = 0$, $A\cdot\nu|_{\partial\Omega} = 0$) in the following. Finally, one has to prescribe h at a single point on $\partial\Omega$ for all $t>0$, for reasons detailed below.

Figure 1 presents a schematic description of a circuit, composed of two superconducting wires, a DC power source and a resistor. Consider first a two-dimensional setting, where h is perpendicular to the plane. Then $\nabla\times h = -\nabla_\perp h$, and hence by (1.1b) and (1.1c) we have that $\partial h/\partial s = J/\kappa^2$ along the interface with the resistor and the power source, where $\partial/\partial s$ denotes the tangential derivative. Outside the circuit, the Maxwell equation $\nabla\times h = 0$ should be satisfied, and hence h must be constant



(a) An example of a 2D circuit: A power source is connected to a resistor by two superconducting wires, h, $\bar{J}$ and κ are defined above and below. Note the different magnetic field values inside and outside the circuit.

(b) Magnification of the interface between the power source and the left wire in fig. 1(a) at the outer boundary. The left wire assumes the role of Ω in (1.1).

inside and outside the circuit. We choose the field inside and outside the circuit to be of equal magnitude opposite directions. This choice can be justified by considering an infinite cylindrical wire with a circular cross section in $\mathbb{R}^3$: in the absence of applied magnetic field, the Maxwell equation $\nabla \times h = 0$ outside the wire possesses the solution

$$h = \frac{h_d}{r} e_\theta \quad r > d \,,$$

where h_d is a constant. Looking at a planar cross-section along the wire's axis of symmetry, we see that the magnetic field on both sides of this cross-section are of equal magnitude and opposite direction.

To avoid discussing the complex multi-connected circuit we consider a single wire separately. Accordingly, Ω in (1.1) may represent any of the wires in figure 1. Consequently, instead of assigning a fixed potential drop over the power source, we need to prescribe the normal current density J on $\partial\Omega_c$. Finally, let $x_0 \in \partial\Omega_i$. Then, if x_0 is on the outer boundary, we set

$$h(x_0, t) = -\frac{\bar{J}d}{2\kappa^2} \quad \forall t > 0 \,. \tag{1.2}$$

Otherwise, if x_0 lies on the inner part of $\partial\Omega_i$, then $h(x_0, t) = \frac{\bar{J}d}{2\kappa^2}$.

We now let $\kappa \to \infty$ in (1.1). Denote the leading order behaviour of the magnetic field by h_0. From (1.1b) together with (1.2) we formally obtain that

$$\begin{cases} \nabla \times h_0 = 0 & \text{in } \Omega \times \mathbb{R} \\ h_0 = 0 & \text{on } \partial\Omega_i \times \mathbb{R} \end{cases} .$$

Consequently, $h_0 = 0$. Similarly, since we have chosen the Coloumb gauge, we also have $A_0 = 0$ for the leading order vector potential.

The above formal expansion gives rise to a simplified model, where the magnetic field is assumed to be negligible, i.e., $A \equiv 0$. Note, that given that spatial coordinates in (1.1) were scaled with respect to the coherence length ξ, in the limit $\kappa = \lambda/\xi \gg 1$, the domain's size is much smaller than the penetration depth λ which is the length-scale characterizing variations in the magnetic field (cf. [3]). The domain size need not be fixed, as long as $\bar{J}d/\kappa^2 \ll 1$ we obtain $A_0 = 0$ for the leading order vector potential.

Note, that while we expect $h \to 0$ as $\kappa \to \infty$, $\kappa^2 \nabla \times h$ need not vanish in that limit. Rather, it tends to a divergence-free vector, representing the overall current in the wire. As this current is a priory unknown, we proceed by taking the divergence of (1.1b) and then setting $A = 0$. We obtain

$$\frac{\partial \psi}{\partial t} + i\phi\psi = \Delta\psi + \psi\left(1 - |\psi|^2\right) \qquad \text{in } \Omega \times \mathbb{R}_+ \tag{1.3a}$$

$$\sigma\Delta\phi = \nabla \cdot [\Im(\bar{\psi}\nabla\psi)] \qquad \text{in } \Omega \times \mathbb{R}_+ \tag{1.3b}$$

$$\psi = 0 \qquad \text{on } \partial\Omega_c \times \mathbb{R}_+ \tag{1.3c}$$

$$-\sigma\frac{\partial \phi}{\partial \nu} = J \qquad \text{on } \partial\Omega_c \times \mathbb{R}_+ \tag{1.3d}$$

$$\frac{\partial \psi}{\partial \nu} = 0 \qquad \text{on } \partial\Omega_i \times \mathbb{R}_+ \tag{1.3e}$$

$$\frac{\partial \phi}{\partial \nu} = 0 \qquad \text{on } \partial\Omega_i \times \mathbb{R}_+ \tag{1.3f}$$

$$\psi(x,0) = \psi_0 \qquad \text{in } \Omega\,. \tag{1.3g}$$

Consider now the steady-state version of (1.3) in a one-dimensional setting. A one-dimensional wire is assume to lie in $\mathbb{R}_+$. Then,

$$\begin{cases} -\psi'' + i\phi\psi - \psi(1 - |\psi|^2) = 0 & \text{in } \mathbb{R}_+ \\ -\sigma\phi'' + \Im\{\psi'\bar{\psi}\}' = 0 & \text{in } \mathbb{R}_+ \\ \psi(0) = 0 & -\sigma\phi'(0) = J \\ |\psi| \to \rho_\infty & \text{as } x \to \infty \\ \phi \to 0 & \text{as } x \to \infty\,. \end{cases} \tag{1.4}$$

In the above, the current J is a constant. The boundary condition at $x = 0$ represents an interface with the normal state. As $x \to \infty$ a fully superconducting state is prescribed. For this simplified model it is well-known [20] that the perfectly superconducting state is given by

$$\psi_s = \rho_\infty e^{i\alpha x} \quad ; \quad \phi \equiv 0\,, \tag{1.5}$$

with $\alpha = [1-\rho_\infty^2]^{1/2}$, and

$$J^2 = \rho_\infty^4 (1-\rho_\infty^2)\,. \tag{1.6}$$

Thus, in (1.4), J and ρ_∞ must satisfy (1.6). It is easy to show, as $0 \leq \rho_\infty \leq 1$, that the values of J for which (1.6) can be satisfied are limited to the interval $J \in [0, J_c]$ where

$$J_c = \max_{\rho_\infty \in [0,1]} \rho_\infty^2 \sqrt{1-\rho_\infty^2} = \left[\frac{4}{27}\right]^{1/2}, \tag{1.7}$$

where for $J = J_c$ we have $\rho_\infty^2 = 2/3$. This critical current is well known to Physicists and has been noted in various textbooks [20, 9].

Using the polar representation $\psi = \rho e^{i\chi}$ we obtain that whenever $\rho \neq 0$.

$$-\rho'' + \frac{(\sigma\phi' + J)^2}{\rho^3} - \rho(1-\rho^2) = 0 \quad \text{in } \mathbb{R}_+ \tag{1.8a}$$

$$-\sigma\phi'' + \rho^2\phi = 0 \quad \text{in } \mathbb{R}_+ \tag{1.8b}$$

$$\rho(0) = 0 \tag{1.8c}$$

$$\rho \xrightarrow[x\to\infty]{} \rho_\infty \tag{1.8d}$$

$$\phi'(0) = -\frac{J}{\sigma} \tag{1.8e}$$

$$\phi \xrightarrow[x\to\infty]{} 0 \tag{1.8f}$$

In a similar manner, if (ρ, ϕ) is a solution of (1.8) then for any χ satisfying

$$\chi' = \frac{(\sigma\phi' + J)}{\rho^2}\,,$$

$(\rho e^{i\chi}, \phi)$ is a solution of (1.4). In the sequel, we shall therefore confine the discussion to (strong) solutions of (1.8) for which $\rho > 0$ for all $x > 0$. We note, however, that for a solution (ρ, ϕ) of (1.8), if for some $x_0 >$ we have $\rho(x_0) > 0$, then ρ must be positive for all $x > x_0$. This is because by (1.8c,d) we have that $\sigma\phi' + J > 0$ for all $x > x_0$, and hence the positivity of ρ in (x_0, ∞) follows from (1.8a).

With different domains and boundary conditions (1.3) has attracted significant interest among Physicists [12, 13, 5, 21] and Mathematicians alike [17, 15, 18, 18, 3]. A different simplification of (1.1) was derived by Du & Gray [6], for the same limit $(\kappa \to \infty)$ but with J and σ of $\mathcal{O}(\kappa^2)$. The focus of this work is mainly on solutions of (1.8). Any such solution represents a transition layer between the normal state at

$x = 0$ and the purely superconducting state, or (1.5), as $x \to \infty$. It is reasonable to expect that for large domains of higher dimensions, the behaviour near an interface with the normal state would be similar in the long-time limit of (1.3).

The first question we raise in this work is: for which values of J and σ will solutions of (1.8) exist? For an infinite wire residing in $\mathbb{R}$, with boundary conditions similar to (1.8d,f) satisfied as $x \to \infty$, solutions exist whenever $J \in [0, J_c]$ for all positive σ. However, besides the obvious point that (1.5) does not include the necessary effect of the interface with the normal state, the fact that J_c is independent of σ seems odd. First, it lies in stark contrast with the linear dependence on σ of the critical current at which the normal state looses its stability[12][3]. Furthermore, it is very unlikely that metals of very small normal conductivity would be able to carry the same amount of superconducting current as a metal with much higher conductivity. Such a current is expected to generate a large potential drop between $x = 0$ and $x \sim \mathcal{O}(1)$. This large potential drop, as we demonstrate here, destroys superconductivity altogether.

Once the existence of solutions of (1.8) for some J and σ is established, the question of their stability naturally arises. Due to technical difficulties we resort here to the simpler problem of studying the stability of (1.5). The stability of (1.5) has been briefly addressed in [12], where without much explanation the authors arrive at the conclusion that it is stable for $\rho^2 > 2/3$ and unstable for $\rho^2 < 2/3$. It is not clear in which setting this statement is true (if it is true at all).

We take here an alternative approach, prescribing zero potential drop, but in return allow for a time-dependent current. We first obtain global stability results for a problem similar to (1.3), but instead of (1.3d) require $\phi = 0$. In addition, we prescribe $\psi = f$ in (1.3c) with $f \neq 0$. The values of ψ (or f) on $\partial\Omega_c$ should match those of the fully superconducting state. We then apply the results for general domains to long (though finite) wires. We prove that if there is a global attractor, it is not (1.5) but another equilibrium solution of (1.8) corresponding to a vanishingly small current in the long wire limit.

In the next section we deal with the small σ limit. We show that the maximal current for which solutions of (1.8) exist is of $\mathcal{O}(\sigma^{1/4})$. We also evaluate formally the leading order term of the critical current. In § 3 we consider the large σ limit. We prove existence of solutions for (1.4), when σ is sufficiently large, for any fixed J. In § 4 we prove existence of solutions for (1.4) for sufficiently small J, and in § 5 we consider the stability of (1.5) in a bounded domain with zero potential drop between the conducting surfaces. Finally, in the last section we summarize the main findings of this work, and offer a few additional insights into the problem.

2 Small σ

In the following we demonstrate that as $\sigma \to 0$ the maximal current for which solutions for (1.8) exist diminishes together with σ. To this end we first prove the following uniform bounds on ρ and ρ'

Lemma 2.1. *Any solution (ρ, ϕ) of (1.8) must satisfy*

$$\rho(x) < 1 \quad \forall x \in \mathbb{R}_+ \tag{2.1a}$$

$$|\rho'(x)| < \sqrt{\frac{2}{3}} \quad \forall x \in \mathbb{R}_+ \,. \tag{2.1b}$$

Proof. By (1.8a), ρ cannot have a maximum greater or equal than 1. The proof of (2.1a) then follows from (1.8c,d). To prove (2.1b) we set

$$H = \frac{1}{2}\Big[|\rho'|^2 + \frac{(\sigma\phi' + J)^2}{\rho^2} + \rho^2 - \frac{1}{2}\rho^4\Big] \,. \tag{2.2}$$

It is easy to show that

$$H' = (\sigma\phi' + J)\phi \,. \tag{2.3}$$

By (1.8b,d) we have that $\phi > 0$ and $\phi' > \phi'(0)$, and hence $H' > 0$ for all $x \in \mathbb{R}_+$.

In view of (1.6) and (1.8b,c,d) we have that

$$\lim_{x\to\infty} H(x) = \frac{J^2}{\rho_\infty^2} + \rho_\infty^2 - \frac{1}{2}\rho_\infty^4 = 2\rho_\infty^2 - \frac{3}{2}\rho_\infty^4 \,. \tag{2.4}$$

Since H is increasing we obtain

$$|\rho'|^2 + \frac{(\sigma\phi' + J)^2}{\rho^2} \le 2\rho_\infty^2 - \frac{3}{2}\rho_\infty^4 - \rho^2 + \frac{1}{2}\rho^4 \,.$$

With the aid of (2.1a) the lemma easily follows. □

We now prove the main result of this section

Proposition 2.1. *Let*

$$S(\sigma) = \{J \in \mathbb{R}_+ \,|\, \exists(\rho,\phi) \in C^2(\mathbb{R}_+) \times C^2(\mathbb{R}_+) \text{ satisfying } (1.8)\} \,.$$

Then, $\exists C > 0$ such that

$$\sup S(\sigma) \le C\sigma^{1/4} \,. \tag{2.5}$$

Proof. Let $J \in S(\sigma)$ and let (ρ, ϕ) denote the solution of (1.8) corresponding to J and σ. Let $\delta > 0$. Set

$$x_\delta = \inf\{t \in \mathbb{R}_+ \mid \inf_{x \in (t,\infty)} \rho(x) \geq \delta\}\,.$$

Clearly, x_δ is finite for every $\delta < \rho_\infty$ in view of (1.8c). Furthermore, in view of (1.8b,d) we have

$$-\frac{J}{\sigma} < \phi'(x_\delta) < 0\,.$$

Consider then the problem

$$\begin{cases} -\phi_m'' + \delta^2 \phi_m = 0 & x > x_\delta \\ \phi_m'(x_\delta) = -\frac{J}{\sigma} & \phi \xrightarrow[x \to 0]{} 0 \end{cases}. \tag{2.6}$$

From the maximum principle and (1.8b,d) it follows that $\phi_m' < \phi' < 0$. Hence,

$$0 < J^2 - (\sigma\phi' + J)^2 < J^2 - (\sigma\phi_m' + J)^2\,.$$

We can now solve (2.6) to obtain

$$0 < J^2 - (\sigma\phi' + J)^2 \leq J^2\big[1 - \big(1 - e^{-\delta\sigma^{-1/2}(x - x_\delta)^2}\big)^2\big]\,.$$

Substituting in (1.8a) we then obtain

$$0 < -\rho'' + \frac{J^2}{\rho^3} - \rho(1 - \rho^2) < \frac{2J^2}{\rho^3} e^{-\delta\sigma^{-1/2}(x - x_\delta)^2}\,. \tag{2.7}$$

Multiplying (2.7) by ρ' and integrating over (x_δ, ∞) we obtain

$$-\frac{1}{2}\Big[|\rho'|^2 + \frac{J^2}{\rho^2} + \rho^2 - \frac{1}{2}\rho^4\Big]_{x_\delta}^\infty \leq 2J^2 \int_{x_\delta}^\infty \frac{|\rho'|}{\rho^3} e^{-\delta\sigma^{-1/2}(x - x_\delta)^2} dx\,.$$

By (2.1) and the fact that $\rho(x_\delta) = \delta < \rho_\infty < 1$ we obtain that

$$\frac{J^2}{\delta^2} \leq \frac{J^2}{\rho_\infty^2} + \rho_\infty^2 - \frac{1}{2}\rho_\infty^4 + \frac{2\sqrt{2}\sigma^{1/2}}{\delta^4} J^2\,.$$

Hence, we obtain that

$$J^2 \leq \Big(2\rho_\infty^2 - \frac{3}{2}\rho_\infty^4\Big)\delta^2 + \frac{2\sqrt{2}\sigma^{1/2}}{\delta^2} J^2\,. \tag{2.8}$$

We now choose

$$\delta^2 = \frac{1}{2}\min(\rho_\infty^2, 8\sqrt{2}\sigma^{1/2})\,.$$

If $\rho_\infty^2 < 8\sqrt{2}\sigma^{1/2}$ the proposition is proved in view of (1.6). Otherwise, we have by (2.8) that

$$J^2 \le \sqrt{2}\Big(2\rho_\infty^2 - \frac{3}{2}\rho_\infty^4\Big)\sigma^{1/2}\,,$$

from which (2.5) easily follows. □

We conclude this section by formally obtaining the leading order term of the critical current in the limit $\sigma \to 0$ [1]. Formally, we assume the existence of a boundary layer near $x = 0$ where ϕ converges exponentially fast to 0. Unlike ϕ, there is no reason to expect that ρ would vary along an accelerated length scale, especially in view of (2.1b).

Thus, we assume that $\rho \sim \alpha x$ within the boundary layer where $\alpha = \rho'(0)$. The equation for ϕ (1.8b) assumes then the form

$$\begin{cases} -\sigma\phi'' + \alpha^2 x^2\phi = 0 & \text{in } \mathbb{R}_+ \\ \phi'(0) = -\frac{J}{\sigma} & \phi \xrightarrow[x\to\infty]{} 0 \end{cases}. \tag{2.9}$$

We then introduce the scaled coordinate $\xi = \alpha^{1/2}\sigma^{-1/4}x$ and function

$$\Phi(\xi) = \frac{\alpha^{1/2}\sigma^{3/4}}{J}\phi(x)\,.$$

The rescaled form of (2.9) is then

$$\begin{cases} -\Phi'' + \xi^2\Phi = 0 & \text{in } \mathbb{R}_+ \\ \Phi'(0) = -1 & \Phi \xrightarrow[\xi\to\infty]{} 0 \end{cases}. \tag{2.10}$$

One can easily express Φ in terms of the parabolic cylinder function $U(0,\xi)$ (cf. [2] for the precise definition). Hence, we shall treat Φ as known in the sequel.

To obtain an approximate solution of (1.8) it is necessary to obtain α which in turn, depends on J. With the aid of (2.2), (2.3), and (2.4) we obtain that

$$\int_0^\infty (\sigma\phi' + J)\phi\,dx = \frac{J^2}{\alpha\sigma^{1/2}}\int_0^\infty (\Phi' + 1)\Phi\,d\xi = 2\rho_\infty^2 - \frac{3}{2}\rho_\infty^4 - \alpha^2\,.$$

Define the following constant, which can be evaluated in principle,

$$A = \int_0^\infty (\Phi' + 1)\Phi\,d\xi\,,$$

to obtain

$$\alpha^2 + \frac{AJ^2}{\alpha\sigma^{1/2}} = 2\rho_\infty^2 - \frac{3}{2}\rho_\infty^4 \sim \frac{1}{2}\,. \tag{2.11}$$

Where the last asymptotic approximation relies on the fact that $J \sim \mathcal{O}(\sigma^{1/4})$ in view of (2.5), considering the solution of (1.6) for which $\rho_\infty^2 > 2/3$ [(2.11) can be applied to obtain the asymptotic behaviour of this critical current for the other solution of (1.6) as well]. The left hand side of (2.11) is minimal for

$$\alpha = \left[\frac{AJ^2}{2\sigma^{1/2}}\right]^{1/3}.$$

It follows that as $\sigma \to 0$

$$\sup S(\sigma) \sim \sqrt{\frac{2}{A}}\left(\frac{1}{6}\right)^{3/4}\sigma^{1/4}.$$

Of interest is also the potential drop for this maximal current which can be easily calculated to obtain

$$\phi(0) = [3A\sigma]^{-1/2}\Phi(0).$$

3 Large σ

In the following we prove that for every $J \in (0, J_c)$, for sufficiently large σ, there exists a solution for (1.8) . This result lies in stark contrast with the case of small σ presented in the previous section, where the maximal value of J for which solutions of (1.8) can exist diminishes as σ tends to zero. For every $J \in (0, J_c)$ there are two values of ρ_∞ for which (1.6) is satisfied. We prove in this section existence of solutions for solutions of (1.6) satisfying $\rho_\infty > \sqrt{2/3}$.

Existence of solutions is proved in the following by applying an iterative scheme. To prove existence of solutions for each step in the scheme we use shooting arguments. As a first step consider then the following modified version of (1.8b,e,f)

$$\begin{cases} -\sigma v'' + f^2 v = 0 & x > 0 \\ v'(0) = -\frac{J}{\sigma} & v \xrightarrow[x\to\infty]{} 0 \end{cases}, \tag{3.1}$$

where f must satisfy

$$f(0) = 0\ ;\ 0 < f'(0) \leq \frac{1}{\sqrt{2}}\ ;\ \|f'\|_{L^\infty(\mathbb{R}_+)} \leq \sqrt{\frac{2}{3}} \tag{3.2a}$$

$$f \geq \rho_\infty \quad \forall x \geq x_0(\rho_\infty) \tag{3.2b}$$

$$0 < f < 1 \quad \forall x \in \mathbb{R}_+ . \tag{3.2c}$$

It is easy to show that (3.1) possesses a unique solution. The following lemma lists a few of the properties of this solution

Lemma 3.1. *Let f satisfy* (3.2). *Then the solution of* (3.1) *satisfies*

$$-\frac{J}{\sigma} < v' < 0 \quad \forall x > 0 \tag{3.3a}$$

$$0 < v(0) \leq \frac{J}{\rho_\infty \sigma^{1/2}} + \frac{J}{\sigma} x_0 \tag{3.3b}$$

$$|\sigma v' + J| \leq C(\rho_\infty, \sigma) x^3 \quad \textit{for sufficiently small } x > 0\,. \tag{3.3c}$$

Proof. (3.3a) is an immediate consequence of the maximum principle. To prove (3.3b) we construct an upper bound for v

$$\begin{cases} -\sigma \bar{v}'' + \rho_\infty^2 \bar{v} = 0 & x > x_0 \\ \bar{v}'(x_0) = -\frac{J}{\sigma} & \bar{v} \xrightarrow[x\to\infty]{} 0 \end{cases}.$$

The solution of the above problem is given by

$$\bar{v}(x) = \frac{J}{\rho_\infty \sigma^{1/2}} \exp\{-\sigma^{-1/2} \rho_\infty (x - x_0)\}\,, \tag{3.4}$$

and hence by (3.3a), (3.2b) and the maximum principle we have that

$$v(x_0) \leq \bar{v}(x_0) = \frac{J}{\rho_\infty \sigma^{1/2}}\,.$$

To complete the proof of (3.3b) we use (3.3a) once again.

The proof of (3.3c) follows from the bound

$$|(\sigma v' + J)'| = |\sigma v''| = |f^2 v| \leq 2v(0)|f'(0)|^2 x^2\,,$$

which is valid in some sufficiently small semi-neighborhood of $x = 0$. Integrating the above between 0 and x yields (3.3c). Note that by (3.3b) the upper bound of $v(0)$ is independent of the specific form of f and depends only on ρ_∞ and σ. □

Consider now the following problem

$$\begin{cases} -u'' + \frac{(\sigma v' + J)^2}{u^3} - u(1 - u^2) = 0 & \text{in } \mathbb{R}_+ \\ u(0) = 0 & u \xrightarrow[x\to\infty]{} \rho_\infty \end{cases}, \tag{3.5}$$

where v is a solution of (3.1). The next step on our way to establish existence of solutions of (1.8) for sufficiently large σ is to prove existence of solutions for (3.5) in the same range of σ values.

Proposition 3.1. *For sufficiently large σ, there exists a solution of* (3.5) *satisfying* (3.2) *(with f replaced by u).*

We prove proposition 3.1 using shooting arguments. To this end we consider solutions of the following initial value problem

$$\begin{cases} -u_\alpha'' + \frac{(\sigma v'+J)^2}{u_\alpha^3} - u_\alpha(1-u_\alpha^2) = 0 & \text{for } x>0 \\ u_\alpha(0) = 0 \\ u_\alpha'(0) = \alpha \end{cases} . \tag{3.6}$$

The above problem possesses a unique solution in some positive semi-neighborhood of $x = 0$ in view of (3.3c). We attempt now to prove that there exists $\alpha \in \mathbb{R}_+$ such that $u_\alpha \to \rho_\infty$ as $x \to \infty$.

We start by demonstrating that we can confine α to an appropriate subset of $\mathbb{R}_+$.

Lemma 3.2. *Let*

$$g(s) = s^2 - \frac{1}{2}s^4 .$$

and

$$\alpha_0 = g(s_0) \text{ where } s_0 = \frac{1+\rho_\infty}{2} . \tag{3.7}$$

Then, for all $\alpha > \alpha_0$ *there exists* $x_1(\alpha)$ *for which the solution of* (3.6) *satisfies* $u_\alpha(x_1) = \rho_\infty$ *and* $u_\alpha'(x_1) > 0$*. Furthermore, let*

$$x_0 = \frac{\rho_\infty}{\sqrt{g(s_0) - g(\rho_\infty)}} . \tag{3.8}$$

Then, $x_1(\alpha) \le x_0$ *for all* $\alpha \ge \alpha_0$*.*

Proof. Let u_α be the solution of (3.6) for some $\alpha \ge \alpha_0$. By (3.6) we have that

$$u_\alpha'' + u_\alpha(1-u_\alpha^2) \ge 0 .$$

Hence, as along as $u_\alpha' > 0$ we have that

$$\left(|u_\alpha'|^2 + g(u_\alpha)\right)' \ge 0 .$$

Consequently,

$$|u_\alpha'|^2 + g(u_\alpha) \ge \alpha^2 > g(s_0) \tag{3.9}$$

as long as $u_\alpha' > 0$. In view of the above and since $g(s)$ is increasing on $(0,1)$, it follows that $u_\alpha' > 0$ in some right semi-neighborhood of $x = 0$ where $u_\alpha < s_0$. In particular, this is true when $u_\alpha \in (0, \rho_\infty]$. Furthermore, by (3.9) we have that

$$u_\alpha \in (0,\rho_\infty] \Rightarrow u_\alpha' \ge \sqrt{g(s_0) - g(\rho_\infty)} , \tag{3.10}$$

from which the lemma easily follows. □

We now define two subsets of (α_0, ∞). Let

$$S_1 = \left\{\alpha \in (\alpha_0, \infty) \,\middle|\, \exists x_2 > x_1 : \, u_\alpha(x_2) = 1, \, u'_\alpha(x_2) > 0, \, u_\alpha\big((x_1, x_2)\big) = (\rho_\infty, 1)\right\}$$

and

$$S_2 = \left\{\alpha \in (\alpha_0, \infty) \,\middle|\, \exists x_2 > x_1 : \, u_\alpha(x_2) = \rho_\infty, \, u'_\alpha(x_2) < 0, \, u_\alpha\big((x_1, x_2)\big) \subset (\rho_\infty, 1)\right\}.$$

In the above definitions x_1 denotes, as before, the first point where $u_\alpha(x_1) = \rho_\infty$.

Next, we derive a few properties of S_1 and S_2.

Lemma 3.3. *The sets S_1 and S_2 are both open and satisfy $S_1 \cap S_2 = \emptyset$.*

Proof. Openness of S_1 and S_2 follows immediately from continuity of solutions of ordinary differential equations with respect to the initial conditions. The proof that S_1 and S_2 are disjoint follows from the fact that u_α cannot have a maximum where $u_\alpha \geq 1$. □

Lemma 3.4. *$S_1 \neq \emptyset$.*

Proof. We show that $(1/\sqrt{2}, \infty) \subset S_1$. Let $\alpha > 1/\sqrt{2}$. Then, in every right semi-neighborhood of $x = 0$ where u'_α is positive we have by (3.9) that

$$u'_\alpha \geq \sqrt{\alpha^2 - \frac{1}{2}}\,.$$

Hence, there exists

$$x_2 < \left[\alpha^2 - \frac{1}{2}\right]^{-1/2}$$

at which $u_\alpha(x_2) = 1$. Since u_α is increasing on (x_1, x_2) we obtain that $\alpha \in S_1$. □

Note that lemmas 3.2, 3.3, and 3.4, are correct for any value of σ. The next lemma is the only place in the proof of proposition 3.1 where we need to assume a sufficiently large value of σ.

Lemma 3.5. *For every fixed $\alpha \in (\alpha_0, 1/\sqrt{2})$ there exists $\sigma_0(\alpha)$ such that for all $\sigma > \sigma_0$ we have $\alpha \in S_2$ (hence $S_2 \neq \emptyset$).*

Proof. We first obtain a lower bound for u_α/f. Clearly by (3.2) we have that

$$f \leq \sqrt{\frac{2}{3}}x\,.$$

Let x_1 denote the leftmost point where $u_\alpha = \rho_\infty$. By the above and (3.10) we have that

$$x \leq x_1 \Rightarrow u_\alpha \geq f\sqrt{\frac{3}{2}[g(s_0) - g(\rho_\infty)]}\,. \tag{3.11}$$

For $x > x_1$, as long as $u_\alpha > \rho_\infty$ we have that

$$u_\alpha \geq \rho_\infty f \,. \tag{3.12}$$

Note that if there exists $x_2 > x_1$ such that $u_\alpha(x_2) = \rho_\infty$ then the lemma is proved if we show that $u'_\alpha(x_2) < 0$. In such a case, without any loss of generality we assume that x_2 is the leftmost point where $u_\alpha = \rho_\infty$. Clearly, we must then have $u'_\alpha(x_2) \leq 0$. However, if $u'_\alpha(x_2) = 0$ then by (3.6), (1.6), and (3.3a) we have that $u''_\alpha(x_2) < 0$, or that x_2 is a local maximum - a contradiction.

Suppose then, for a contradiction, that there exists $\{\sigma_k\}_{k=1}^\infty$ such that $\sigma_k \uparrow \infty$ $u_\alpha^k(x) = u_\alpha(x,\sigma_k) > \rho_\infty$ for every $x > x_1^k$ (where $u_\alpha^k(x_1^k) = \rho_\infty$ and $x < x_1^k \Rightarrow u_\alpha^k < \rho_\infty$) in the maximal right interval of existence. Let then x_2^k denote the point where $u_\alpha^k(x_2^k) = 1$. Otherwise, if $\rho_\infty < u_\alpha^k < 1$ for all $x_1^k < x$ we set $x_2^k = \infty$. Combining (3.11) and (3.12) we thus have

$$x < x_2^k \Rightarrow u_\alpha^k \geq C_0 f \quad \text{where } C_0 = \min(\sqrt{2[g(s_0) - g(\rho_\infty)]}, \rho_\infty) \,. \tag{3.13}$$

We next obtain an upper bound for $(\sigma_k v'_k + J)|u_\alpha^k|^{-2}$, where $v_k(x) = v(x,\sigma_k)$. By (3.1) and (3.13) we have

$$x < x_2^k \Rightarrow \sigma_k v'_k + J = \int_0^x f^2 v_k \, dt \leq \frac{1}{C_0^2} \int_0^x (u_\alpha^k)^2 v_k(t) \, dt \,.$$

By (3.10) u_α^k is increasing on $(0, x_1^k]$. Consequently, we have that

$$x \leq x_1^k \Rightarrow \frac{\sigma_k v'_k + J}{(u_\alpha^k)^2} \leq \frac{1}{C_0^2} \int_0^x v_k(t) \, dt \,.$$

For $x \in (x_1^k, x_2^k)$ we have $u_\alpha^k(t) \leq u_\alpha^k(x)/\rho_\infty$ for all $t \in (0,x)$, which, combined with the above inequality, yields

$$x < x_2^k \Rightarrow \frac{\sigma_k v'_k + J}{(u_\alpha^k)^2} \leq \frac{1}{C_0^2 \rho_\infty^2} \int_0^x v_k(t) \, dt \,.$$

We now employ (3.3b) to obtain that

$$x < x_2^k \Rightarrow \frac{\sigma_k v'_k + J}{(u_\alpha^k)^2} \leq \frac{C}{\sigma_k^{1/2}} x \,, \tag{3.14}$$

where C is independent of σ_k and x.

Let U denote the solution of

$$\begin{cases} U'' + U(1 - U^2) = 0 & \text{in } \mathbb{R}_+ \\ U(0) = 0 & U'(0) = \alpha \end{cases} \,. \tag{3.15}$$

It is easy to show that for all $\alpha \in (\alpha_0, 1/\sqrt{2})$, U is periodic and satisfies $U = \rho_\infty$ infinitely many times in $\mathbb{R}_+$. Denote the leftmost two of them by $\tilde{x}_1$ and $\tilde{x}_2$ (naturally, both of them are independent of σ). By (3.14) we have that

$$x < x_2^k \Rightarrow |(u_\alpha^k)'' + u_\alpha^k(1 - |u_\alpha^k|^2)| \leq \frac{C}{\sigma_k^{1/2}} x \,.$$

Set $w = u_\alpha^k - U$. We obtain

$$\begin{cases} x < x_2^k \Rightarrow |w'' + w(1 - |u_\alpha^k|^2 - U u_\alpha^k - U^2)| \leq \frac{C}{\sigma_k^{1/2}} x \\ w(0) = w'(0) = 0 \end{cases} . \tag{3.16}$$

We argue from here by bootstraping. For some fixed $\alpha \in [\alpha_0, 1/\sqrt{2})$ let $a_k = a(\sigma_k)$ be the leftmost point in $\mathbb{R}_+$ where $|w| = \sigma_k^{-1/4}$. Clearly $a_k > 0$ since $w(0) = 0$. Note that by (3.15) we have that

$$|U'|^2 - \frac{1}{2}(1 - U^2)^2 = \alpha^2 - \frac{1}{2} \,,$$

and hence

$$1 - U^2 \geq (1 - 2\alpha^2)^{1/2} \,.$$

Consequently, since α is fixed it follows that for sufficiently large k we have $a_k \leq x_2^k$.

We can now use (3.16) and lemma 4.1 in [11] to obtain

$$x \leq a_k \Rightarrow |w(x)| \leq e^{2x} \frac{C}{\sigma_k^{1/2}} x^2 \,.$$

For $x = a_k$ we obtain, since $w(a_k) = \sigma_k^{-1/4}$, that

$$a_k^2 e^{2a_k} \geq C \sigma_k^{1/4} \,,$$

which clearly implies that $a_k \to \infty$. For sufficiently large k we thus obtain that $\tilde{x}_2 + 1 \leq a_k \leq x_2^k$. This leads, however, to a contradiction since it is easy to show that for sufficiently large k, u_α^k must obtain the value ρ_∞ at least twice in $[0, \tilde{x}_2 + 1/2]$, a fact which follows from the inequality $|u_\alpha^k - U| \leq \sigma_k^{-1/4}$.

□

Proof of proposition 1. Since S_1 and S_2 are two open disjoint sets, we conclude that $S_1 \cup S_2 \neq (\alpha_0, \infty)$. Consequently, there exists $\tilde{\alpha} \in (\alpha_0, \infty)$ such that the solution of (3.6) must satisfy at least one of the following requirement

$$\exists x_2 > x_1 : \; u_\alpha(x_2) = 1 \,, \; u_\alpha'(x_2) = 0 \,, \; u_\alpha([x_1, x_2]) = [\rho_\infty, 1] \tag{3.17a}$$

$$\exists x_2 > x_1 : \; u_\alpha(x_2) = \rho_\infty \,, \; u_\alpha'(x_2) = 0 \,, \; u_\alpha\big((x_1, x_2)\big) \subset (\rho_\infty, 1) \tag{3.17b}$$

$$\rho_\infty < u_\alpha < 1 \quad \forall x_1 < x \tag{3.17c}$$

It is easy to show that neither (3.17a) nor (3.17b) can exist: a solution of (3.6) satisfying $u_\alpha(x_2) = 1$, $u'(x_2) = 0$ must have a local minimum at x_2 in view of (3.6a), and hence (3.17a) is impossible. In a similar manner, by (3.6a), (3.3a), and (1.6), if $u(x_2) = \rho_\infty$ and $u'(x_2) = 0$, then x_2 must be a local maximum point of u. Therefore, no solution of (3.6) can satisfy (3.17b) as well.

It follows then, that a solution of (3.6) satisfying (3.17c) must exist. Denote it by $\tilde{u}$. To complete the proof of the proposition we need only show that $\tilde{u} \to \rho_\infty$ as $x \to \infty$. Denote by $u_v : \mathbb{R}_+ \to (\rho_\infty, 1)$ the solution of

$$u_v^4 - u_v^6 = (\sigma v' + J)^2 \,.$$

Since $(\sigma v' + J)$ is monotone increasing on $\mathbb{R}_+$ and tends to J as $x \to \infty$, it follows that $u_v \downarrow \rho_\infty$ as $x \to \infty$. Suppose then, for a contradiction that $\tilde{u}$ has a local minimum at a point ξ where $\tilde{u}(\xi) \geq u_v(\xi)$. Since u_v is strictly decreasing, and since $\tilde{u}$ cannot have a maximum where $\tilde{u} > u_v$, $\tilde{u}$ must be monotone increasing for all $x > \xi$. Furthermore, we must also have by (3.6a) that $\tilde{u}'' > 0$ for all $x > \xi$, which clearly contradicts (3.17c).

By (3.6a) $\tilde{u}$ cannot have a minimum where $\rho_\infty < \tilde{u} < u_v$. Consequently, we must have either that u is monotone increasing in (x_1, ∞), or else that it must have a single maximum point $\xi > x_1$ and then decrease for all $x > \xi$. It is easy to rule out the former: if $\tilde{u}$ is increasing for $x > x_1$, then, since u_v is decreasing, there must exist some $x_3 > x_1$ such that $\tilde{u} > u_v$ for all $x > x_3$. Then $\tilde{u}$ must be convex in (x_3, ∞), contradicting (3.17c).

The above discussion proves by elimination that $\tilde{u}$ is decreasing for sufficiently large x and hence must converge to a value greater or equal than ρ_∞. Thus, since $v' \to 0$ as $x \to \infty$, and since u'' must tend to zero as well in that limit, we must have that

$$\lim_{x\to\infty} \tilde{u} = \rho_\infty \,.$$

□

Before proving existence of solutions for (1.8) for sufficiently large σ we need to establish the following bound

Lemma 3.6. *Any solution of* (3.5) *satisfies* $\|u'\|_{L^\infty(\mathbb{R}_+)} \leq \sqrt{2/3}$.

Proof. The proof is similar to that of (2.1b). Set

$$H_v = \frac{1}{2}\Big[|u'|^2 + \frac{(\sigma v' + J)^2}{u^2} + u^2 - \frac{1}{2}u^4\Big] \,.$$

It follows then that

$$H_v' = \frac{f^2}{u^2} v(\sigma v' + J) > 0 \,.$$

We proceed from here in exactly the same manner as in the proof of (2.1b). □

We prove existence of solutions for (1.8) by considering the limit of an iterative scheme. We thus define $\{u_k, \tilde{v}_k\}_{k=1}^{\infty}$ as the solution of

$$u_0 = \begin{cases} \frac{1}{2}\rho_\infty x & 0 \le x \le 2 \\ \rho_\infty & x > 2 \end{cases}, \tag{3.18a}$$

and

$$\begin{cases} -u_k'' + \frac{(\sigma\tilde{v}_k' + J)^2}{u_k^3} - u(1-u^2) = 0 & \text{in } \mathbb{R}_+ \\ -\sigma\tilde{v}_k'' + u_{k-1}^2\tilde{v}_k = 0 & \text{in } \mathbb{R}_+ \\ u_k(0) = 0 & u_k \xrightarrow[x\to\infty]{} \rho_\infty \\ \tilde{v}_k'(0) = -\frac{J}{\sigma} & \tilde{v}_k \xrightarrow[x\to\infty]{} 0 \end{cases}, \tag{3.18b}$$

for all $k \ge 1$.

Before we can consider the limit as $k \to \infty$ we need to show first that the definition of $\{u_k, \tilde{v}_k\}_{k=1}^{\infty}$ is proper.

Lemma 3.7. *For sufficiently large σ there exists a solution of* (3.18b) *for all $k \ge 1$.*

Proof. Since u_0 satisfies (3.2), we obtain the existence of u_1 from proposition 3.1. We then need to show that u_1 satisfies (3.2) as well. Recall that by the shooting arguments we have applied in order to establish that existence we have

1. $u_1'(0) \in (\alpha_0, 1/\sqrt{2}]$ by lemmas 3.2 and 3.4.
2. By lemma 3.6 we have $\|u_1'\|_{L^\infty(\mathbb{R}_+)} \le \sqrt{\frac{2}{3}}$.
3. $u_1 < 1$ by the maximum principle.
4. $0 < u_1$ since by lemma 3.2 $u_1' > 0$ for all $x < x_1$, and $u > \rho_\infty$ for all $x > x_1$ by (3.17c).
5. $x_1 < x_0$ where x_0 is given by (3.8).

Invoking inductive arguments we assume that u_{k-1} exists and satisfy (3.2). Existence of $(u_k, \tilde{v}_k)$ then follows from proposition 3.1. The fact that u_k must satisfy (3.2) is proved in exactly the same manner as in the case $k = 1$. $\square$

We can now prove existence of solution for (1.8).

Theorem 1. *For sufficiently large σ there exists a solution of* (1.8) *with $\rho_\infty^2 \ge 2/3$.*

Proof. We first prove that $\{u_k, \tilde{v}_k\}_{k=1}^{\infty}$ is compact in $[L^\infty(0,a)]^2$ for every positive a. It follows from the maximum principle, lemma 3.6, and (3.3) that $\{u_k, \tilde{v}_k\}_{k=1}^{\infty}$ is bounded in $[C^1(0,a)]^2$ for every $a > 0$. Compactness then follows from the Arzela-Ascoli Theorem.

By (3.18b) we obtain that a a subsequence of $\{u_k, \tilde{v}_k\}_{k=1}^{\infty}$ which is a Cauchy sequence in $[L^\infty(0,a)]^2$ must be a Cauchy sequence in $[C^2(0,a)]^2$ as well. Denote then a partial limit of $\{u_k, \tilde{v}_k\}_{k=1}^{\infty}$ by (ρ, ϕ). In view of the afore-mentioned $[C^2(0,a)]^2$ convergence, (ρ, ϕ) must satisfy (1.8a), (1.8b), (1.8c) and (1.8e) .

Clearly, the $\tilde{v}_k's$ are all positive and bounded from above for $x > x_0$ by $\bar{v}$ which is given by (3.4). Consequently, $\phi \leq \bar{v}$ for all $x > x_0$ and since ϕ is positive it must satisfy (1.8f). By (3.17c), $u_k \in (\rho_\infty, 1)$ for all $x > x_0$. Hence, $\rho \in [\rho_\infty, 1]$ for all $x > x_0$. We proceed in a similar manner as in the proof of proposition 3.1. By the same arguments used there, ρ cannot have a local minimum for all $x > x_0$ and hence must converge to ρ_∞ as $x \to \infty$. The theorem is proved. $\square$

We conclude this section by presenting a formal asymptotic approximation of the leading order behaviour of (ρ, ϕ) in the limit $\sigma \to \infty$. We assume an $\mathcal{O}(1)$ boundary layer, and an outer solution which varies along an $\mathcal{O}(\sigma^{-1/2})$ length scale. Consider first the outer solution. Introducing the scaling

$$\eta = x/\sigma^{1/2} \quad ; \quad \Phi = \sigma^{1/2}\phi$$

into (1.8a,b) we obtain

$$\begin{cases} -\sigma^{-1}\frac{d^2\rho}{d\eta^2} + \frac{(d\Phi/d\eta + J)^2}{\rho^3} - \rho(1-\rho^2) = 0 \\ -\frac{d^2\Phi}{d\eta^2} + \rho^2\Phi = 0 \end{cases} .$$

Let then

$$\rho = \rho_0 + \mathcal{O}(\sigma^{-1}) \quad ; \quad \Phi = \Phi_0 + \mathcal{O}(\sigma^{-1}) .$$

The equations for the leading order (ρ_0, ϕ_0) assumes the form

$$\begin{cases} \rho_0^4(1-\rho_0^2) = (d\Phi_0/d\eta + J)^2 \\ -\frac{d^2\Phi_0}{d\eta^2} + \rho_0^2\Phi_0 = 0 \end{cases} .$$

The boundary conditions satisfied by ϕ_0 are

$$\phi_0'(0) = J \quad ; \quad \phi_0 \xrightarrow[\eta\to\infty]{} 0 .$$

It follows that either $\rho_0(0) = 0$ or $\rho_0(0) = 1$. We avoid the former case, as the unique solution is $(\rho_0, \phi_0) = (0, J\eta + A)$, where A is an undetermined constant. In the case where $\rho_0(0) = 1$, however, one can easily prove the existence of (ρ_0, ϕ_0) which must clearly satisfy

$$\rho_0 \xrightarrow[\eta\to\infty]{} \rho_\infty .$$

Denote the inner solution by (ρ_i, ϕ_i). It follows from (1.8b) that $\phi_i' \sim J/\sigma$. consequently, to leading order we have

$$\rho_i'' + \rho_i(1 - \rho_i^2) = 0\,, \quad \rho_i(0) = 0$$

To match the inner solution with the outer one, we obtain that

$$\rho_i \xrightarrow[x\to\infty]{} 1\,.$$

From here it is easy to show that ρ_i must satisfy

$$\begin{cases} \rho_i' = \frac{1}{\sqrt{2}}(1 - \rho_i^2) \\ \rho_i(0) = 0 \end{cases}.$$

A uniformly valid approximation can thus be written in the form

$$\rho \sim \rho_0 + \rho_i - 1 \quad \phi(x) \sim \sigma^{-1/2}\Phi_0(\eta)\,.$$

4 Small J -existence

Another case where existence of solutions for (1.8) can be proved for all values of σ is the limit $J \to 0$. Consider first the case $J = 0$. From the maximum principle it follows that $\phi \equiv 0$ in this case. It is easy to show that the solution in this case, which we denote by ρ_0 must satisfy

$$\begin{cases} \rho_0' = \frac{1}{\sqrt{2}}(1 - \rho_0^2) \\ \rho_0(0) = 0 \end{cases}. \tag{4.1}$$

Set now $\phi_0(x) \in H^2(\mathbb{R}_+)$ to be the solution of

$$\begin{cases} -\sigma\phi_0'' + \rho_0^2\phi_0 = 0 \\ \phi_0'(0) = -\frac{J}{\sigma} \end{cases}. \tag{4.2}$$

Clearly, for $J = 0$ we have $\phi_0 \equiv 0$. We then set in (1.8)

$$\rho = \rho_0 + u \quad ; \quad \phi = \phi_0 + v\,,$$

where $(u, v) \in \mathcal{H}$ and

$$\mathcal{H} = \{(u, v) \in H^2(\mathbb{R}_+) \times H^2(\mathbb{R}_+) \,|\, u(0) = 0\,;\, v'(0) = 0\}\,.$$

Finally, let $F : \mathcal{H} \times [0, J_c] \to L^2(\mathbb{R}_+)$ be given by

$$F(u, v, J) = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{Bmatrix} -u'' + \frac{|\sigma(v' + \phi_0') + J|^2}{(\rho_0 + u)^3} - u(1 - 3\rho_0^2 - 3\rho_0 u - u^2) \\ -\sigma v'' + (\rho_0 + u)^2 v + [(\rho_0 + u)^2 - \rho_0^2]\phi_0 \end{Bmatrix}.$$

Theorem 2. *There exists $\epsilon(\sigma) > 0$, such that for all $0 \leq J < \epsilon$ a unique family $(u(J), v(J)) \in \mathcal{H}$ exists, satisfying $u(0) \equiv v(0) \equiv 0$, and $F(u(J), v(J), J) = 0$.*

Proof. It is easy to show that $F(0,0,0) = 0$. To prove existence of solutions of $F = 0$ in some right semi-neighborhood of $J = 0$ we use the implicit function theorem. To this end we evaluate the Frechet derivative of F at $(0,0,0)$

$$DF\Big|_{(0,0,0)} = \begin{Bmatrix} -\frac{d^2}{dx^2} - (1 - 3\rho_0^2) & 0 \\ 0 & -\sigma\frac{d^2}{dx^2} + \rho_0^2 \end{Bmatrix} .$$

Then, we show that DF is an isomorphism from $\mathcal{H}$ to L^2. It is enough to show then that each of the main diagonal elements of DF is invertible. Consider then the operator

$$D_{11}F = -\frac{d^2}{dx^2} - (1 - 3\rho_0^2) \, ,$$

whose domain is $H^2(\mathbb{R}_+) \cap H_0^1(\mathbb{R}_+)$. By Persson's Theorem [8] and the fact that $\rho_0 \to 1$ as $x \to \infty$ the bottom of the essential spectrum of $D_{11}F$ is at 2. Hence, it is suffices to prove that $D_{11}F$ is injective.

Consider then the problem

$$\begin{cases} -\frac{d^2w}{dx^2} - (1 - 3\rho_0^2)w = 0 & \\ w(0) = 0 & w \xrightarrow[x\to\infty]{} 0 \end{cases} . \tag{4.3}$$

Multiplying the above by ρ_0' and integrating by parts, we obtain with the aid of the identity

$$\rho_0'' + \rho_0(1 - \rho_0^2) = 0 \, ,$$

that

$$w'(0) = 0 \, .$$

Consequently, $w \equiv 0$ is the unique solution of (4.3).

Consider next the operator

$$D_{22}F = -\sigma\frac{d^2}{dx^2} + \rho_0^2 \, ,$$

whose domain is the space of all $w \in H^2(\mathbb{R}_+)$ such that $w' \in H_0^1(\mathbb{R}_+)$. To prove that $D_{22}F$ is an isomorphism we first observe that $\inf \sigma_{ess}(D_{22}F) = 1$, and hence it is sufficient to show that $D_{22}F$ is injective in order to prove its invertibility. The injectivity of $D_{22}F$, however, follows immediately from the maximum principle.

In view of the above, we have established that DF is an isomorphism of $\mathcal{H}$ onto $L^2(\mathbb{R}_+) \times L^2(\mathbb{R}_+)$. Since $\partial F/\partial J$ is continuous, we obtain the existence of (u, v), satisfying $F(u, v) = 0$, that are continuously differentiable in J in some right semi-neighborhood of $J = 0$.

$\square$

5 global stability of the fully superconducting state

We start by considering a generalized version of (1.3), allowing for higher dimensions

$$\begin{aligned}
&\frac{\partial\psi}{\partial t} + i\phi\psi = \Delta\psi + \psi\left(1-|\psi|^2\right) && \text{in } \Omega\times\mathbb{R}_+ && (5.1a)\\
&\sigma\Delta\phi = \nabla\cdot[\Im(\bar{\psi}\nabla\psi)] && \text{in } \Omega\times\mathbb{R}_+ && (5.1b)\\
&\psi = f && \text{on } \partial\Omega_c\times\mathbb{R}_+ && (5.1c)\\
&\phi = 0 && \text{on } \partial\Omega_c\times\mathbb{R}_+ && (5.1d)\\
&\frac{\partial\psi}{\partial\nu} = 0 && \text{on } \partial\Omega_i\times\mathbb{R}_+ && (5.1e)\\
&\frac{\partial\phi}{\partial\nu} = 0 && \text{on } \partial\Omega_i\times\mathbb{R}_+ && (5.1f)\\
&\psi(x,0) = \psi_0 && \text{in } \Omega\,. && (5.1g)
\end{aligned}$$

In the above $f\in H^2(\Omega)$ satisfies $\|f\|_{L^\infty(\partial\Omega_c)}\leq 1$. The domain Ω may be an open bounded subset of $\mathbb{R}$, $\mathbb{R}^2$, or $\mathbb{R}^3$. The boundary consists of $\partial\Omega_c$, through which current may enter the sample, and $\partial\Omega_i$ which is adjacent to an insulator. The boundary is assumed to be C^2, and $\partial\Omega_c\neq\emptyset$.

Let

$$\mathcal{H} = \left\{u\in H^2(\Omega)\,\Big|\, u\Big|_{\partial\Omega_c} = 0\,,\ \frac{\partial u}{\partial\nu}\Big|_{\partial\Omega_i} = 0\right\}$$

As $\partial\Omega_c\neq\emptyset$ the operator $\Delta^{-1}: L^2(\Omega)\to\mathcal{H}$ is well defined. Thus, ϕ is obtained from ψ via

$$\phi[\psi] = \frac{1}{\sigma}\Delta^{-1}\big(\nabla\cdot[\Im(\bar{\psi}\nabla\psi)]\big)\,. \tag{5.2}$$

Substituting (5.2) into (5.1) yields

$$\begin{aligned}
&\frac{\partial\psi}{\partial t} + i\phi[\psi]\psi = \Delta\psi + \psi\left(1-|\psi|^2\right) && \text{in } \Omega\times\mathbb{R}_+ && (5.3a)\\
&\psi = f && \text{on } \partial\Omega_c\times\mathbb{R}_+ && (5.3b)\\
&\frac{\partial\psi}{\partial\nu} = 0 && \text{on } \partial\Omega_i\times\mathbb{R}_+ && (5.3c)\\
&\psi(x,0) = \psi_0 && \text{in } \Omega\,. && (5.3d)
\end{aligned}$$

Assuming that $\partial f/\partial\nu = 0$, existence of a unique solution $\psi - f\in C([0,T],\mathcal{H})$ for some $T>0$, can then be proved using standard fixed point arguments: By the maximum principle any solution of (5.1) satisfies $|\psi|\leq 1$ in $\Omega\times(0,T)$, provided that $\|\psi_0\|_\infty\leq 1$ (where $\|\cdot\|_p$ denotes the $L^p(\Omega)$ norm). Thus, truncating the functions $\phi[\psi]$ and $\psi(1-|\psi|^2)$ must appropriately for $|\psi|>1$ we may apply the same technique used in § 9.2.1 of [7] (cf. also [16]).

Let

$$E(\psi) = \frac{1}{2} \int_\Omega \Big[|\nabla \psi|^2 + \frac{1}{2}(1 - |\psi|^2)^2 \Big] \, dx \, . \tag{5.4}$$

For ψ which solves (5.1), and $\phi = \phi[\psi]$ we easily obtain

$$\frac{dE}{dt} = - \int_\Omega \Big[|\psi_t + i\phi\psi|^2 + \sigma |\nabla \phi|^2 \Big] \, dx \, . \tag{5.5}$$

Obviously, $E(t)$ is a strict Liapunov function of (5.9), which must converge as $t \to \infty$. In the following we prove convergence of ψ as well.

Proposition 5.1. *Let $\psi_0 \in H^2(\Omega)$ satisfy (5.1c,e) and $\|\psi_0\|_\infty \le 1$. Then for any solution of* (5.9) *we have that $\bigcup_{t>0} (\psi(\cdot, t), \phi(\cdot, t))$ is precompact in $H^2(\Omega)$. Furthermore, any point on the ω-limit set of $(\psi(\cdot, t), \phi(\cdot, t))$ is given by an equilibrium solution $(\psi^\infty, 0)$ of* (5.1)*.*

Proof. Since $E \ge 0$ is decreasing, it must converge to some $E_\infty \ge 0$ as $t \to \infty$. Hence, there exists $\{\tau_k\}_{k=1}^\infty$ such that $\tau_k \uparrow \infty$ and $|dE/dt(\tau_k)| < M$ for all $k \ge 1$. It follows then by (5.5) and (5.1) that

$$\int_\Omega \Big| \Delta \psi - \psi(|\psi|^2 - 1) \Big|^2 \, dx \Big|_{t = \tau_k} \le M \, .$$

Let $\psi_k = \psi(\tau_k)$. Combining the above with the fact that $|\psi| \le 1$, we obtain from standard elliptic estimates that $\{\psi_k\}_{k=1}^\infty$ is bounded in $H^2(\Omega)$. Consequently by choosing an appropriate subsequence we can say that $\psi_k \to \psi_0^\infty$ in $H^1(\Omega)$, where $\psi_0^\infty - \psi_0 \in \mathcal{H}$.

Denote by S_t the semigroup associated with (5.1) (so that $\psi = S_t(\psi_0)$). Let $\psi^\infty(\cdot, t) = S_t(\psi_0^\infty)$ and let $\phi^\infty(\cdot, t)$ denote the corresponding potential. Suppose, for a contradiction, that for some $t^* > 0$ we have $E(\psi^\infty(t^*)) < E_\infty$. Since solutions of (5.1) are continuous with respect to initial conditions, it follows that

$$\exists \delta > 0 : \, \|\tilde{\psi} - \psi_0^\infty\|_{H^1(\Omega)} < \delta \Rightarrow E\big(S_{t^*}(\tilde{\psi})\big) < E_\infty \, .$$

For sufficiently large k, however, $\|\psi_k - \psi_0^\infty\|_{H^1(\Omega)} < \delta$ and since by (5.5) $E\big(S_{t^*}(\tilde{\psi})\big) \ge E_\infty$ we obtain a contradiction. Hence, $E(\psi^\infty)(t) \equiv E_\infty$ for all $t > 0$.

Since $dE/dt \equiv 0$ for $(\psi, \phi) = (\psi^\infty, \phi^\infty)$, we immediately obtain that $\phi^\infty \equiv 0$ and hence that $\psi_t \equiv 0$ as well. Therefore, $(\psi^\infty, 0)$ is an equilibrium solution of (5.9).

It remains necessary to show that the above compactness holds for every diverging monotone sequence of instances. Let then $\{t_n\}_{n=1}^\infty$ satisfy then $t_n \uparrow \infty$. Set

$$\delta_n = E(t_n - 1) - E_\infty \, .$$

Then, for sufficiently large n

$$\int_{t_n - \delta_n^{1/2}}^{t_n} |dE/dt| \, dt \le \delta_n \, .$$

Consequently, for all $n \in \mathbb{N}$ there exists $\tau_n \in [t_n - \delta_n, t_n]$ such that $|dE/dt(\tau_n)| \leq \delta_n^{1/2}$. By the above established compactness there exists a subsequence $\{\tau_{n_k}\}_{k=1}^{\infty}$ and an equilibrium solution $(\psi^\infty, 0)$ such that $(\psi(\cdot, \tau_{n_k}), \phi(\cdot, \tau_{n_k}) \to (\psi^\infty, 0)$ in $H^1(\Omega)$. As $|dE/dt(\tau_{n_k})| \to 0$ the convergence is also in $H^2(\Omega)$.

Standard continuity arguments for the heat operator (cf. theorem 5 in § 7.1 in [7] for instance), taking into account that both ϕ and ψ are uniformly bounded in $H^1(\Omega)$, in view of (5.2) and the fact that E is decreasing, imply that

$$\|\psi(\cdot, t_{n_k}) - \psi(\cdot, \tau_{n_k})\|_{H^2(\Omega)} \xrightarrow[k\to\infty]{} 0\,.$$

Consequently,

$$\|\psi(\cdot, t_{n_k}) - \psi^\infty(\cdot)\|_{H^2(\Omega)} \xrightarrow[k\to\infty]{} 0$$

Precompactness of $\phi(\cdot, t_{n_k})$ easily follows from (5.2). □

Once precompactness is established, it is possible to prove that the ω-limit set of any solution of (5.1) consists of exactly one point. Before we prove this result we mention here the following result, which follows immediately from the Lojasiewicz-Simon theorem [19, 14].

Lemma 5.1. *Let* $(\psi^\infty, 0)$ *denote an equilibrium solution of* (5.1)*. Then, there exists* $\theta \in (0, 1/2)$ *and* $r > 0$ *such that for all* $\psi \in B(\psi^\infty, r)$ *in* $H^2(\Omega)$ *we have*

$$\|\Delta\psi - \psi(|\psi|^2 - 1)\|_2 \geq |E(\psi) - E(\psi^\infty)|^{1-\theta}\,. \tag{5.6}$$

We can now prove convergence as $t \to \infty$

Theorem 3. *Let* $\psi_0 \in H^2(\Omega)$ *satisfy (5.1c,e) and* $\|\psi_0\|_\infty \leq 1$*. Then the solution of* (5.1) *satisfies*

$$\|\phi\|_2 + \|\psi(\cdot, t) - \psi^\infty\|_2 \xrightarrow[t\to\infty]{} 0\,, \tag{5.7}$$

where $(\psi^\infty, 0)$ *is an equilibrium solution of* (5.1)*.*

Proof. Denote by $\omega(\psi_0)$ the ω-limit set of ψ_0. It is well-known that $\omega(\psi_0)$ is a non-empty compact, connected set of equilibrium solutions of (5.1). Let $\psi^\infty \in \omega(\psi_0)$ (obviously by proposition 5.1 we have $\phi(\psi^\infty) = 0$). Since $\psi^\infty \in \omega(\psi_0)$ there exists $\{t_n\}_{n=1}^{\infty}$ such that

$$\psi(t_n, \cdot) \to \psi^\infty(\cdot) \quad \text{in } L^2(\Omega)\,.$$

We argue from here by bootstraping. Let $0 < \epsilon \ll r$, where r is the same as in lemma 5.1. For sufficiently large n we have

$$\|\psi - \psi^\infty(\cdot)\|_{H^2(\Omega)} \leq \epsilon \quad \frac{[E(\psi(\cdot, t_n)) - E_\infty]^\theta}{\theta} \leq \epsilon\,,$$

where θ is the same as in (5.6). Let then

$$\bar{t} = \sup\{t \geq t_n \,|\, \|\psi(\cdot,s) - \psi^\infty(\cdot)\|_{H^2(\Omega)} \leq r\,,\ \forall s \in [t_n,t]\}\,,$$

and suppose, for a contradiction that $\bar{t}$ is finite.

Unless ψ_0 is itself an equilibrium solution, we must have $E(\psi(\cdot,t)) > E_\infty$ for all $t > 0$, for otherwise it would mean by backward uniqueness that $E \equiv E_\infty$ and $\psi(\cdot,t) \equiv \psi^\infty(\cdot)$. Hence, we can write by (5.5) that

$$-\frac{d}{dt}[E(\psi(\cdot,t)) - E_\infty]^\theta = -\frac{dE(\psi(\cdot,t))}{dt}[E(\psi(\cdot,t)) - E_\infty]^{\theta-1} \geq$$
$$\frac{\theta}{2}\big(\|\psi_t+i\phi\psi\|_2+\sigma\|\nabla\phi\|_2\big)\|\Delta\psi-\psi(|\psi|^2-1)\|_2[E(\psi(\cdot,t))-E_\infty]^{\theta-1} \geq \frac{\theta}{2}\big(\|\psi_t+i\phi\psi\|_2+\sigma\|\nabla\phi\|_2\big)\,.$$

Integrating the above between t_n and $\bar{t}$ yields

$$\theta\int_{t_n}^{\bar{t}}\big(\|\psi_t + i\phi\psi\|_2 + \|\nabla\phi\|_2\big)\,dt \leq 2[E(\psi(\cdot,t_n)) - E_\infty]^\theta\,.$$

By Poincare inequality we then have (recall that $\partial\Omega_c \neq \emptyset$) that

$$\int_{t_n}^{\bar{t}}\|\phi\|_2\,dt \leq C[E(\psi(\cdot,t_n)) - E_\infty]^\theta$$

Hence,

$$\int_{t_n}^{\bar{t}}\|\psi_t\|_2\,dt \leq \int_{t_n}^{\bar{t}}\|\psi_t + i\phi\psi\|_2 + \|\phi\|_2\big)\,dt \leq C[E(\psi(\cdot,t_n)) - E_\infty]^\theta\,. \tag{5.8}$$

However,

$$\|\psi(\cdot,\bar{t}) - \psi^\infty(\cdot)\|_2 \leq \int_{t_n}^{\bar{t}}\|\psi_t\|_2\,dt + \|\psi(\cdot,t_n) - \psi^\infty(\cdot)\|_2 \leq C\epsilon\,.$$

For sufficiently small ϵ we have, in view of the precompactness proved in proposition 5.1, that $\|\psi(\cdot,\bar{t}) - \psi^\infty(\cdot)\|_{H^2(\Omega)} \leq r/2$ – a contradiction. Hence $\bar{t} = \infty$. It follows by (5.8) that

$$\int_{t_n}^{\infty}\|\psi_t\|_2\,dt \leq C[E(\psi(\cdot,t_n)) - E_\infty]^\theta\,.$$

Convergence as $t \to \infty$ can now easily be proved. □

We return back to a one-dimensional setting. Consider the following version of (5.1) in $[-L, L]$ [18, 12].

$$\begin{cases} \frac{\partial\psi}{\partial t} - \psi_{xx} + i\phi\psi - \psi(1-|\psi|^2) = 0 & \text{in } (-L,L)\times\mathbb{R}_+ \\ \sigma\phi_{xx} = \frac{1}{2i}(\bar{\psi}\psi_x - \psi\bar{\psi}_x)_x & \text{in } (-L,L)\times\mathbb{R}_+ \\ \psi(x,0) = \psi_0(x) & \text{on } (-L,L) \\ \psi(-L,t) = \rho_\infty e^{-i\alpha L}\ ;\ \psi(L,t) = \rho_\infty e^{i\alpha L} & \text{in } \mathbb{R}_+ \\ \phi(-L,t) = \phi(L,t) = 0 & \text{in } \mathbb{R}_+ \end{cases} . \tag{5.9}$$

In the above $\rho_\infty \in [0,1]$, $\alpha = \sqrt{1-\rho_\infty^2}$, and $\psi_0 \in H^1([-L,L],\mathbb{C})$ satisfies the boundary conditions at $x = \pm L$.

As in the stationary case (1.4), the overall current is given in this case by

$$J = -\sigma\phi' + \Im\{\bar{\psi}\psi_x\}\,. \tag{5.10}$$

However, in contrast with the stationary case J may depend on t.

It follows from theorem 3 that $(\psi, \phi[\psi])$ converges to an equilibrium solution $(\psi^\infty, 0)$. It is easy to show that (1.5)) is an equilibrium solution of (5.9). For the stationary case (1.4), there exist two different values of ρ_∞ which satisfy (1.6) for any given $J \in [0, J_c)$. Therefore, the question which of them is related to a stable solution of (5.9) naturally arises. In [12] it is stated that $(\psi_s, 0)$ is stable when $\rho_\infty^2 > 2/3$ and unstable when $\rho_\infty^2 < 2/3$.

It is not clear how this conclusion is derived in [12]. For solutions of (5.9) it is simply wrong. As we demonstrate in the sequel $(\psi_s, 0)$ is not the unique equilibrium solution of (5.9). As a matter of fact, for sufficiently large L, ψ_s is never the global minimizer of E over all $\psi - \psi_s \in H_0^1(-L,L)$. This is exactly what the next lemma shows

Lemma 5.2. *Let*

$$\mathcal{E}_L = \inf_{\psi-\psi_s\in H_0^1(-L,L)} E(\psi)\,.$$

Then, there exists $C > 0$, which is independent of L such that $E_L \le C$ for all $L \ge 2$.

Proof. We obtain an upper bound for $\mathcal{E}$ by using the following test function

$$\tilde{\psi} = \begin{cases} -\rho_\infty(x+L-1)e^{-i\alpha L} & -L \le x < -L+1 \\ 0 & -L+1 \le x < -L+2 \\ (x+L-2)e^{i\alpha L} & -L+2 \le x < -L+3 \\ e^{i\alpha L} & -L+3 \le x \le L-1 \\ [1-(1-\rho_\infty)(x-L+1)]e^{i\alpha L} & L-1 < x \le L \end{cases} .$$

Clearly, $\tilde{\psi} - \psi_s \in H_0^1(-L,L)$ and since $E(\tilde{\psi})$ is independent of L for all $L \ge 2$ the lemma is proved. $\square$

Let $J_L = \lim_{t\to\infty} J(t, L)$. We conclude this section by obtaining the asymptotic behaviour of J_L in the limit $L \to \infty$, in the case where the long-time equilibrium solution of (5.9) is the global minimizer of E.

Lemma 5.3. *Let $\psi_L = \rho_L e^{i\chi_L}$ denote the global minimizer of E over all functions ψ such that $\psi - \psi_s \in H_0^1(-L, L)$. Let $J_L = \rho_L^2 \chi_L'$. Then there exists $C > 0$ such that*

$$J_L \le \frac{C}{L} \, .$$

Proof. By (5.10), and since for an equilibrium solution we have $\phi \equiv 0$, we obtain

$$J_L = \rho_L^2 \chi_L' \, .$$

Since

$$\int_{-L}^{L} \frac{J_L}{\rho_L^2} \, dx = 2\alpha L \, ,$$

We have

$$\mathcal{E}_L = E(\psi_L) = 2\alpha L J_L + \int_{-L}^{L} |\rho_L'|^2 + \frac{1}{2}(1-\rho_L^2)^2 \, dx \ge 2\alpha L J_L$$

The lemma is then proved from lemma 5.2. □

6 Concluding Remarks

In the following we briefly summarize the main results of this work, and illuminate a few points that may have escaped the reader's attention.

In the limit $\sigma \to 0$ we have demonstrated in § 2 that for $J > C\sigma^{1/4}$ for some $C > 0$, no solution can exist for (1.8). We have also formally evaluated C. It is clear from § 4 that for sufficiently small J a solution for (1.8) does exist. It is not clear, however, whether a critical value $J_+(\sigma)$ exists such that no solution of (1.8) exists for $J > J_+$, but existence is guaranteed for every $J < J_+$. Alternatively we may say that it is not clear whether the set $\mathcal{S}(\sigma)$ is connected.

In the limit $\sigma \to \infty$ we have, in § 3, established existence of solutions for (1.8) for every fixed $J < J_c$, in the case where $\rho_\infty^2 > 2/3$. We have also obtained formally the leading order behaviour of ψ in this asymptotic case. If we assume the existence of the above critical current J_+, the above existence result means that $J_+ \to J_c$ as $\sigma \to \infty$. It is probably possible to show that $\sigma_c > 0$ exists such that $J_+(\sigma) \equiv J_c$ for all $\sigma > \sigma_c$. A more difficult task is showing that $J_+ < J_c$ for all $\sigma < \sigma_c$ (and perhaps also that J_+ is decreasing there). Further research is necessary: at least one can numerically obtain the value of σ_c and the dependence on σ of J_+

Finally, in § 5 we have proved convergence as $t \to \infty$ of solutions of (5.1) to an equilibrium solution, by first establishing that the Ginzburg-Landau energy is a strict Liapunov function. Then we showed, for a one-dimensional setting in (5.9), that the global minimizer for the energy functional, for long wires, corresponds to a vanishingly small current. Note that (5.9) assumes zero potential drop along the wire. An important question that has to be addressed is whether for fixed current (and time-dependent potential drop) the solution converges, as $t \to \infty$ to an equilibrium solution. If so, it is easy to show that the unique equilibrium in such a case would be (1.5). However, (5.4) is no longer a Liapunov function [16]. Therefore, it is possible in this case that one can establish that the statements in [12] regarding the stability of (1.5) are correct. Linear stability analysis can provide some insight into this problem.

Acknowledgments. The author was supported by NSF grant DMS 0604467. The author wishes to thank Professor Jacob Rubinstein for his contribution to § 2 and for his comments on the manuscript.

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