SUPERCONDUCTIVITY NEAR THE NORMAL STATE IN A HALF-PLANE UNDER THE ACTION OF A PERPENDICULAR ELECTRIC CURRENT AND AN INDUCED MAGNETIC FIELD II : THE LARGE CONDUCTIVITY LIMIT

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ABSTRACT. We consider the linearized Ginzburg-Landau equation in the half-plane, in the presence of an electric current, perpendicular to the boundary, and the magnetic field it induces. In a previous work we have considered the same problem in the limit of small normal conductivity. In the present contribution we consider the large normal conductivity limit, which is more frequently encountered in experiments than the other limit. Like in the previous work we obtain an approximation of the critical current where the normal state looses its stability. We find that this critical current is determined by the ground state of the anharmonic oscillator

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1. INTRODUCTION

1.1. Former results.

Consider a superconductor placed at a temperature lower than the critical one. If an electric current is applied through the sample it will induce a magnetic field, and as is well-understood from numerous experimental observations, a sufficiently strong current will force the superconductor to arrive at the normal state. If the current is then lowered, the normal state would lose stability and the sample would become superconducting again. In addition to experimental observations a similar pattern of behaviour has also been obtained theoretically by analyzing the stability of the normal state for the time dependent Ginzburg-Landau system, but with the induced magnetic field neglected [13, 3].

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In a previous contribution [4] we analyzed the stability of the normal state in the presence of an electric current which induces a magnetic field, but in the absence of boundaries. We offered an analysis of a two-dimensional setting, i.e, in $\mathbb{R}^2$, which is the simplest case where one can consider the effect of a magnetic field induced by a current, with boundary effects ignored. We found in [4] that the normal state is always stable, irrespectively of current intensity. This result is in line with those obtained for a reduced model where the induced magnetic field has been neglected [13, 3] .

The effect of boundaries in the absence of magnetic field has been first analyzed by considering a one-dimensional problem on $\mathbb{R}_+$ with a Dirichlet boundary condition which stands for a normal/superconducting interface [13, 3]. Due to the boundary's effect the normal state looses its stability for currents that are weaker than a certain critical value. It has been proved in [3] that the critical current for a large bounded three-dimensional domain is bounded from above by the one-dimensional value. Furthermore, for currents below the critical one a short-time instability was proved [3] (the question whether the normal state is unstable for such domains and currents is still open).

In another contribution [5] we introduced the effect of boundaries in the limit of small normal conductivity. We showed that it has a similar effect to the one found in [13, 3], i.e., the normal state looses its stability for currents lower than a critical value. Moreover, it was found that as the normal conductivity tends to 0, the critical current converges to the value obtained for the simplified model, where the magnetic field is neglected [13, 3, 17, 16, 18].

Assuming that a magnetic field described by $\mathcal{H}^e$ is perpendicularly applied to the sample, the time-dependent Ginzburg-Landau system can be written as follows (see for instance [6, 7, 8, 9, 13, 19, 21]):

$$\begin{cases} \partial_t\psi + i\kappa\Phi\psi = \nabla^2_{\kappa\mathbf{A}}\psi + \kappa^2(1-|\psi|^2)\psi\,, & \text{in } \mathbb{R}_+\times\mathbb{R}^2_+, \\ \kappa^2\mathrm{curl}\,^2\mathbf{A} + \sigma(\partial_t\mathbf{A} + \nabla\Phi) = \kappa\,\mathrm{Im}\,(\bar\psi\nabla_{\kappa\mathbf{A}}\psi) + \kappa^2\mathrm{curl}\,\mathcal{H}^e\,, & \text{in } \mathbb{R}_+\times\mathbb{R}^2_+\,, \\ \psi = 0, \qquad -\frac{\sigma}{\kappa^2}\frac{\partial\Phi}{\partial\nu} = J\,, & \text{on } \partial\mathbb{R}^2_+\,, \end{cases} \tag{1.1}$$

where ψ is the order parameter, $\mathbf{A}$ is the magnetic potential, Φ is the electric potential, the Ginzburg-Landau parameter of the superconductor is denoted by κ, the normal conductivity of the sample is denoted by σ, the magnitude of the dimensionless electric current is denoted by J, and the magnetic field is symbolized by $\mathcal{H}^e$. We note that spatial coordinates are scaled with respect to the coherence length, and not, as one often encounters, with respect to the penetration depth. This is why a factor of κ^2 appears in the second equation of (1.1). The half-plane $\mathbb{R}^2_+$ is defined in the following manner

$$\mathbb{R}^2_+ = \{(x,y)\in\mathbb{R}^2\,:\,y>0\}\,.$$

The triplet $(\psi,\mathbf{A},\Phi)$ should also satisfy an initial condition at $t=0$.

A solution $(\psi,\mathbf{A},\Phi)$ is called a normal state solution if $\psi\equiv 0$. From (1.1) we see that if $(0,\mathbf{A},\Phi)$ is a time-independent normal state solution then $(\mathbf{A},\Phi)$ satisfies the following equation

$$\kappa^2\mathrm{curl}\,^2\mathbf{A} + \sigma\nabla\Phi = \kappa^2\mathrm{curl}\,\mathcal{H}^e \quad \text{in } \mathbb{R}^2_+\,. \tag{1.2}$$

By taking the divergence of (1.2) we obtain

$$\begin{cases} \Delta\Phi = 0 & \text{in } \mathbb{R}^2_+, \\ \frac{\partial\Phi}{\partial\nu} = -\frac{\kappa^2 J}{\sigma} & \text{on } \partial\mathbb{R}^2_+. \end{cases} \tag{1.3}$$

Since we expect solutions of (1.3) to represent the electric potential at the normal state near the boundary of a large bounded domain, we look for solutions with bounded gradient (or $|\nabla\Phi| \in L^\infty(\mathbb{R}^2_+)$). Assuming that the current is of constant magnitude J along $\partial\mathbb{R}^2_+$, and that its direction is always perpendicular to it, we obtain that the unique solution which obeys these assumption is given by

$$\Phi = \frac{\kappa^2 J}{\sigma} y\,. \tag{1.4}$$

Assuming further that the applied magnetic field is, like the current, of constant magnitude as well, we obtain

$$\mathcal{H}^e = h\hat{\mathbf{i}}_z,$$

throughout the entire sample. Here $\hat{\mathbf{i}}_x, \hat{\mathbf{i}}_y$ and $\hat{\mathbf{i}}_z$ denote the canonical basis in $\mathbb{R}^3$. Hence, we consider an applied magnetic field which is perpendicular to the sample. Under these additional assumptions equation (1.2) admits the following solution

$$\mathbf{A} = \frac{1}{2J}(Jx+h)^2\hat{\mathbf{i}}_y\,. \tag{1.5}$$

Thus, $(0, \mathbf{A}, \Phi)$ is a normal state solution of (1.1). Note that the magnetic field

$$\mathbf{H} = \operatorname{curl}\mathbf{A} = (Jx+h)\hat{\mathbf{i}}_z\,,$$

is the sum of the constant applied magnetic field $h\hat{\mathbf{i}}_z$ and a linear term induced by the electric current.

The linearization of (1.1) near the normal state solution $(0, \mathbf{A}, \Phi)$ obtained above yields a linear equation

$$\begin{cases} \partial_t\psi + \frac{i\kappa^3 Jy}{\sigma}\psi = \Delta\psi - \frac{i\kappa}{J}(Jx+h)^2\partial_y\psi - (\frac{\kappa}{2J})^2(Jx+h)^4\psi + \kappa^2\psi & \text{in } \mathbb{R}_+\times\mathbb{R}^2_+, \\ \psi = 0 & \text{on } \partial\mathbb{R}^2_+\,. \end{cases} \tag{1.6}$$

Applying the transformation

$$(t,x,y) \to (t, x - \frac{h}{J}, y)$$

we obtain

$$\begin{cases} \partial_t\psi + \frac{i\kappa^3 Jy}{\sigma}\psi = \Delta\psi - i\kappa Jx^2\partial_y\psi - \Big((\frac{\kappa J}{2})^2x^4 - \kappa^2\Big)\psi & \text{in } \mathbb{R}_+\times\mathbb{R}^2_+, \\ \psi = 0 & \text{on } \partial\mathbb{R}^2_+. \end{cases} \tag{1.7}$$

We assume $J > 0$ in the sequel. Otherwise we may consider the complex conjugate of (1.6). Hence, we can rescale x, y and t by applying

$$t \to (\kappa J)^{2/3}t, \qquad (x,y) \to (\kappa J)^{1/3}(x,y), \tag{1.8}$$

yielding

$$\begin{cases} \partial_t u = -(\mathcal{A}_{0,c} - \lambda) u & \text{in } \mathbb{R}_+ \times \mathbb{R}^2_+, \\ u = 0 & \text{on } \partial \mathbb{R}^2_+, \end{cases} \tag{1.9}$$

where $\mathcal{A}_{0,c}$ is the differential operator defined by

$$\mathcal{A}_{0,c} = D_x^2 + (D_y - \frac{1}{2}x^2)^2 + icy\,, \tag{1.10}$$

with

$$D_x = -i\partial_x, \qquad D_y = -i\partial_y, \qquad c \in \mathbb{R}_+\,,$$

and

$$c = \frac{\kappa^2}{\sigma}, \quad \lambda = \lambda_0 \equiv \frac{\kappa^{4/3}}{J^{2/3}}, \quad u(x,y,t) = \psi((\kappa J)^{-1/3}x, (\kappa J)^{-1/3}y, (\kappa J)^{-2/3}t)\,. \tag{1.11}$$

The operator $\mathcal{A}_{0,0}$ will be denoted by $\mathcal{A}_0$ for simplicity.

While the operator $\mathcal{A}_{0,c}$ is defined on smooth functions only, we have already proved in [5] that it can be extended into an operator $\mathcal{A}_c^+$ whose domain is given by

$$D(\mathcal{A}_c^+) = \{u \in \tilde{V} \,:\, \mathcal{A}_c^+ u \in L^2(\mathbb{R}^2_+, \mathbb{C})\}\,, \tag{1.12}$$

where

$$\tilde{V} = H_0^{1,\text{mag}}(\mathbb{R}^2_+, \mathbb{C}) \cap L^2(\mathbb{R}^2_+\,,\, y\, dxdy)\,,$$

$L^2(\mathbb{R}^2_+, \mathbb{C})$ denotes the L^2 space of complex-valued functions, and $H_0^{1,\text{mag}}(\mathbb{R}^2_+, \mathbb{C})$ is the closure of $C_0^\infty(\mathbb{R}^2_+, \mathbb{C})$ under the norm

$$u \longmapsto \sqrt{\|u\|^2 + \|D_x u\|^2 + \|(D_y - \frac{x^2}{2})u\|^2}.$$

Here and thereafter $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ denote the L^2 norm and inner product on $\mathbb{R}^2_+$:

$$\|u\| = \|u\|_{L^2(\mathbb{R}^2_+)} = \Big(\int_{\mathbb{R}^2_+} |u|^2 dx\Big)^{1/2}, \qquad \langle u, v\rangle = \int_{\mathbb{R}^2_+} u\bar{v}\, dx\,.$$

The L^2 norms and the associated inner products in both $L^2(\mathbb{R})$ and $L^2(\mathbb{R}, \mathbb{C})$ are denoted by $\|\cdot\|_{L^2(\mathbb{R})}$ and $\langle\cdot,\cdot\rangle_{L^2(\mathbb{R})}$.

Once the definition of the extended operator $\mathcal{A}_c^+$ has been formulated, we may write

$$\mathcal{A}_c^+ = D_x^2 + (D_y - \frac{1}{2}x^2)^2 + icy\,. \tag{1.13}$$

Note that $\mathcal{A}_c^+$ is not self-adjoint. Furthermore, we have that

$$(\mathcal{A}_c^+)^* = \mathcal{A}_{-c}^+\,.$$

In the present contribution we analyze the spectrum of $\mathcal{A}_c^+$, denoted by $\sigma(\mathcal{A}_c^+)$, in the limit $c \to 0$. The following proposition, whose proof is straightforward, has already been stated in [5]:

Proposition 1.1. *For any $c \neq 0$, $\mathcal{A}_c^+$ has a compact resolvent. Moreover, if $E_0(s)$ denotes the ground state energy of the anharmonic oscillator*

$$\mathcal{M}_s := -\frac{d^2}{dx^2} + \Big(\frac{x^2}{2} - s\Big)^2 \,, \tag{1.14}$$

and if

$$E_0^* = \inf_{s\in\mathbb{R}} E_0(s) \,, \tag{1.15}$$

then

$$\sigma(\mathcal{A}_c^+) \subset \{\lambda \in \mathbb{C} \,,\, \operatorname{Re}\lambda \geq E_0^*\} \,. \tag{1.16}$$

1.2. **Main result.**

Our main result is the following:

Theorem 1.2. *There exists $c_0 > 0$ such that if $0 < |c| \leq c_0$, then*

$$\sigma(\mathcal{A}_c^+) \neq \emptyset \,.$$

Furthermore, there exists $\mu(c) \in \sigma(\mathcal{A}_c^+)$ which admits, as $c \to 0$, the following expansion:

$$\mu(c) \;=\; E_0^* + |c|^{2/3}\lambda_2 + o(|c|^{2/3}) \,, \tag{1.17}$$

where E_0^* is introduced in (1.15), and λ_2 is a spectral invariant which will be given by (3.24).

Remark 1.3. *We expect that $\mu(c)$ corresponds to the eigenvalue with smallest real part. Note further that the proof we bring for (1.17) provides a more precise error estimate for it.*

Relying on (1.11), (1.16), and (1.17) we can estimate the critical current.

Corollary 1.4. *Let J_c denote the critical current, such that for $|J| < J_c$ in (1.7), the normal state is unstable. Then, for all $|c| \leq c_0$ we have*

$$\frac{\kappa^2}{(E_0^*)^{3/2}}\Big(1 - \frac{3\lambda_2}{2E_0^*}|c|^{2/3} + o(|c|^{2/3}\Big) \leq J_c \leq \frac{\kappa^2}{(E_0^*)^{3/2}} \,. \tag{1.18}$$

From the results of [5] we thus obtain that

$$J_c \sim \frac{\kappa^2}{(E_0^*)^{3/2}} \quad \text{as } c \to 0; \qquad \frac{J_c}{\sigma} \sim \frac{|a_1|}{2} \quad \text{as } c \to \infty \,,$$

where a_1 is the rightmost zero point of the Airy's function [1]. Hence, the critical current is determined, to leading order, in the limit $c \to 0$, by setting $c = 0$ in (1.10). In contrast, in the limit $c \to \infty$ the leading order is obtained by erasing the magnetic potential $x^2/2$ from (1.10). In the language of Physics, the normal state looses its stability, in the large conductivity limit, since the magnetic field generated by the current reaches a critical level, whereas in the small conductivity limit it is the drop in the electric potential that produces the instability. Nevertheless, although (1.18) is highly intuitive from a physical point of view, considerable effort is necessary, as can be seen later, to derive it even formally.

Since a standard perturbation expansion fails near the boundary, we first need to derive the behaviour of eigenfunctions of $\mathcal{A}_c^+$ for $y = \mathcal{O}(1)$. We obtain this behaviour in Section 2.

In Section 3 we obtain an approximation for an eigenvalue and an eigenfunction formally, relying on the results of Section 2. Section 4 includes some preliminary estimates we need in order to prove the formal expansion. In Section 5, we prove the "outer" expansion, i.e., the behaviour of the eigenmode for $y = \mathcal{O}(|c|^{-1/3})$, whereas in Section 6 we prove some "inner" estimates for $y = \mathcal{O}(1)$. Finally, in Section 7 we complete the proof of (1.17).

2. The Boundary Layer

Consider the operator $\mathcal{A}_c^+$ defined in (1.13). Let (λ, v) denote an eigenpair, i.e. an eigenvalue λ of $\mathcal{A}_c^+$ and one corresponding eigenfunction v. For bounded y and small c, it appears reasonable to estimate (λ, v) by (E_0^*, u), where E_0^* is the bottom of the spectrum of the self-adjoint operator $\mathcal{A}_0^+$ (which is defined in (1.13) by setting $c = 0$ and $\lambda = E_0^*$), and $u(x,y) = \phi^*(x)e^{is^*y}$ is an associated L^∞ "eigenfunction".

In this section we obtain an auxiliary result that provides us with the asymptotic behavior of u as $y \to \infty$. In addition, with the notation introduced around (1.15), let $E_0(s)$ and $E_1(s)$ denote the first and second eigenvalues of the operator $\mathcal{M}_s$ (see (1.14)). Let s^* be the unique point where $E_0(s)$ is minimal (cf. [12])

$$E_0(s^*) = E_0^* \,. \tag{2.1}$$

Let $\phi_0(x,s)$ denote the corresponding normalized positive eigenfunction associated with $E_0(s)$, i.e.,

$$\mathcal{M}_s \phi_0(x,s) = E_0(s)\phi_0(x,s) \,. \tag{2.2}$$

Set

$$\phi^*(x) = \phi_0(x, s^*) \tag{2.3}$$

and

$$E_1^* = \inf_{s\in\mathbb{R}} E_1(s) \,.$$

Let $\hat{\mathbf{i}}_x$ and $\hat{\mathbf{i}}_y$ denote the unit vectors in the positive x- and y-axes. For convenience, we introduce a new $\mathcal{A}_0^{\text{new}}$ deduced from $\mathcal{A}_0$ by using a conjugation by $\exp(iys^*)$:

$$\mathcal{A}_0^{\text{new}} := -\Big(\nabla - i\big(\frac{x^2}{2} - s^*\big)\hat{\mathbf{i}}_y\Big)^2 \,. \tag{2.4}$$

To simplify notation, we omit from now on the reference "new" and write simply $\mathcal{A}_0$.

Theorem 2.1. *Let $f \in \mathcal{S}(\mathbb{R}, \mathbb{C})$. Then there exists a unique pair $(u, \alpha) \in H^2_{\text{loc}}(\overline{\mathbb{R}^2_+}, \mathbb{C}) \times \mathbb{C}$ such that*

(i) *u satisfies*

$$(\mathcal{A}_0 - E_0^*)u = 0 \quad \text{in } \mathbb{R}^2_+ \,, \tag{2.5a}$$

$$u = f \quad \text{on } \partial\mathbb{R}^2_+ \,; \tag{2.5b}$$

(ii) *$u - \alpha\phi^* \in L^2(\mathbb{R}^2_+, \mathbb{C})$ and for any $k \geq 1$, there exists a constant $C(k) > 0$ such that, for $y \geq 1$,*

$$\|u(\cdot, y) - \alpha\phi^*(\cdot)\|_{L^2(\mathbb{R})} \leq \frac{C(k)}{y^k} \,. \tag{2.5c}$$

To prove the existence of a solution for (2.5) we convert it first to an inhomogeneous problem in $H^2_{\rm loc}(\overline{\mathbb{R}^2_+},\mathbb{C})$ with trace 0 at $y=0$. To this end we define the cutoff function $\chi\in C^\infty(\mathbb{R}_+,[0,1])$ such that

$$\chi(y)=\begin{cases}1 & \text{if } y<1\,,\\ 0 & \text{if } y>2\,.\end{cases} \tag{2.6}$$

Set then

$$u=w+\chi(y)f(x)\,. \tag{2.7}$$

For the equivalent inhomogeneous problem we thus look for $w\in H^2_{\rm loc}(\mathbb{R}^2_+,\mathbb{C})$ and $\alpha\in\mathbb{C}$, which satisfy

$$\begin{cases}(\mathcal{A}_0-E_0^*)w=g & \text{in } \mathbb{R}^2_+\,,\\ w=0 & \text{on } \partial\mathbb{R}^2_+\,,\end{cases} \tag{2.8}$$

together with (2.5c). One can obtain the precise form of g by substituting (2.7) into (2.5a). Nevertheless, in the sequel, we need only the fact that $g\in L^2(\mathbb{R}^2_+,\mathbb{C})$ and that it is supported on the set

$$\{(x,y)\in\mathbb{R}^2: 0<y<2\}\,.$$

Let $\mathcal{S}_R=\mathbb{R}\times(0,R)$ and denote by $\|\cdot\|_2$ the $L^2(\mathcal{S}_R)$ norm. Let $\mathcal{A}_0^R$ denote the Dirichlet realization (obtained by application of Lax-Milgram's theorem) in $\mathcal{S}_R$ of the differential operator

$$-\Big(\nabla-i\big(\frac{x^2}{2}-s^*\big)\hat{\mathbf{i}}_y\Big)^2\,.$$

We construct such a function w which solves (2.8) for some $\alpha\in\mathbb{C}$, as a limit, as R tends to infinity, of solutions w^R in the domain of $\mathcal{A}_0^R$ of

$$(\mathcal{A}_0^R-E_0^*)w^R=g\,. \tag{2.9}$$

We first need to make the obvious observation that solutions for (2.9) do exist for $R\geq 2$.

Lemma 2.2. *Given $g\in L^2(\mathbb{R}^2_+,\mathbb{C})$ which vanishes on $\mathbb{R}^2_+\setminus\mathcal{S}_2$, there exists, for all $R\geq 2$, a unique solution $w^R\in L^2(\mathcal{S}_R,\mathbb{C})$ for* (2.9)*.*

Proof. It is easy to show, using for instance Theorem 4 in [15] and the compactness of $[0,R]$, that the Dirichlet realization $\mathcal{A}_0^R$ of $\mathcal{A}_0$ in $\mathcal{S}_R$ has a compact resolvent, and hence, has a discrete spectrum in $\mathbb{R}_+=\{x\in\mathbb{R}: x\geq 0\}$. Introduce

$$\mu_0(R)=\inf_{R>0}\sigma\big(\mathcal{A}_0^R\big)\,.$$

To complete the proof of the lemma we now show that

$$\mu_0(R)>E_0^*\,,\quad \text{for all } R>0\,. \tag{2.10}$$

As a quick aside, we mention that by domain monotonicity and comparison with the problem in $\mathbb{R}^2$ it easily follows that $\mu_0(R)$ is decreasing with R and that

$$\mu_0(R)\geq E_0^*\quad\text{for all } R>0\,.$$

Furthermore, strict monotonicity can be proven employing the Hadamard formula (which provides an explicit expression for $\mu_0'(R)$). Below we give a direct proof, avoiding the use of the Hadamard formula.

Denote by w_0 the eigenfunction of $\mathcal{A}_0^R$ associated with $\mu_0(R)$ with unity L^2 norm. Clearly,

$$\begin{aligned} 0 =&\langle w_0, (\mathcal{A}_0-\mu_0)w_0\rangle_{L^2(\mathcal{S}_R)} \\ =&\big\|\big(\nabla - i\big[\frac{x^2}{2}-s^*\big]\hat{\mathbf{i}}_y\big)w_0\big\|^2_{L^2(\mathcal{S}_R)} - \mu_0(R)\|w_0\|^2_{L^2(\mathcal{S}_R)}\,. \end{aligned} \tag{2.11}$$

Set then

$$\tilde{w}_0(x,y) = \begin{cases} w_0(x,y)\,, & \text{if } 0<y<R\,, \\ 0\,, & \text{otherwise,} \end{cases} \tag{2.12}$$

and let its partial Fourier transform in y be given by

$$\hat{w}_0(x,s) = \mathcal{F}[\tilde{w}_0] = \frac{1}{\sqrt{2\pi}}\int_0^R w_0(x,y)e^{-isy}dy\,. \tag{2.13}$$

Moreover since $w_0\in H_0^{1,\mathrm{mag}}(\mathcal{S}_R)$ we have that $\widehat{\partial_y w_0}(x,s) = is\hat{w}_0$. Hence in terms of the partial Fourier transform, (2.11) takes the form

$$\int_{\mathbb{R}^2}\Big\{|\hat{w}_0'|^2 + \Big(\frac{x^2}{2}-s^*-s\Big)^2|\hat{w}_0|^2 - \mu_0(R)|\hat{w}_0|^2\Big\}\,dxds = 0\,,$$

where $\hat{w}_0' = \partial\hat{w}_0/\partial x$.

Next we set

$$W(s) = \|\hat{w}_0(\cdot,s)\|_{L^2(\mathbb{R})}\not\equiv 0\,.$$

Since

$$\int_{\mathbb{R}}\Big\{|\hat{w}_0'|^2 + \Big(\frac{x^2}{2}-s^*-s\Big)^2|\hat{w}_0|^2\Big\}\,dx \geq E_0(s^*+s)W^2(s)\,,$$

we have that

$$\int_{\mathbb{R}}[E_0(s^*+s)-\mu_0(R)]W^2(s)\,ds\leq 0\,.$$

From the above inequality and the fact (cf. [12]) that $E_0(s)$ has a unique minimum E_0^* at s^*, we see that

$$(E_0^*-\mu_0(R))\int_{\mathbb{R}}W^2(s)\,ds < \int_{\mathbb{R}}[E_0(s^*+s)-\mu_0(R)]W^2(s)\,ds\leq 0\,.$$

Hence, $\mu_0(R)>E_0^*$ and the lemma is proved. ∎

Next we prove some bounds on w^R that are uniform in R.

Lemma 2.3. *There exists a constant $C>0$ such that, for any $R\geq 2$ and $g\in L^2(\mathbb{R}^2_+,\mathbb{C})$ which vanishes on $\mathbb{R}^2_+\setminus\mathcal{S}_2$, then the solution w^R of* (2.9) *admits the following decomposition:*

$$w^R(x,y) = v^R(x,y) + b_R(y)\phi^*(x)\,, \tag{2.14a}$$

where ϕ^* is given in (2.3)*, and v^R and b_R satisfy the estimates*

$$\|v^R\|_{L^2(\mathcal{S}_R)}\leq C\,\|g\|_{L^2(\mathcal{S}_2)}\,, \tag{2.14b}$$

$$\|b_R'\|_{L^2(0,R)}\leq C\,\|g\|_{L^2(\mathcal{S}_2)}\,. \tag{2.14c}$$

Furthermore,

$$\Big\|\frac{\partial w^R}{\partial x}\Big\|_{L^2(\mathcal{S}_R)} \leq C\,\|g\|_{L^2(\mathcal{S}_2)}\,, \tag{2.14d}$$

and, for all $L \geq \max\{1, 2\sqrt{s^*}\}$,

$$\Big\|\frac{\partial w^R}{\partial y}\Big\|_{L^2((-L,L)\times(0,R))} \leq C\,L^2\,\|g\|_{L^2(\mathcal{S}_2)}\,. \tag{2.14e}$$

Finally, for all $l>0$*, we have*

$$\|w^R\|_{L^2(\mathcal{S}_l)} \leq C\,(1+l^{3/2})\,\|g\|_{L^2(\mathcal{S}_2)}\,. \tag{2.14f}$$

Proof. The construction of v^R and b_R will be provided in Step 4 below. To simplify our notation we drop the superscript R throughout the proof. For instance w^R is denoted by w.

Multiplying (2.9) by $\bar{w}$ and integrating over $\mathcal{S}_R$ yields

$$\Big\|\big(\nabla - i\big[\frac{x^2}{2}-s^*\big]\hat{\mathbf{i}}_y\big)w\Big\|^2_{L^2(\mathcal{S}_R)} - E_0^*\|w\|^2_{L^2(\mathcal{S}_R)} = \operatorname{Re}\langle w,g\rangle_{L^2(\mathcal{S}_2)}\,. \tag{2.15}$$

The right hand term is an integral over $\mathcal{S}_2$ because g is supported only there. Denote by $\hat{w}(x,s)$ the partial Fourier transform of $w(x,y)$ as defined above. In Fourier space, the above identity takes the form

$$\|\hat{w}'\|^2_{L^2(\mathbb{R}^2)} + \Big\|\big(\frac{x^2}{2}-s^*-s\big)\hat{w}\Big\|^2_{L^2(\mathbb{R}^2)} - E_0^*\|\hat{w}\|^2_{L^2(\mathbb{R}^2)} = \operatorname{Re}\langle \hat{w},\hat{g}\rangle_{L^2(\mathbb{R}^2)}\,. \tag{2.16}$$

We next introduce the following orthogonal decomposition in $L^2(\mathbb{R}^2)$ for $\hat{w}$:

$$\begin{cases} \hat{w}(x,s) = \hat{w}_{\|,s}(x,s) + \hat{w}_{\perp,s}(x,s)\,, \\ \hat{w}_{\|,s}(x,s) = \hat{b}(s^*+s)\phi_0(x,s^*+s)\,, \\ \hat{b}(s^*+s) = \mathbf{1}_{[-1,1]}(s)\langle \hat{w}(\cdot,s),\phi_0(\cdot,s^*+s)\rangle_{L^2(\mathbb{R})}\,, \end{cases} \tag{2.17}$$

where $\mathbf{1}_{[-1,+1]}(s)$ is the characteristic function of the interval $[-1,1]$. The notation $\hat{w}_{\perp,s}$ and $\hat{w}_{\|,s}$ indicates the orthogonality of the two components in $L^2(\mathbb{R})$ for all $s\in[-1,1]$, i.e.,

$$\int_{\mathbb{R}} \hat{w}_{\|,s}(x,s)\overline{\hat{w}}_{\perp,s}(x,s)\,dx = 0\,. \tag{2.18}$$

We save the notation $w_\perp$ and $w_\|$ for a different type of decomposition employed in Section 5. Since $\hat{w}_{\|,s}(x,s)$ is supported in $\mathbb{R}\times[-1,1]$, we obtain using (2.18) that

$$\begin{aligned} &\int_{\mathbb{R}^2} |\hat{w}_{\|,s}+\hat{w}_{\perp,s}|^2\,dxds = \int_{\mathbb{R}^2}\{|\hat{w}_{\|,s}|^2 + |\hat{w}_{\perp,s}|^2 + 2\operatorname{Re}\hat{w}_{\|,s}\bar{\hat{w}}_{\perp,s}\}\,dxds \\ &= \int_{-1}^1\int_{\mathbb{R}} |\hat{w}_{\|,s}|^2\,dxds + \int_{\mathbb{R}^2}|\hat{w}_{\perp,s}|^2\,dxds + 2\operatorname{Re}\int_{-1}^1\int_{\mathbb{R}}\hat{w}_{\|,s}\bar{\hat{w}}_{\perp,s}\,dxds \\ &= \int_{-1}^1\int_{\mathbb{R}} |\hat{w}_{\|,s}|^2\,dxds + \int_{\mathbb{R}^2}|\hat{w}_{\perp,s}|^2\,dxds\,. \end{aligned} \tag{2.19}$$

Step 1. We first prove that

$$\begin{aligned}
&\int_{-1}^{1}[E_0(s^*+s)-E_0^*]\|\hat{w}_{\|,s}(\cdot,s)\|^2_{L^2(\mathbb{R})}\,ds\\
&\quad+\min\Big((E_1^*-E_0),\inf_{|s|>1}(E_0(s+s^*)-E_0^*)\Big)\int_{\mathbb{R}}\|\hat{w}_{\perp,s}\|^2\,ds\\
&\leq|\mathrm{Re}\,\langle\hat{w},\hat{g}\rangle_{L^2(\mathbb{R}^2)}|\,,
\end{aligned}\tag{2.20}$$

which can alternatively be phrased, using some of the properties of E_0 which were derived in [12], in the form

$$\begin{aligned}
&\textstyle\int_{-1}^{1}[E_0(s^*+s)-E_0^*]\|\hat{w}_{\|,s}(\cdot,s)\|^2_{L^2(\mathbb{R})}\,ds\\
&\quad+[\min(E_1^*,E_0(1+s^*),E_0(-1+s^*))-E_0^*]\textstyle\int_{\mathbb{R}}\|\hat{w}_{\perp,s}\|^2\,ds\\
&\qquad\leq\ |\mathrm{Re}\,\langle\hat{w},\hat{g}\rangle_{L^2(\mathbb{R}^2)}|\,.
\end{aligned}\tag{2.21}$$

By (2.16) and (2.17) we have

$$\begin{aligned}
&\|\hat{w}'\|^2_{L^2(\mathbb{R}^2)}+\Big\|\big(\frac{x^2}{2}-s^*-s\big)\hat{w}\Big\|^2_{L^2(\mathbb{R}^2)}\\
&=\int_{\mathbb{R}^2}\Big\{|\hat{w}'_{\|,s}(x,s)|^2+\big(\frac{x^2}{2}-s^*-s\big)^2|\hat{w}_{\|,s}(x,s)|^2\Big\}\,dxds\\
&\quad+\int_{\mathbb{R}^2}\Big\{|\hat{w}'_{\perp,s}(x,s)|^2+\big(\frac{x^2}{2}-s^*-s\big)^2|\hat{w}_{\perp,s}(x,s)|^2\Big\}\,dxds\\
&\quad+2\mathrm{Re}\int_{\mathbb{R}^2}\Big\{\hat{w}'_{\|,s}(x,s)\bar{\hat{w}}'_{\perp,s}(x,s)+\big(\frac{x^2}{2}-s^*-s\big)^2\hat{w}_{\|,s}(x,s)\bar{\hat{w}}_{\perp,s}(x,s)\Big\}\,dxds\,.
\end{aligned}\tag{2.22}$$

From the definition of $E_0(\cdot)$ and its positivity, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^2}\{|\hat{w}'_{\|,s}(x,s)|^2+\big(\frac{x^2}{2}-s^*-s\big)^2|\hat{w}_{\|,s}(x,s)|^2\}\,dxds\\
&=\int_{\mathbb{R}^2}E_0(s^*+s)|\hat{w}_{\|,s}(x,s)|^2\,dxds=\int_{-1}^{1}\int_{\mathbb{R}}E_0(s^*+s)|\hat{w}_{\|,s}(x,s)|^2\,dxds\,.
\end{aligned}$$

For each fixed $s\in[-1,1]$, we see from (2.17) and (2.18) that $\hat{w}_{\perp,s}(\cdot,s)$ is orthogonal to the eigenfunction $\phi_0(\cdot,s^*+s)$ associated with the first eigenvalue $E_0(s^*+s)$ (recall that $E_1(s^*+s)$ is the second eigenvalue). Hence, we have

$$\int_{\mathbb{R}}\Big\{|\hat{w}'_{\perp,s}(x,s)|^2+\big(\frac{x^2}{2}-s^*-s\big)^2|\hat{w}_{\perp,s}(x,s)|^2\Big\}dx\geq E_1(s^*+s)\int_{\mathbb{R}}|\hat{w}_{\perp,s}(x,s)|^2\,dx\,.$$

Consequently,

$$\begin{aligned}
&\int_{\mathbb{R}^2}\Big\{|\hat{w}'_{\perp,s}(x,s)|^2+\big(\frac{x^2}{2}-s^*-s\big)^2|\hat{w}_{\perp,s}(x,s)|^2\Big\}dxds\\
&\geq\int_{-1}^{1}E_1(s^*+s)\int_{\mathbb{R}}|\hat{w}_{\perp,s}(x,s)|^2\,dxds\\
&\quad+\int_{|s|>1}E_0(s^*+s)\int_{\mathbb{R}}|\hat{w}_{\perp,s}(x,s)|^2\,dxds\,.
\end{aligned}$$

From (2.17) and (2.2) we see that

$$\begin{aligned}
&\mathrm{Re}\int_{\mathbb{R}^2}\Big\{\hat w'_{\|,s}(x,s)\bar{\hat w}'_{\perp,s}(x,s)+\big(\frac{x^2}{2}-s^*-s\big)^2\hat w_{\|,s}(x,s)\bar{\hat w}_{\perp,s}(x,s)\Big\}\,dxds\\
=&\mathrm{Re}\int_{-1}^{1}\hat b(s^*+s)ds\int_{\mathbb{R}}\Big\{\phi_0'(x,s^*+s)\bar{\hat w}'_{\perp,s}(x,s)+\big(\frac{x^2}{2}-s^*-s\big)^2\phi_0(x,s^*+s)\bar{\hat w}_{\perp,s}(x,s)\Big\}\,dx\\
=&0\,.
\end{aligned}$$

Substituting the above into (2.22) yields

$$\begin{aligned}
\|\hat w'\|^2_{L^2(\mathbb{R}^2)}&+\Big\|\big(\frac{x^2}{2}-s^*-s\big)\hat w\Big\|^2_{L^2(\mathbb{R}^2)}\\
&\geq\int_{-1}^{1}\int_{\mathbb{R}}\big\{E_0(s^*+s)|\hat w_{\|,s}(x,s)|^2+E_1(s^*+s)|\hat w_{\perp,s}(x,s)|^2\big\}\,dxds\\
&\quad+\int_{|s|>1}E_0(s^*+s)\int_{\mathbb{R}}|\hat w_{\perp,s}(x,s)|^2\,dxds\,.
\end{aligned}$$

We can now use the above inequality in conjunction with (2.16) and (2.19) to obtain

$$\begin{aligned}
&\int_{-1}^{1}[E_0(s^*+s)-E_0^*]\|\hat w_{\|,s}(\cdot,s)\|^2_{L^2(\mathbb{R})}+[E_1(s^*+s)-E_0^*]\|\hat w_{\perp,s}\|_{L^2(\mathbb{R})}\,ds\\
&\quad+\int_{|s|>1}(E_0(s^*+s)-E_0^*)\|\hat w_{\perp,s}\|^2\,ds\\
&\qquad\leq\ |\mathrm{Re}\,\langle\hat w,\hat g\rangle_{L^2(\mathbb{R}^2)}|\,,
\end{aligned}$$

implying that

$$\begin{aligned}
&\int_{-1}^{1}[E_0(s^*+s)-E_0^*]\|\hat w_{\|}(\cdot,s)\|^2_{L^2(\mathbb{R})}\,ds\\
&\quad+\min\Big((E_1^*-E_0),\inf_{|s|>1}(E_0(s+s^*)-E_0^*)\Big)\int_{\mathbb{R}}\|\hat w_{\perp,s}\|^2\,ds\\
&\qquad\leq\ |\mathrm{Re}\,\langle\hat w,\hat g\rangle_{L^2(\mathbb{R}^2)}|\,,
\end{aligned}$$

which readily yields (2.20).

Step 2. Next we show that there exists a constant $\widehat{C}$ such that:

$$\|s\hat b(s^*+\cdot)\|^2_{L^2(\mathbb{R})}+\|\hat w_{\perp,s}\|^2_{L^2(\mathbb{R}^2)}\leq\ \widehat{C}\,\|w\|_{L^2(\mathcal{S}_2)}\|g\|_{L^2(\mathcal{S}_2)}\,. \tag{2.23}$$

Recall that g is supported in $\mathcal{S}_2$. Hence,

$$|\mathrm{Re}\,\langle\hat w,\hat g\rangle_{L^2(\mathbb{R}^2)}|=\ |\mathrm{Re}\,\langle w,g\rangle_{L^2(\mathbb{R}^2)}|=\ |\mathrm{Re}\,\langle w,g\rangle_{L^2(\mathcal{S}_2)}|\leq\|w\|_{L^2(\mathcal{S}_2)}\|g\|_{L^2(\mathcal{S}_2)}\,.$$

From [12] we learn that there exists $C_0>0$, such that we have

$$E_0(s^*+s)-E_0^*\geq C_0\,s^2\quad\text{for all } s\in[-1,1]\,. \tag{2.24}$$

Consequently, we have

$$\begin{aligned}&\int_{-1}^{1}\int_{\mathbb{R}}[E_0(s^*+s)-E_0^*]|\hat{w}_{\|,s}(x,s)|^2\,dxds \geq C_0\,\int_{-1}^{1}\int_{\mathbb{R}}s^2|\hat{w}_{\|,s}(x,s)|^2\,dxds\\ =&C_0\,\int_{-1}^{1}\int_{\mathbb{R}}s^2|\hat{b}(s^*+s)\phi_0(x,s^*+s)|^2\,dxds = C_0\,\int_{-1}^{1}|s\hat{b}(s^*+s)|^2\,ds\\ =&C_0\,\|s\hat{b}(s^*+\cdot)\|^2_{L^2(\mathbb{R})}\,.\end{aligned}$$

Let

$$C_1=\min\{C_0,E_1^*-E_0^*,E_0(-1+s^*)-E_0^*,E_0(1+s^*)-E_0^*\}\,.$$

Then we use (2.21) to get

$$C_1\left[\|s\hat{b}(s^*+\cdot)\|^2_{L^2(\mathbb{R})}+\|\hat{w}_{\perp,s}\|^2_{L^2(\mathbb{R}^2)}\right]\leq\|w\|_{L^2(\mathcal{S}_2)}\|g\|_{L^2(\mathcal{S}_2)}\,,$$

from which (2.23) follows.

Step 3. Let $w_{\|,s}$ and $w_{\perp,s}$ respectively denote the inverse partial Fourier transform of $\hat{w}_{\|,s}$ and $\hat{w}_{\perp,s}$. We next attempt to control $\frac{\partial}{\partial y}|w_{\perp,s}|$.

Computing the left-hand-side of (2.16) as in (2.22) we obtain that

$$\int_{\mathbb{R}^2}\left\{|\hat{w}'_{\perp,s}|^2+\big(\frac{x^2}{2}-s^*-s\big)^2|\hat{w}_{\perp,s}|^2-E_0^*|\hat{w}_{\perp,s}|^2\right\}dxds\leq\,\left|\langle w,g\rangle_{L^2(\mathcal{S}_2)}\right|.$$

Note that, unlike w, the support of $w_{\perp,s}$ extends beyond $\mathcal{S}_R$ to the entire plane. Thus,

$$\Big\|\big(\frac{x^2}{2}-s^*-s\big)\hat{w}_{\perp,s}\Big\|_{L^2(\mathbb{R}^2)}=\Big\|\frac{\partial w_{\perp,s}}{\partial y}-i\big(\frac{x^2}{2}-s^*\big)w_{\perp,s}\Big\|_{L^2(\mathbb{R}^2)}.$$

Consequently, we have

$$\Big\|\frac{\partial w_{\perp,s}}{\partial y}-i\big(\frac{x^2}{2}-s^*\big)w_{\perp,s}\Big\|^2_{L^2(\mathbb{R}^2)}\leq\,\left|\langle w,g\rangle_{L^2(\mathcal{S}_2)}\right|+E_0^*\,\|w_{\perp,s}\|^2_{L^2(\mathbb{R}^2)}\,. \tag{2.25}$$

With the aid of Kato's inequality we then obtain

$$\begin{aligned}\Big\|\frac{\partial|w_{\perp,s}|}{\partial y}\Big\|^2_{L^2(\mathbb{R}^2)}&\leq\Big\|\frac{\partial w_{\perp,s}}{\partial y}-i\big(\frac{x^2}{2}-s^*\big)w_{\perp,s}\Big\|^2_{L^2(\mathbb{R}^2)}\\ &\leq\|w\|_{L^2(\mathcal{S}_2)}\|g\|_{L^2(\mathcal{S}_2)}+E_0^*\,\|\hat{w}_{\perp,s}\|^2_{L^2(\mathbb{R}^2)}\,.\end{aligned} \tag{2.26}$$

Combining the above inequality with (2.23) yields

$$\|s\hat{b}(s^*+\cdot)\|^2_{L^2(\mathbb{R})}+\|\hat{w}_{\perp,s}\|^2_{L^2(\mathbb{R}^2)}+\Big\|\frac{\partial|w_{\perp,s}|}{\partial y}\Big\|^2_{L^2(\mathbb{R}^2)}\leq\,C_3\,\|w\|_{L^2(\mathcal{S}_2)}\|g\|_{L^2(\mathcal{S}_2)}\,. \tag{2.27}$$

Note that $\frac{\partial w_{\|,s}}{\partial y}$ is in $L^2(\mathbb{R}^2,\mathbb{C})$. In fact, as its partial Fourier transform is given by

$$\mathcal{F}\Big\{\frac{\partial w_{\|,s}}{\partial y}\Big\}=i\,s\,\hat{b}(s^*+s)\phi_0(x,s^*+s)\,,$$

it follows by (2.27) that

$$\Big\|\frac{\partial w_{\|,s}}{\partial y}\Big\|_{L^2(\mathbb{R}^2)}=\|s\hat{b}(s^*+\cdot)\phi_0(\cdot,s^*+\cdot)\|_{L^2(\mathbb{R}^2)}=\|s\hat{b}(s^*+\cdot)\|_{L^2(\mathbb{R})}\,. \tag{2.28}$$

We now use the continuity of the trace on $y=0$ of $|w_{\perp,s}| \in H^1(\mathbb{R}^2)$ in order to obtain

$$\|w_{\perp,s}(\cdot,0)\|^2_{L^2(\mathbb{R})} \leq \|w_{\perp,s}\|^2_{L^2(\mathbb{R}^2)} + \Big\|\frac{\partial |w_{\perp,s}|}{\partial y}\Big\|^2_{L^2(\mathbb{R}^2)}\,.$$

As

$$w_{\|,s}(x,0) + w_{\perp,s}(x,0) = 0\,,$$

we also have that

$$\|w_{\|,s}(\cdot,0)\|^2_{L^2(\mathbb{R})} = \|w_{\perp,s}(\cdot,0)\|^2_{L^2(\mathbb{R})} \leq \|w_{\perp,s}\|^2_{L^2(\mathbb{R}^2)} + \Big\|\frac{\partial |w_{\perp,s}|}{\partial y}\Big\|^2_{L^2(\mathbb{R}^2)}\,.$$

Let $l>0$. As for all $y\in[0,l]$

$$|w_{\|,s}(x,y)| = |w_{\|,s}(x,0)| + \int_0^y \frac{\partial |w_{\|,s}|}{\partial \eta}(x,\eta)d\eta \leq |w_{\|,s}(x,0)| + \sqrt{y}\Big[\int_0^l \Big|\frac{\partial |w_{\|,s}|}{\partial y}(x,y)\Big|^2 dy\Big]^{1/2}\,,$$

we have, for all $y\in[0,l]$, that

$$\|w_{\|,s}(\cdot,y)\|^2_{L^2(\mathbb{R})} \leq 2\,\|w_{\|,s}(\cdot,0)\|^2_{L^2(\mathbb{R})} + l^2\,\Big\|\frac{\partial |w_{\|,s}|}{\partial y}\Big\|^2_{L^2(\mathcal{S}_l)}$$

Consequently

$$\|w_{\|,s}\|^2_{L^2(\mathcal{S}_l)} \leq l\left\{2\|w_{\perp,s}\|^2_{L^2(\mathbb{R}^2)} + 2\,\Big\|\frac{\partial |w_{\perp,s}|}{\partial y}\Big\|^2_{L^2(\mathbb{R}^2)} + l^2\Big\|\frac{\partial |w_{\|,s}|}{\partial y}\Big\|^2_{L^2(\mathcal{S}_l)}\right\}\,. \tag{2.29}$$

Using (2.28) we have

$$\Big\|\frac{\partial |w_{\|,s}|}{\partial y}\Big\|^2_{L^2(\mathcal{S}_l)} \leq \Big\|\frac{\partial w_{\|,s}}{\partial y}\Big\|^2_{L^2(\mathbb{R}^2)} = \|s\hat{w}_{\|,s}\|^2_{L^2(\mathbb{R}^2)} = \|s\hat{b}(s^*+\cdot)\|^2_{L^2(\mathbb{R})}\,. \tag{2.30}$$

Thus combining the above inequality and (2.29) yields

$$\begin{aligned}\|w\|_{L^2(\mathcal{S}_l)} &\leq \|w_{\|,s}\|_{L^2(\mathcal{S}_l)} + \|w_{\perp,s}\|_{L^2(\mathcal{S}_l)}\\ &\leq (\sqrt{2l}+1)\,\|w_{\perp,s}\|_{L^2(\mathbb{R}^2)} + \sqrt{2l}\,\Big\|\frac{\partial |w_{\perp,s}|}{\partial y}\Big\|_{L^2(\mathbb{R}^2)} + l^{3/2}\|s\hat{b}(s^*+\cdot)\|_{L^2(\mathbb{R})}\,.\end{aligned} \tag{2.31}$$

Substituting the above, with $l=2$, into (2.27) we obtain that

$$\begin{aligned}&\|s\hat{b}(s^*+\cdot)\|^2_{L^2(\mathbb{R})} + \|\hat{w}_{\perp,s}\|^2_{L^2(\mathbb{R}^2)} + \Big\|\frac{\partial |w_{\perp,s}|}{\partial y}\Big\|^2_{L^2(\mathbb{R}^2)}\\ &\leq 2\sqrt{2}C_3\Big(\|w_{\perp,s}\|_{L^2(\mathbb{R}^2)} + \Big\|\frac{\partial |w_{\perp,s}|}{\partial y}\Big\|_{L^2(\mathbb{R}^2)} + \|s\hat{b}(s^*+\cdot)\|_{L^2(\mathbb{R})}\Big)\|g\|_{L^2(\mathcal{S}_2)}\,,\end{aligned}$$

from which we easily obtain that

$$\|s\hat{b}(s^*+\cdot)\|_{L^2(\mathbb{R})} + \|\hat{w}_{\perp,s}\|_{L^2(\mathbb{R}^2)} + \Big\|\frac{\partial |w_{\perp,s}|}{\partial y}\Big\|_{L^2(\mathbb{R}^2)} \leq C_4\,\|g\|_{L^2(\mathcal{S}_2)}\,. \tag{2.32}$$

From (2.32) and (2.31), we then get (2.14f).

Next we use (2.15) to obtain

$$\Big\|\frac{\partial w}{\partial x}\Big\|^2_{L^2(\mathcal{S}_R)} \leq \Big\|\big(\nabla - i\big[\frac{x^2}{2} - s^*\big]\hat{\mathbf{i}}_y\big)w\Big\|^2_{L^2(\mathcal{S}_R)} = E_0^*\,\|w\|^2_{L^2(\mathcal{S}_R)} + \mathrm{Re}\,\langle w, g\rangle_{L^2(\mathcal{S}_2)}$$
$$\leq (E_0^* + \frac{1}{2})\|w\|^2_{L^2(\mathcal{S}_R)} + \frac{1}{2}\|g\|^2_{L^2(\mathcal{S}_R)}.$$

From the above and (2.14f) we get (2.14e).

Step 4. We now prove (2.14b,c). To this end we decompose $\hat{w}$ yet another time

$$\hat{w}(x,s) = \hat{w}_{\perp,s}(x,s) + \hat{b}(s^*+s)\phi^*(x) + \hat{b}(s^*+s)[\phi_0(x,s^*+s) - \phi^*(x)]\,. \tag{2.33}$$

As there exists C such that

$$\|\phi^*(\cdot) - \phi_0(\cdot, s^*+s)\|_{L^2(\mathbb{R})} \leq \, C\,|s| \quad \text{for all } s \in [-1,1]\,,$$

we obtain using (2.32) that

$$\|\hat{b}(s^*+s)[\phi_0(x,s^*+s) - \phi^*(x)]\|_{L^2(\mathbb{R}^2)} \leq \, C_5\,\|s\hat{b}(s^*+\cdot)\|_{L^2(\mathbb{R})} \leq CC_4\|g\|_{L^2(\mathcal{S}_2)}\,.$$

Thus, by setting v^R to be the inverse Fourier transform of

$$\hat{v}^R(x,s) = \hat{w}_{\perp,s}(x,s) + \hat{b}(s^*+s)[\phi_0(x,s^*+s) - \phi^*(x)]\,,$$

(2.14b) is readily satisfied.

Reintroducing the reference to R, let $b_R(y)$ denote the inverse Fourier transform of $\hat{b}(s^*+s)$. As

$$\|b_R'\|_{L^2(0,R)} \leq \|b_R'\|_{L^2(\mathbb{R})} = \|s\hat{b}(s^*+\cdot)\|_{L^2(\mathbb{R})}\,,$$

(2.14c) readily follows from (2.32).

Step 5. To complete the proof of the lemma, we make the obvious observation that, by (2.25) and (2.32)

$$\Big\|\frac{\partial w_{\perp,s}}{\partial y} - i\big(\frac{x^2}{2} - s^*\big)w_{\perp,s}\Big\|_{L^2((-L,L)\times(0,R))} \leq \big|\langle w, g\rangle_{L^2(\mathcal{S}_2)}\big|^{1/2} + (E_0^*)^{1/2}\|w_{\perp,s}\|_{L^2(\mathbb{R}^2)}\,.$$

Hence,

$$\Big\|\frac{\partial w_{\perp,s}}{\partial y}\Big\|_{L^2((-L,L)\times(0,R))}$$
$$\leq \frac{1}{2}\|w\|_{L^2(\mathcal{S}_2)} + \frac{1}{2}\|g\|_{L^2(\mathcal{S}_2)} + (E_0^*)^{1/2}\|w_{\perp,s}\|_{L^2(\mathbb{R}^2)} + \|\big(\frac{x^2}{2} - s^*\big)w_{\perp,s}\Big\|_{L^2((-L,L)\times(0,R))}.$$

Then we use (2.32) and (2.14f) to obtain the existence of a constant C such that, for any $L > \max\{1, 2\sqrt{s^*}\}$,

$$\Big\|\frac{\partial w_{\perp,s}}{\partial y}\Big\|_{L^2((-L,L)\times(0,R))} \leq C\,L^2\|g\|_{L^2(\mathcal{S}_2)}\,.$$

The above, together with (2.30) and (2.32), proves (2.14e) and thus completes the proof of the lemma. ∎

Once (2.14) is obtained, there is no further necessity to discuss (2.9) in $\mathcal{S}_R$. We consider then the limit $R \to +\infty$.

Lemma 2.4. *With the notation of Lemma 2.3, there exist $C > 0$ and, for any $g \in L^2(\mathbb{R}^2_+, \mathbb{C})$ which is supported on $\mathcal{S}_2$, a sequence $\{R_k\}_{k\in\mathbb{N}}$ tending to $+\infty$ and functions w_∞, v_∞ and b_∞ such that the following claims are true:*

(i) *We have $w^{R_k} \to w_\infty$ and $v_{R_k} \to v_\infty$ strongly in $H^1_{\mathrm{loc}}(\overline{\mathbb{R}^2_+}, \mathbb{C})$, and $b_{R_k} \to b_\infty$ strongly in $H^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{C})$ as $k \to \infty$.*

(ii) *w_∞ is a solution of*

$$\begin{cases} (\mathcal{A}_0 - E_0^*)w_\infty = g & \text{in } \mathbb{R}^2_+\,, \\ w_\infty = 0 & \text{on } \partial\mathbb{R}^2_+\,. \end{cases} \tag{2.34}$$

(iii) *w_∞ admits the representation*

$$w_\infty(x,y) = v_\infty(x,y) + b_\infty(y)\phi^*(x)\,, \tag{2.35a}$$

where v_∞ and $b_\infty(y)$ satisfy the estimates

$$\|v_\infty\|_{L^2(\mathbb{R}^2_+)} \le C\,\|g\|_{L^2(\mathcal{S}_2)}\,, \tag{2.35b}$$

$$\|b'_\infty\|_{L^2(\mathbb{R}_+)} \le C\,\|g\|_{L^2(\mathcal{S}_2)}\,. \tag{2.35c}$$

(iv) *For all $L > 1$ we have*

$$\Big\|\frac{\partial w_\infty}{\partial y}\Big\|_{L^2((-L,L)\times\mathbb{R}_+)} \le C\,L^2\|g\|_{L^2(\mathcal{S}_2)}. \tag{2.35d}$$

Proof. By (2.14), the family $\{|w^R| : R \ge 2\}$ is uniformly bounded in $H^1_{\mathrm{loc}}(\overline{\mathbb{R}^2_+}, \mathbb{C})$, and hence, by standard elliptic estimates, since $g \in L^2(\mathbb{R}^2_+, \mathbb{C})$ and w_R is a solution of (2.9) , the family $\{|w^R| : R \ge 2\}$ is uniformly bounded in $H^2_{\mathrm{loc}}(\overline{\mathbb{R}^2_+}, \mathbb{C})$. Hence there exists a sequence $\{R_k\}_{k=1}^\infty$ with $R_k \to +\infty$ as $k \to +\infty$ such that w^{R_k} converges weakly in $H^2_{\mathrm{loc}}(\overline{\mathbb{R}^2_+}, \mathbb{C})$ and strongly in $H^1_{\mathrm{loc}}(\overline{\mathbb{R}^2_+}, \mathbb{C})$ to some function $w_\infty \in H^2_{\mathrm{loc}}(\overline{\mathbb{R}^2_+}, \mathbb{C})$. In particular $w_\infty(\cdot, 0) = 0$ holds in the sense of trace in $H^{1/2}_{\mathrm{loc}}(\partial\mathbb{R}^2_+, \mathbb{C})$. Hence w_∞ solves (2.34).

In view of (2.14b,c,f), b^{R_k} is bounded in $H^1_{\mathrm{loc}}(\mathbb{R}_+)$ and, by moving to a subsequence (still denoted by $\{R_k\}$), there exists some $b_\infty \in H^1_{\mathrm{loc}}(\mathbb{R}_+)$ such that $b^{R_k} \to b_\infty$ weakly in $H^1_{\mathrm{loc}}(\mathbb{R}_+)$ and strongly in $L^2_{\mathrm{loc}}(\mathbb{R}_+)$. Furthermore, for every $l > 0$ we have that

$$\|b_\infty\|_{H^1(0,l)} \le \liminf_{k\to\infty} \|b^{R_k}\|_{H^1(0,l)}\,.$$

Hence, in view of (2.14c), there exists $C > 0$ such that for every $l > 0$

$$\|b'_\infty\|_{L^2(0,l)} \le C\,\|g\|_{L^2(\mathcal{S}_2)}\,.$$

The proof of (2.35a,b,c) now follows from the strong convergence in $L^2_{\mathrm{loc}}(\mathbb{R}^2_+)$ of w^{R_k} and $b^{R_k}\phi^*$. To prove (2.35d) we use the H^1_{loc} convergence of w^{R_k} and (2.14e). ∎

Remark 2.5. *Note that, in view of (2.14f), $w_\infty \in L^2(\mathcal{S}_l)$ for any $l > 0$. In fact, there exists $C > 0$ such that for every $l > 0$ we have*

$$\|w_\infty\|_{L^2(\mathcal{S}_l)} \le C(1 + l^{3/2})\|g\|_{L^2(\mathcal{S}_2)}\,. \tag{2.36}$$

For convenience we drop the subscript ∞ in the sequel and represent w_∞ by w. Let $f \in \mathcal{S}(\mathbb{R})$ and χ be given by (2.6). Let further w denote a solution of (2.8) with

$$g = (\mathcal{A}_0 - E_0^*)\big(\chi(y)f(x)\big)\,,$$

which satisfies (2.35) and (2.36). Then, u – the corresponding solution of (2.5) – which is obtained via (2.7), satisfies (2.35) and (2.36) as well. In the following we show that this u satisfies (2.5c). To this end we need to prove that $b(y)$ converges to a positive constant as $y \to +\infty$. Note that the estimates of $\|b'\|_{L^2(\mathbb{R}^2)}$ we currently have are insufficient in order to prove such a convergence. To close this gap we need a decay estimate of $\|\frac{\partial w}{\partial y}\|_{L^2((-L,L)\times(R,\infty))}$ as $R \to +\infty$. The following lemma – a rather standard estimate for solutions of (2.5a) – constitutes a preliminary step towards this end.

Lemma 2.6. *There exist C and, for any $k \geq 1$, $C(k)$ such that any solution u of (2.5a) satisfies, for any $l > 2$ and any $L \geq 2$,*

$$\|x^k u\|_{L^2(\mathbb{R}\times(l-1,l+1))} \leq C(k)\, \|u\|_{L^2(\mathbb{R}\times(l-2,l+2))}\,, \tag{2.37a}$$

$$\|u(\cdot,l)\|_{L^2(\mathbb{R}\setminus[-L,L])} \leq \frac{C(k)}{L^k}\, \|u\|_{L^2(\mathbb{R}\times(l-2,l+2))}\,, \tag{2.37b}$$

$$\Big\|\frac{\partial u}{\partial y}\Big\|_{L^2(\mathbb{R}\times(l-1,l+1))} \leq C\, \|u\|_{L^2(\mathbb{R}\times(l-2,l+2))}\,. \tag{2.37c}$$

Proof. The arguments we apply here are similar to those used in the proof of Lemma 5.2 in [5]. Let χ be given by (2.6). Then, set

$$\chi_r(x) = \chi\Big(\frac{|x|}{r}\Big), \qquad \eta(x,y) = [1 - \chi(|x| - L)]\chi(|y - l|)\,.$$

Let $l > 2$. Multiplying (2.5a) by $x^{2k}\chi_r^2\eta^2\bar{u}$, which vanishes near $\partial\mathbb{R}^2_+$, and integrating over $\mathbb{R}^2_+$ yields

$$\Big\|\big(\nabla - i\big[\frac{x^2}{2} - s^*\big]\hat{\mathbf{i}}_y\big)(x^k\chi_r\eta u)\Big\|^2_{L^2(\mathbb{R}^2_+)} - \Big\|u\nabla(\chi_r x^k\eta)\Big\|^2_{L^2(\mathbb{R}^2_+)} = E_0^*\Big\|ux^k\chi_r\eta\Big\|^2_{L^2(\mathbb{R}^2_+)}\,. \tag{2.38}$$

We now use [15, Theorem 4] with

$$\mathbf{A} = (0, \frac{x^2}{2} - s^*), \qquad B = \operatorname{curl}\mathbf{A} = x$$

to get, for any $\phi \in H^1_0(\mathbb{R}^2_+, \mathbb{C})$,

$$\int_{\mathbb{R}^2_+} \Big|\big(\nabla - i\big[\frac{x^2}{2} - s^*\big]\hat{\mathbf{i}}_y\big)\phi\Big|^2\, dxdy \geq \Big|\int_{\mathbb{R}^2_+} x|\phi|^2\, dxdy\Big|\,. \tag{2.39}$$

We first choose in (2.39)

$$\phi = \xi_+(x)x^k\chi_r\eta u\,,$$

where ξ_+ is a Heavyside function, i.e.

$$\xi_+(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}\,,$$

to obtain

$$\int_{\mathbb{R}^2_+\cap\{x>0\}}\Big|\big(\nabla - i\big[\frac{x^2}{2}-s^*\big]\hat{\mathbf{i}}_y\big)x^k\chi_r\eta u\Big|^2\,dxdy \geq \int_{\mathbb{R}^2_+\cap\{x>0\}} x|x^k\chi_r\eta u|^2\,dxdy\,.$$

We next substitute into (2.39) a different choice of ϕ

$$\phi = -\xi_-(x)x^k\chi_r\eta u\,,$$

where $\xi_-(x) = 1-\xi_+(x)$, to obtain

$$\int_{\mathbb{R}^2_+\cap\{x<0\}}\Big|\big(\nabla - i\big[\frac{x^2}{2}-s^*\big]\hat{\mathbf{i}}_y\big)x^k\chi_r\eta u\Big|^2\,dxdy \geq \int_{\mathbb{R}^2_+\cap\{x<0\}} |x||x^k\chi_r\eta u|^2\,dxdy\,.$$

Summarizing the above pair of inequalities together yields

$$\int_{\mathbb{R}^2_+}\Big|\big(\nabla - i\big[\frac{x^2}{2}-s^*\big]\hat{\mathbf{i}}_y\big)x^k\chi_r\eta u\Big|^2\,dxdy \geq \int_{\mathbb{R}^2_+} |x^{k+1}\chi_r\eta u|^2\,dxdy\,.$$

Substituting the above inequality into (2.38) leads to

$$\begin{aligned}\Big\|x^{k+1/2}u\eta\Big\|^2_{L^2(\mathbb{R}\times(l-1,l+1))} &\leq \int_{\mathbb{R}^2_+}|x^{k+1}\chi_r\eta u|^2\,dxdy\\ \leq &\Big\|u\nabla(\chi_r x^k\eta)\Big\|^2_{L^2(\mathbb{R}^2_+)} + E_0^*\Big\|ux^k\chi_r\eta\Big\|^2_{L^2(\mathbb{R}^2_+)} \leq C_1(k,r)\Big\|ux^k\Big\|^2_{L^2(\mathbb{R}\times(l-2,l+2))}\,,\end{aligned}$$

where $C_1(k,r)$ is uniformly bounded for $r\geq 2$. Taking the limit $r\to\infty$ we obtain (2.37a) by invoking inductive arguments.

To prove (2.37b) we let $r\to+\infty$ in (2.38) to obtain

$$\begin{aligned}\Big\|\big(\partial_y - i\big[\frac{x^2}{2}-s^*\big]\big)(x^k\eta u)\Big\|^2_{L^2(\mathbb{R}^2_+)} &\leq \Big\|\big(\nabla - i\big[\frac{x^2}{2}-s^*\big]\hat{\mathbf{i}}_y\big)(x^k\eta u)\Big\|^2_{L^2(\mathbb{R}^2_+)}\\ =&\Big\|u\nabla(x^k\eta)\Big\|^2_{L^2(\mathbb{R}^2_+)} + E_0^*\Big\|ux^k\eta\Big\|^2_{L^2(\mathbb{R}^2_+)} \leq C_2(k)\,\Big\|ux^k\Big\|^2_{L^2(\mathbb{R}\times(l-2,l+2))}\,.\end{aligned}$$

Hence

$$\begin{aligned}\Big\|x^k\frac{\partial(\eta u)}{\partial y}\Big\|^2_{L^2(\mathbb{R}^2_+)} \leq& 2\Big\|\big(\partial_y - i\big[\frac{x^2}{2}-s^*\big]\big)(x^k\eta u)\Big\|^2_{L^2(\mathbb{R}^2_+)} + 2\Big\|\big[\frac{x^2}{2}-s^*\big](x^k\eta u)\Big\|^2_{L^2(\mathbb{R}^2_+)}\\ \leq& 2C_2(k)\Big\|ux^k\Big\|^2_{L^2(\mathbb{R}\times(l-2,l+2))} + 2\Big\|\big[\frac{x^2}{2}-s^*\big](x^k\eta u)\Big\|^2_{L^2(\mathbb{R}\times(l-2,l+2))}\,.\end{aligned}$$

Then, by (2.37a) it follows that

$$\Big\|x^k\frac{\partial(\eta u)}{\partial y}\Big\|^2_{L^2(\mathbb{R}^2_+)} \leq C_3(k)\|u\|^2_{L^2(\mathbb{R}\times(l-2,l+2))}\,, \tag{2.40}$$

where $C_3(k) = 2[C_2(k)+2][C(k)+C(k+2)]$. Hence,

$$\Big\|\frac{\partial u}{\partial y}\Big\|^2_{L^2(\mathbb{R}\setminus[-L,L]\times(l-1,l+1))} \leq \frac{C_3(k)}{L^{2k}}\|u\|^2_{L^2(\mathbb{R}\times(l-2,l+2))}\,. \tag{2.41}$$

Let $y_1\in[l-1,l+1]$ satisfy

$$\|u(\cdot,y_1)\|_{L^2(\mathbb{R}\setminus[-L,L])} = \min_{y\in[l-1,l+1]}\|u(\cdot,y)\|_{L^2(\mathbb{R}\setminus[-L,L])}\,. \tag{2.42}$$

Clearly, there exists C_4, such that for any $L\geq 2$, $l>2$, and $y\in[l-1,l+1]$ we have

$$\int_{\{|x|>L\}}|u(x,y)-u(x,y_1)|^2\,dx\leq C_4\left\|\frac{\partial u}{\partial y}\right\|^2_{L^2(\mathbb{R}\setminus[-L,L]\times(l-1,l+1))}. \tag{2.43}$$

Furthermore, by (2.37a)

$$\|u(\cdot,y_1)\|_{L^2(\mathbb{R}\setminus[-L,L])}\leq\|u\|_{L^2(\mathbb{R}\setminus[-L,L]\times(l-1,l+1))}\leq\frac{C(k)}{L^k}\|u\|^2_{L^2(\mathbb{R}\times(l-2,l+2))}. \tag{2.44}$$

Combining (2.42), (2.43), and (2.44) yields (2.37b).

The proof of (2.37c) easily follows from (2.40) with $k=0$, by letting $L\to\infty$. ∎

We can now prove the decay estimate of $\frac{\partial w}{\partial y}$. Recall that we have omitted the subscript ∞ in (2.35a) and thus

$$w(x,y)=v(x,y)+b(y)\phi^*(x).$$

Lemma 2.7. *There exists $l_0>0$ and for any $k\geq 1$ a constant $C(k)$ such that, for every $l>l_0$ and $L\geq 2$, we have*

$$\left\|\frac{\partial w}{\partial y}\right\|^2_{L^2((-L,L)\times(kl+2,+\infty))}\leq\frac{C(k)L^4}{l^k}. \tag{2.45}$$

Proof. Step 1. Let $l>8$, and let K denote the integer part of $l/8$. Clearly,

$$\sum_{k=0}^{K-1}\int_{\mathbb{R}\times(l/2+4k,l/2+4(k+1))}\Big[|\phi^*(x)|^2|b'(y)|^2+|v(x,y)|^2\Big]dxdy$$
$$\leq\int_{\mathbb{R}\times(l/2,l)}\Big[|\phi^*(x)|^2|b'(y)|^2+|v(x,y)|^2\Big]dxdy.$$

Then, applying (2.35) to b and v in order to estimate the right hand side of the above inequality, yields the existence of C_1 such that, for any l,

$$K\min_{0\leq k\leq K-1}\int_{\mathbb{R}\times(l/2+4k,l/2+4(k+1))}\Big[|\phi^*(x)|^2|b'(y)|^2+|v(x,y)|^2\Big]dxdy\leq C_1\,\|g\|_{L^2(\mathbb{R}^2_+)}.$$

Consequently, for any l, there exists $y_0\in(l/2,l)$ such that

$$\int_{\mathbb{R}\times(y_0-1,y_0+3)}\Big[|\phi^*(x)|^2|b'(y)|^2+|v(x,y)|^2\Big]\,dxdy\leq\frac{C_1}{K}\|g\|_{L^2(\mathbb{R}^2_+)}\leq\frac{C_2}{l}\|g\|_{L^2(\mathbb{R}^2_+)}, \tag{2.46}$$

with $C_2=8C_1$.

Step 2. Next we set

$$u_1(x,y)=w(x,y)-b(y_0)\phi^*(x). \tag{2.47}$$

Clearly, u_1 must satisfy

$$\begin{cases}(\mathcal{A}_0-E_0^*)u_1=0 & \text{in } \mathbb{R}\times(y_0,+\infty),\\ u_1(x,y_0)=w(x,y_0)-b(y_0)\phi^* & \text{in } \mathbb{R}.\end{cases}$$

In order to facilitate the application of the estimates (2.35) we decompose w even further by writing

$$u_1(x,y)=w_1(x,y)+\chi(y-y_0)[w(x,y)-b(y_0)\phi^*(x)], \tag{2.48}$$

to obtain

$$\begin{cases} (\mathcal{A}_0 - E_0^*)w_1 = g_1 & \text{in } \mathbb{R}\times(y_0,+\infty), \\ w_1(x,y_0) = 0 & \text{in } \mathbb{R}, \end{cases} \tag{2.49}$$

where

$$\begin{aligned} g_1(x,y) =& \Big[-\chi''(y-y_0) + 2i\chi'(y-y_0)\big(\frac{x^2}{2} - s^*\big)\Big]\big(w(x,y) - b(y_0)\phi^*(x)\big) \\ & - 2\chi'(y-y_0)\frac{\partial w(x,y)}{\partial y}, \end{aligned}$$

and χ is the cut-off function defined in (2.6).

We now estimate $\|g_1\|_{L^2(\mathbb{R}^2_+)}$. Note that $\chi'(y-y_0)$ and $\chi''(y-y_0)$ are supported in the interval $[y_0+1, y_0+2]$. Hence

$$\|g_1\|_{L^2(\mathbb{R}^2_+)} = \|g_1\|_{L^2(\mathbb{R}\times(y_0,y_0+2))}.$$

Clearly, by (2.46), there exist C_3 and C_4, such that, for any l,

$$\begin{aligned} &\big\|\chi''(y-y_0)[w - b(y_0)\phi^*]\big\|^2_{L^2(\mathbb{R}^2_+)} \le C_3\|w - b(y_0)\phi^*\|^2_{L^2(\mathbb{R}\times(y_0,y_0+2))} \\ \le& 2C_3\Big[\|v\|^2_{L^2(\mathbb{R}\times(y_0,y_0+2))} + \|b - b(y_0)\|^2_{L^2(y_0,y_0+2)}\Big] \le \frac{C_4}{l}\,. \end{aligned}$$

We apply the estimates in Lemma 2.6 to u_1. Then using the exponential decay of ϕ^*, as $x\to\infty$, we have that

$$\begin{aligned} &\big\|\chi'(y-y_0)(\frac{x^2}{2} - s^*)(w - b(y_0)\phi^*)\big\|^2_{L^2(\mathbb{R}^2_+)} \\ \le& C_5\Big\|(\frac{x^2}{2} - s^*)(w - b(y_0)\phi^*)\Big\|^2_{L^2(\mathbb{R}\times(y_0+1,y_0+2))} \\ =& C_5\Big\|(\frac{x^2}{2} - s^*)u_1\Big\|^2_{L^2(\mathbb{R}\times(y_0+1,y_0+2))} \le C_6\|u_1\|^2_{L^2(\mathbb{R}\times(y_0-1,y_0+3))} \le \frac{C_7}{l}\,. \end{aligned}$$

Finally, again applying Lemma 2.6 to u_1 we have

$$\begin{aligned} &\Big\|\chi'(y-y_0)\frac{\partial w}{\partial y}\Big\|^2_{L^2(\mathbb{R}^2_+)} = \Big\|\chi'(y-y_0)\frac{\partial}{\partial y}[w(x,y) - b(y_0)\phi^*(x)]\Big\|^2_{L^2(\mathbb{R}\times(y_0+1,y_0+2))} \\ \le& C_5\Big\|\frac{\partial}{\partial y}[w(x,y) - b(y_0)\phi^*(x)]\Big\|^2_{L^2(\mathbb{R}\times(y_0+1,y_0+2))} \\ =& C_5\Big\|\frac{\partial u_1}{\partial y}\Big\|^2_{L^2(\mathbb{R}\times(y_0+1,y_0+2))} \le C_8\|u_1\|^2_{L^2(\mathbb{R}\times(y_0-1,y_0+3))} \le \frac{C_9}{l}\,. \end{aligned}$$

Consequently,

$$\|g_1\|^2_{L^2(\mathbb{R}^2_+)} \le \frac{C_{10}}{l}\,. \tag{2.50}$$

For the function w_1 defined in (2.48) which is a solution of (2.49) with $g = g_1$, we apply the estimate (2.35d) to w_1 and use (2.50) to obtain the existence of C_{11} such that, for any l,

$$\Big\|\frac{\partial w_1}{\partial y}\Big\|^2_{L^2((-L,L)\times(y_0,+\infty))} \le \frac{C_{11}}{l}L^4\,. \tag{2.51}$$

Finally by (2.47) and (2.48) we obtain that for $y > y_0$

$$w(x,y) = b(y_0)\phi^*(x) + \chi(y-y_0)[w(x,y) - b(y_0)\phi^*(x)] + w_1(x,y)\,.$$

Hence, for all $y > l+2$ we have

$$\frac{\partial w}{\partial y} = \frac{\partial w_1}{\partial y},$$

which, with the aid of (2.51), completes the proof of the lemma for $k = 1$.

The proof of (2.45) for all $k > 1$ can be easily obtained by invoking inductive arguments. ∎

Proof of Theorem 2.1.

Let u be the solution of (2.5a,b) obtained via (2.7) from a solution w of (2.8) which satisfies (2.45). Recall that to complete the proof we need to show that there exists a unique $\alpha \in \mathbb{C}$ such that $u - \alpha\phi^* \in L^2(\mathbb{R}^2_+, \mathbb{C})$ and obeys the condition (2.5c).

Step 1. We first show that $\|w(\cdot,y)\|_{L^2(\mathbb{R})}$ is convergent in $L^2(-L,L)$ for every finite L as $y \to +\infty$. Let then $z > kl+2$ for some $k \in \mathbb{N}$, and let M denote the integer part of $z - kl - 2$. Set

$$\delta = \frac{z-kl-2}{M+1}.$$

For $z > kl+3$, we clearly have, $1/2 \leq \delta \leq 1$. Using (2.45) with $l = \frac{z-2}{k}$ we then have

$$\|w(\cdot,z+\delta) - w(\cdot,z)\|^2_{L^2(-L,L)} = \int_{-L}^{L} \Big| \int_z^{z+\delta} \frac{\partial w}{\partial y}\, dy \Big|^2 dx$$

$$\leq \delta \Big\| \frac{\partial w}{\partial y} \Big\|^2_{L^2((-L,L)\times(z,z+\delta))} \leq \frac{\delta C(k) L^4}{(\frac{z-2}{k})^k} = [C(k)k^k(\frac{z}{z-2})^k]\frac{\delta L^4}{z^k} \leq \frac{\delta C'(k)L^4}{z^k}\,,$$

where $C'(k)$ depends only on k. Consequently,

$$\begin{aligned} &\|w(\cdot,z) - w(\cdot,kl+2)\|_{L^2(-L,L)} \\ \leq &\sum_{n=0}^{M} \|w(\cdot,kl+2+(n+1)\delta) - w(\cdot,kl+2+n\delta)\|_{L^2(-L,L)} \\ \leq &C_1 \sum_{n=0}^{M} \frac{\sqrt{\delta C(k)}L^2}{(kl+2+n\delta)^{k/2}} \leq \frac{C_1(k)L^2}{l^{k/2-1}}\,. \end{aligned} \tag{2.52}$$

It follows from (2.52) that $w(\cdot,y)$ is convergent in $L^2(-L,L)$ as $y \to \infty$. Denote its limit by $W(x)$. By (2.52) there exists, for all $m \geq 1$, a constant $C_2(m)$ such that

$$\|w(\cdot,y) - W(\cdot)\|_{L^2(-L,L)} \leq \frac{C_2(m)L^2}{y^m}\,. \tag{2.53}$$

To obtain W we make use of (2.35a). Set

$$\begin{cases} a(y) = b(y) + \langle v, \phi^* \rangle, \\ V(x,y) = v(x,y) - \langle v(\cdot,y), \phi^*(\cdot)\rangle_{L^2(-L,L)}\phi^*(x). \end{cases}$$

Decompose then W in the form

$$W=\alpha\phi^*+W_\perp\,,$$

where

$$\alpha=\langle W,\phi^*\rangle_{L^2(-L,L)}\,.$$

Clearly,

$$\begin{aligned}&\|V(\cdot,y)-W_\perp(\cdot)\|_{L^2(-L,L)}\\ =&\|[w(\cdot,y)-W(\cdot)]-\langle w(\cdot,y)-W(\cdot),\phi^*(\cdot)\rangle\phi^*(\cdot)\|_{L^2(-L,L)}\\ \leq&\|w(\cdot,y)-W(\cdot)\|_{L^2(-L,L)}\leq\frac{C_2(m)L^2}{y^m}\,.\end{aligned}$$

Since $V\in L^2(\mathbb{R}^2_+,\mathbb{C})$, there exists a sequence $\{y_k\}_{k\in\mathbb{N}}$ such that

$$\|V(\cdot,y_k)\|_{L^2(-L,L)}\to 0\quad\text{as } k\to+\infty\,.$$

Consequently, we have $W_\perp\equiv 0$. Thus,

$$W=\alpha\phi^*\,. \tag{2.54}$$

Step 2. To complete the proof of (2.5c) we need to prove the convergence of $w(\cdot,y)-W(\cdot)$ in $L^2(\mathbb{R},\mathbb{C})$ and its L^2 norm decays as $y\to+\infty$. Since $w-W$ is a bounded solution of (2.5a), we apply Lemma 2.6 to $u=w-W$ and find that for any $k\geq 1$ there exist constants $C(k)>0$ and $L(k)$ such that, for $L\geq L(k)$,

$$\|w(\cdot,y)-W(\cdot)\|_{L^2(\mathbb{R}\setminus[-L,L])}\leq\frac{C(k)}{L^k}\|w-W\|_{L^2((-L,L)\times(y-2,y+2))}\leq\frac{C_2(m)C(k)L^2}{L^ky^m}\,. \tag{2.55}$$

Combining (2.53) and (2.55) with $L=y$ and with $k=2$ we have, for sufficiently large y that

$$\|w(\cdot,y)-W(\cdot)\|_{L^2(\mathbb{R})}\leq\frac{C_3(m)}{y^m}\,.$$

This proves the convergence of $w(\cdot,y)-W(\cdot)$ to 0 in $L^2(\mathbb{R},\mathbb{C})$ as $y\to+\infty$.

Step 3. We now prove uniqueness of the solutions of (2.5). Suppose that there exists some $\tilde{u}$ satisfying (2.5a,b), but it tends to $\beta\phi^*$ in (2.5c) for some $\beta\in\mathbb{C}$. Then $U=u-\tilde{u}$ satisfies (2.5) with $f=0$ in (2.5b) and $\gamma=\alpha-\beta$ instead of α in (2.5c). Let

$$\psi=\begin{cases}\frac{\bar\gamma}{|\gamma|}(U-\gamma\phi^*) & \text{if } \gamma\neq 0\,,\\ U & \text{if } \gamma=0\,.\end{cases}$$

Then, $\psi\in H^{2,\mathrm{mag}}(\mathbb{R}^2_+,\mathbb{C})$ and satisfies

$$(\mathcal{A}_0-E_0^*)\psi=0\quad\text{in } \mathbb{R}^2_+\,, \tag{2.56a}$$

$$\psi=|\gamma|\phi^*\quad\text{on } \partial\mathbb{R}^2_+\,. \tag{2.56b}$$

Let further

$$\tilde\psi=\begin{cases}\psi(y) & \text{if } y\geq 0\,,\\ \bar\psi(-y) & \text{if } y<0\,.\end{cases}$$

Clearly, $\tilde{\psi} \in H^{1,\mathrm{mag}}(\mathbb{R}^2, \mathbb{C})$. Denote then

$$\nabla_{\mathbf{A}} = \nabla - i(\frac{x^2}{2} - s^*)\hat{\mathbf{i}}_y \,,$$

and

$$\chi_R(y) = \chi\Big(\frac{|y|}{R}\Big),$$

where χ is defined in (2.6). Evidently,

$$\|\chi_R\tilde{\psi}\|^2_{L^2(\mathbb{R}^2)} = 2\|\chi_R\psi\|^2_{L^2(\mathbb{R}^2_+)}$$

and

$$\begin{aligned}&\|\nabla_{\mathbf{A}}(\chi_R\tilde{\psi})\|^2_{L^2(\mathbb{R}^2)} = 2\|\nabla_{\mathbf{A}}(\chi_R\psi)\|^2_{L^2(\mathbb{R}^2_+)}\\ =&2\big[\|\nabla_{\mathbf{A}}(\chi_R U)\|^2_{L^2(\mathbb{R}^2_+)} + \|\gamma\nabla_{\mathbf{A}}(\chi_R\phi^*)\|^2_{L^2(\mathbb{R}^2_+)}\big] - 4\mathrm{Re}\,\langle\nabla_{\mathbf{A}}(\chi_R U), \gamma\nabla_{\mathbf{A}}(\chi_R\phi^*)\rangle_{L^2(\mathbb{R}^2_+)}\,.\end{aligned}$$

We now compute the various terms on the right hand side of the above equality. Since U satisfies (2.5a,b) with $U = 0$ on $\partial\mathbb{R}^2_+$, we integrate by parts to obtain

$$\|\nabla_{\mathbf{A}}(\chi_R U)\|^2_{L^2(\mathbb{R}^2_+)} = \|U\nabla\chi_R\|^2_{L^2(\mathbb{R}^2_+)} + E_0^*\|\chi_R U\|^2_{L^2(\mathbb{R}^2_+)}\,.$$

From the definition of ϕ^* and the orthogonality of ϕ^* and $(\frac{x^2}{2} - s^*)\phi^*$ in $L^2(\mathbb{R}, \mathbb{C})$ we learn that

$$\|\nabla_{\mathbf{A}}(\chi_R\phi^*)\|^2_{L^2(\mathbb{R}^2_+)} = \|\phi^*\nabla\chi_R\|^2_{L^2(\mathbb{R}^2_+)} + E_0^*\|\chi_R\phi^*\|^2_{L^2(\mathbb{R}^2_+)}\,,$$

and that

$$\begin{aligned}&\mathrm{Re}\,\langle\nabla_{\mathbf{A}}(\chi_R U), \nabla_{\mathbf{A}}(\chi_R\phi^*)\rangle_{L^2(\mathbb{R}^2_+)}\\ =&E_0^*\mathrm{Re}\,\langle\chi_R U, \chi_R\phi^*\rangle_{L^2(\mathbb{R}^2_+)} - \mathrm{Re}\,\langle\chi_R U, [\chi_R'' - 2i(\frac{x^2}{2} - s^*)\chi_R']\phi^*\rangle_{L^2(\mathbb{R}^2_+)}\,.\end{aligned}$$

Since

$$\big|\mathrm{Re}\,\langle\chi_R U, \chi_R''\phi^*\rangle_{L^2(\mathbb{R}^2_+)}\big| \leq \frac{C_4}{R}\,,$$

and since

$$\begin{aligned}&\big|\mathrm{Re}\,\langle\chi_R U, 2i(\frac{x^2}{2} - s^*)\chi_R'\phi^*\rangle_{L^2(\mathbb{R}^2_+)}\big|\\ \leq&\big|\mathrm{Re}\,\langle\chi_R\phi^*, 2i(\frac{x^2}{2} - s^*)\chi_R'\phi^*\rangle_{L^2(\mathbb{R}^2_+)}\big| + |\gamma|\big|\mathrm{Re}\,\langle\chi_R\psi, 2i(\frac{x^2}{2} - s^*)\chi_R'\phi^*\rangle_{L^2(\mathbb{R}^2_+)}\big|\\ \leq&C_5\,R^{-1/2}\,,\end{aligned}$$

it then follows the existence of C_6, such that, for $R \geq 2$,

$$\Big|\|\nabla_A(\chi_R\tilde{\psi})\|^2_{L^2(\mathbb{R}^2)} - E_0^*\|\chi_R\tilde{\psi}\|^2_{L^2(\mathbb{R}^2)}\Big| \leq C_6\,R^{-1/2}\,.$$

From the above estimate, we easily obtain that

$$\|\nabla_A(\tilde{\psi})\|^2_{L^2(\mathbb{R}^2)} - E_0^*\|\tilde{\psi}\|^2_{L^2(\mathbb{R}^2)} = 0\,.$$

In Fourier space the above identity yields

$$\int_{\mathbb{R}^2}(E_0(s) - E_0^*)|\widehat{\tilde{\psi}}|^2\,dxds \leq 0\,,$$

therefore $\widehat{\tilde{\psi}} \equiv 0$, and hence $\psi \equiv 0$.

In the case where $\gamma = 0$ it follows that $U = 0$. If $\gamma \neq 0$ then since $\psi \equiv 0$, it follows that $U = \gamma\phi^*$. However, since $U = 0$ on $\partial\mathbb{R}^2_+$ we must have $\gamma = 0$, hence a contradiction. Thus, the uniqueness of solutions of (2.5) is proved.

Step 4. Finally, from (2.36) we get, for all $l > 0$ and $\beta \in \mathbb{C}$, that $w - \beta\phi^* \in L^2(\mathcal{S}_l, \mathbb{C})$. By (2.7) we thus have that $u - \alpha\phi^* \in L^2(\mathcal{S}_l, \mathbb{C})$. Hence, in view of (2.5c), it is readily verified $u - \alpha\phi^* \in L^2(\mathbb{R}^2_+, \mathbb{C})$. ∎

Remark 2.8. *Let* (u, α) *be as in* (2.5)*. Then,*

$$u - \alpha\phi^* \in H^{1,\mathrm{mag}}(\mathbb{R}^2_+, \mathbb{C}) . \tag{2.57}$$

Proof. Let $P_0 = (x_0, y_0)$ and set $\tilde{w} = (u - \alpha\phi^*)$. Then,

$$D_x^2\tilde{w} + \Big(D_y - \frac{1}{2}x^2\Big)^2\tilde{w} = 0 . \tag{2.58}$$

Multiplying (2.58) by $\chi^2(|x - P_0|)\bar{\tilde{w}}$, where the cutoff function χ is defined by (2.6), we obtain via integration by parts, the fact that $\tilde{w} \in L^2(\mathbb{R}^2_+, \mathbb{C})$, and (2.5c), that for every $k \geq 1$ there exists $C_k > 0$ such that, for any P_0,

$$\Big\|\Big(\nabla - i\frac{1}{2}x^2\hat{i}_y\Big)\tilde{w}\Big\|_{L^2(B(P_0,1),\mathbb{C})} \leq \frac{C_k}{y_0^k} ,$$

where $B(P_0, 1)$ denotes the disk with center P_0 and radius 1. Hence

$$\Big\|u - \alpha\phi^*\Big\|_{H^{1,\mathrm{mag}}(B(P_0,1),\mathbb{C})} \leq \frac{C_k}{y_0^k} . \tag{2.59}$$

Similarly (2.37b), (2.58) and the fact that $\phi^* \in \mathcal{S}(\mathbb{R})$ permit us to obtain the existence, for every $k \geq 1$, of some $D_k > 0$, such that, for any P_0,

$$\Big\|u - \alpha\phi^*\Big\|_{H^{1,\mathrm{mag}}(B(P_0,1),\mathbb{C})} \leq \frac{D_k}{|x_0|^k} . \tag{2.60}$$

Combining (2.60) and (2.59) yields

$$\Big\|u - \alpha\phi^*\Big\|^2_{H^{1,\mathrm{mag}}(B(P_0,1),\mathbb{C})} \leq \frac{C_k D_k}{y_0^k|x_0|^k} . \tag{2.61}$$

Now, we take k large, and cover $\mathbb{R}^2_+$ with countably many balls $\{B(P_{ij}, 1)\}$, where $P_{ij} = (x_i, y_j)$ and both x_i and y_j are non-zero, and such that on each ball we have (2.61) with $P_0 = P_{ij}$. Taking sum in i, j we find that

$$\Big\|u - \alpha\phi^*\Big\|^2_{H^{1,\mathrm{mag}}(\mathbb{R}^2_+)} \leq C_k D_k \sum_{i,j=1}^{\infty} \frac{1}{|x_i|^k y_j^k} < \infty .$$

Here we have chosen the integer k and the sequence of points (x_i, y_j) such that the series $\sum_{i=1}^{\infty} \frac{1}{|x_i|^k}$ and $\sum_{j=1}^{\infty} \frac{1}{|y_j|^k}$ converge. Consequently, we obtain (2.57). ∎

3. Construction of the Quasimode

We now return to the analysis of the spectrum of the operator $\mathcal{A}_c^+$ as $c \to 0$. In the following we write $\mathcal{A}_c^+$ as

$$\mathcal{A}_c^+ = \mathcal{A}_0 + icy\,,$$

where $\mathcal{A}_0$ is the operator defined in (2.4). Since the resolvent of $\mathcal{A}_c^+$ is compact, we need only consider the existence of an eigenpair (λ, v), $v \neq 0$, which solves the equation

$$\mathcal{A}_c^+ v = \lambda v\,, \tag{3.1}$$

with $v \in \mathcal{D}(\mathcal{A}_c^+)$. An obvious lower bound for $\operatorname{Re}\{\sigma(\mathcal{A}_c^+)\}$ is E_0^* (see (1.15) and (1.16)). This follows easily from the fact that

$$\operatorname{Re}\langle u, \mathcal{A}_c^+ u\rangle = \operatorname{Re}\langle u, \mathcal{A}_0 u\rangle\,.$$

Formal perturbation theory (keeping in mind that for $c = 0$ the spectrum is continuous) suggests that the leftmost eigenvalue, in $\mathbb{C}$, of $\mathcal{A}_c^+$ tends to E_0^* as $c \to 0$. Based on this natural guess, we look for a formal asymptotic expansion of this eigenvalue in fractional powers of c, with E_0^* as its leading order term and a corresponding approximate eigenfunction or quasimode. Set then

$$c = \epsilon^3. \tag{3.2}$$

We construct the quasimode separately in two different zones. In the outer zone we have $y^{-1} = \mathcal{O}(\epsilon)$, whereas inside the inner zone is $y = \mathcal{O}(1)$. Naturally, we expect the two asymptotic expansions to match through an intermediate domain (or the overlap domain as it is often called [14]). Thus, for every term (or order) in the expansion, we present first the outer expansion which is then followed by the corresponding inner expansion.

Outer expansion: $\mathcal{O}(1)$ balance.

We first apply the gauge transformation

$$v \to v \exp(-is^* y)$$

where s^* is introduced in (2.1), and rewrite the equation (3.1) as

$$-\partial_x^2 v - \partial_y^2 v + 2i\Big(\frac{x^2}{2} - s^*\Big)\partial_y v + \Big(\frac{x^2}{2} - s^*\Big)^2 v + i\epsilon^3 y v = \lambda v\,.$$

Then, we adopt the outer zone scaling

$$\eta = \epsilon y \tag{3.3}$$

and write (3.1) in the form

$$-\frac{\partial^2 v}{\partial x^2} + \Big(\frac{x^2}{2} - s^*\Big)^2 v - \lambda v = \epsilon^2\Big[\frac{\partial^2 v}{\partial \eta^2} - i\eta v\Big] - 2i\epsilon\Big(\frac{x^2}{2} - s^*\Big)\frac{\partial v}{\partial \eta}\,. \tag{3.4}$$

Presuming the formal expansion

$$\begin{cases} v = v_0 + \epsilon v_1 + \epsilon^2 v_2 + \mathcal{O}(\epsilon^3)\,, \\ \lambda = \lambda_0 + \epsilon\lambda_1 + \epsilon^2\lambda_2 + \mathcal{O}(\epsilon^3)\,, \end{cases} \tag{3.5}$$

we obtain the leading order balance

$$(\mathcal{M}_{s^*} - \lambda_0)v_0 = 0\,,$$

where, for $s \in \mathbb{R}$, $\mathcal{M}_s$ is defined (see (1.14)) by

$$\mathcal{M}_s = -\frac{\partial^2}{\partial x^2} + \Big(\frac{x^2}{2} - s\Big)^2 .$$

It readily follows from the equality

$$\mathcal{M}_{s^*}\phi_0(x,s^*) = E_0^*\phi_0(x,s^*)\,, \tag{3.6}$$

(see (2.2) and (2.3)) that we can look for a pair (v_0, λ_0) in the form

$$v_0(x,\eta) = \phi_0(x,s^*)\psi_0(\eta), \quad \lambda_0 = E_0^*\,. \tag{3.7}$$

Inner expansion: $\mathcal{O}(1)$ balance.

In the inner zone we keep the original coordinates (x, y). We denote the inner solution by u, and assume for it the analogous of (3.5) with v_j replaced by u_j

$$u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \mathcal{O}(\epsilon^3)\,. \tag{3.8}$$

Write (3.1) in the form

$$\mathcal{A}_0 u - \lambda u = -i\epsilon^3\, y\, u\,.$$

Using the above-obtained $\lambda_0 = E_0^*$, we obtain the leading order balance

$$\begin{cases} (\mathcal{A}_0 - E_0^*)u_0 = 0 & \text{in } \mathbb{R}^2_+\,, \\ u_0 = 0 & \text{on } \partial\mathbb{R}^2_+\,, \\ u_0 \sim \psi_0(0)\phi^*(x) & \text{as } 1 \ll y\,. \end{cases} \tag{3.9}$$

The last condition is obtained by matching u_0 with v_0 in the overlap zone. Here we recall from (2.3), that $\phi^*(x) = \phi_0(x, s^*)$. Obviously, for $\eta \ll 1$,

$$v_0 \sim \psi_0(0)\phi^*\,.$$

Hence, the leading order terms would match for

$$1 \ll y \ll \frac{1}{\epsilon}\,.$$

From the uniqueness part of Theorem 2.1 (namely if (u, α) is a solution of (2.5) with $u = 0$ on $\partial\mathbb{R}^2_+$, then $(u, \alpha) = (0, 0)$), it follows that

$$u_0 \equiv 0 \quad \text{and} \quad \psi_0(0) = 0\,. \tag{3.10}$$

Outer expansion: $\mathcal{O}(\epsilon)$ balance.

The outer $\mathcal{O}(\epsilon)$ balance takes the form

$$\begin{aligned}(\mathcal{M}_{s^*} - \lambda_0)v_1 &= \lambda_1 v_0 - 2i\Big(\frac{x^2}{2} - s^*\Big)\frac{\partial v_0}{\partial \eta} \\ &= \lambda_1\phi^*(x)\psi_0(\eta) - 2i\Big(\frac{x^2}{2} - s^*\Big)\phi^*(x)\psi_0'(\eta)\,.\end{aligned} \tag{3.11}$$

Here we have used (3.7). We now multiply the above by $\phi^*(x)$, integrate over x, then integrate by parts, and use (3.7) to obtain, for any $\eta > 0$,

$$\lambda_1\psi_0(\eta) - 2i\psi_0'(\eta)\int_{\mathbb{R}}\Big(\frac{x^2}{2} - s^*\Big)|\phi^*(x)|^2\,dx = 0\,. \tag{3.12}$$

It is well known (see [10]) that the integral on the left-hand-side of (3.12) vanishes and hence we must have $\lambda_1 = 0$. Moreover, we obtain by differentiating the equality (2.2) with respect to s, letting $s = s^*$ and using the fact that $E_0'(s^*) = 0$,

$$(\mathcal{M}_{s^*} - E_0^*)\phi_s^* = 2(\frac{x^2}{2} - s^*)\phi^* \,, \tag{3.13}$$

where

$$\phi_s^*(x) = \frac{\partial \phi_0}{\partial s}(x, s^*)\,. \tag{3.14}$$

We thus obtain that the general solution of (3.11) in $L^2(\mathbb{R})$ is given by

$$v_1(x,\eta) = -i\psi_0'(\eta)\phi_s^*(x) + \psi_1(\eta)\phi^*(x)\,, \tag{3.15}$$

where $\psi_1(\eta)$ is any function of η, to be determined later. Notice that, unlike v_0, v_1 cannot satisfy the boundary condition at $y = 0$ unless $\psi_0'(0) = 0$. From the next order balance we derive, however, that this would mean that $\psi_0 \equiv 0$ for all positive η, turning the leading order balance into the trivial solution. To avoid the failure of our asymptotic scheme it is therefore essential that a proper inner expansion is introduced at this order.

Inner expansion: $\mathcal{O}(\epsilon)$ balance.

For the inner expansion we obtain at the $\mathcal{O}(\epsilon)$ order the problem

$$(\mathcal{A}_0 - E_0^*)u_1 = 0 \qquad \text{in } \mathbb{R}^2_+\,, \tag{3.16a}$$

$$u_1 = 0 \qquad \text{on } \partial\mathbb{R}^2_+\,, \tag{3.16b}$$

$$u_1 \sim \psi_1(0)\phi^*(x) + \psi_0'(0)[y\phi^*(x) - i\phi_s^*(x)] \qquad \text{as } y \gg 1\,. \tag{3.16c}$$

The last condition is obtained by matching the inner expansion, through the overlap zone, with the outer expansion. In fact, since for η of order ϵ, the outer solution admits, using (3.7) and (3.15), the expansion

$$\begin{aligned} v_0(x,\eta) + \epsilon v_1(x,\eta) =& v_0(x,0) + \eta\frac{\partial v_0}{\partial \eta}(x,0) + \epsilon v_1(x,0) + \mathcal{O}(\epsilon^2) \\ =& \epsilon\big[\psi_0'(0)\big(y\phi^*(x) - i\phi_s^*(x)\big) + \psi_1(0)\phi^*(x)\big] + \mathcal{O}(\epsilon^2)\,, \end{aligned}$$

we readily obtain (3.16c).

It is easily verified that

$$(x,y) \mapsto y\phi^*(x) - i\phi_s^*(x)$$

is a solution of (3.16a). (More generally it is the consequence of the commutation of the operators $(y + D_s)$ and $D_x^2 + (\frac{x^2}{2} - s - D_y)^2$.) We thus define

$$w_1(x,y) = u_1(x,y) - \psi_0'(0)[y\phi^*(x) - i\phi_s^*(x)]\,, \tag{3.17}$$

to obtain the following problem for w_1

$$(\mathcal{A}_0 - E_0^*)w_1 = 0 \qquad \text{in } \mathbb{R}^2_+\,, \tag{3.18a}$$

$$w_1 = i\psi_0'(0)\phi_s^*(x) \qquad \text{on } \partial\mathbb{R}^2_+\,, \tag{3.18b}$$

$$w_1 \sim \psi_1(0)\phi^*(x) \qquad \text{as } y \to +\infty\,. \tag{3.18c}$$

It is easy to show that $\phi_s^* \in \mathcal{S}(\mathbb{R},\mathbb{C})$ and hence we can use Theorem 2.1 to deduce the existence of some $\alpha_1 \in \mathbb{C}$, such that if

$$\psi_1(0) = i\alpha_1\psi_0'(0)\,, \tag{3.19}$$

then there exists $w_1 \in H^2_{\mathrm{loc}}(\overline{\mathbb{R}^2_+},\mathbb{C})$ satisfying (3.18).

Once (w_1,α_1) is obtained, then we have determined u_1 (by (3.17)) and a relation between $\psi_1(0)$ and $\psi_0'(0)$ (by (3.19)). We emphasize that α_1 is independent of $\psi_0'(0)$.

Outer expansion: $\mathcal{O}(\epsilon^2)$ balance.

The next order outer balance is given by

$$\begin{aligned}(\mathcal{M}_{s^*}-\lambda_0)v_2 &= \lambda_2 v_0 - 2i\Big(\frac{x^2}{2}-s^*\Big)\frac{\partial v_1}{\partial\eta}+\frac{\partial^2 v_0}{\partial\eta^2}-i\eta v_0\\ =&\Big[(\lambda_2-i\eta)\psi_0(\eta)-2i\Big(\frac{x^2}{2}-s^*\Big)\psi_1'(\eta)+\psi_0''(\eta)\Big]\phi^*(x)-2\Big(\frac{x^2}{2}-s^*\Big)\psi_0''(\eta)\phi_s^*\,.\end{aligned} \tag{3.20}$$

In deriving the second line in (3.20) we used (3.7) and (3.15). Multiplying it by $\phi^*(x)$, integrating over x and then integrating by parts we obtain, after some manipulation, the solvability condition

$$\begin{aligned}\psi_0''(\eta)+(\lambda_2-i\eta)\psi_0(\eta)-2i\psi_1'(\eta)\int_{\mathbb{R}}\Big(\frac{x^2}{2}-s^*\Big)|\phi^*(x)|^2\,dx\\ =2\,\psi_0''(\eta)\int_{\mathbb{R}}\Big(\frac{x^2}{2}-s^*\Big)\phi^*(x)\phi_s^*(x)\,dx\,.\end{aligned}$$

The integral on the left-hand-side is zero (see (3.12) above). To obtain the integral on the right-hand-side one needs to differentiate

$$E_0'(s) = -2\int(\frac{x^2}{2}-s)\phi_0(x,s)^2dx\,. \tag{3.21}$$

and set $s=s^*$:

$$2\int_{\mathbb{R}}\Big(\frac{x^2}{2}-s^*\Big)\phi^*(x)\phi_s^*(x)\,dx = -\frac{1}{2}E_0''(s^*)+1\,.$$

Let

$$\beta = \frac{1}{2}E_0''(s^*)\,, \tag{3.22}$$

and recall from [12] that β is positive. Then, we have

$$\begin{cases}-\beta\psi_0''(\eta)+(i\eta-\lambda_2)\psi_0(\eta)=0\,, & \eta>0\,,\\ \psi_0(0)=0\,.\end{cases} \tag{3.23}$$

Applying the transformation

$$\eta_1 = \beta^{1/3}\eta$$

to (3.23), we obtain (cf. [3])

$$\lambda_2 = \beta^{1/3}(-a_1)e^{i\pi/3}\,, \tag{3.24}$$

and the corresponding eigenfunction is given by

$$\psi_0(\eta) = C_0\,Ai(\beta^{1/3}e^{i\pi/6}\eta+a_1)\,, \tag{3.25}$$

where $a_1 \in \mathbb{R}_-$ is the rightmost zero of Airy's function. The constant $C_0 > 0$ is determined by requiring $\|\psi_0\|_{L^2(\mathbb{R}_+)} = 1$. Once C_0 is obtained, we can evaluate $\psi_0'(0)$ which, in turn, determines $\psi_1(0)$ (via (3.19)) and w_1.

Next, we look for the general solution in $L^2(\mathbb{R})$ of (3.20). Let

$$\phi_{ss}^*(x) = \frac{\partial^2 \phi_0}{\partial s^2}(x, s^*)\,.$$

We have the identity

$$(\mathcal{M}_{s^*} - \lambda_0)\phi_{ss}^* = 4\Big(\frac{x^2}{2} - s^*\Big)\phi_s^* - 2(1-\beta)\phi^*\,, \tag{3.26}$$

which is obtained by differentiating (2.2) twice with respect to s and then letting $s = s^*$. Using (3.7), (3.15), (3.23) and (3.26), we can show that the general solution of (3.20) is given by

$$v_2(x,\eta) = -\frac{1}{2}\phi_{ss}^*(x)\psi_0''(\eta) - i\phi_s^*(x)\psi_1'(\eta) + \psi_2(\eta)\phi^*(x)\,, \tag{3.27}$$

where ψ_2 is any function ($\psi_2 = 0$ is, therefore, a legitimate choice for our approximate solution). In fact since ψ_0 satisfies (3.23), we can write (3.20) as

$$(\mathcal{M}_{s^*} - \lambda_0)v_2 = \Big[-2i\Big(\frac{x^2}{2} - s^*\Big)\psi_1'(\eta) + (1-\beta)\psi_0''(\eta)\Big]\phi^*(x) - 2\Big(\frac{x^2}{2} - s^*\Big)\psi_0''(\eta)\phi_s^*\,.$$

Write

$$v_2 = z_2 - \frac{1}{2}\phi_{ss}^*(x)\psi_0''(\eta) - i\phi_s^*(x)\psi_1'(\eta)\,.$$

Using (3.13) and (3.26) we reduce (3.20) to

$$(\mathcal{M}_{s^*} - \lambda_0)z_2 = 0\,.$$

Hence $z_2 = \psi_2(\eta)\phi^*(x)$, which validates (3.27).

Inner expansion: $\mathcal{O}(\epsilon^2)$ balance.

From the outer expansion we obtain the following problem for u_2

$$(\mathcal{A}_0 - E_0^*)u_2 = \lambda_2 u_0\,, \qquad \text{in } \mathbb{R}^2_+, \tag{3.28a}$$

$$u_2 = 0\,, \qquad \text{on } \partial\mathbb{R}^2_+, \tag{3.28b}$$

$$u_2 \sim \psi_0''(\eta)\Big[\frac{1}{2}y^2\phi^* - iy\phi_s^* - \frac{1}{2}\phi_{ss}^*\Big] + \psi_1'(0)[y\phi^* - i\phi_s^*] + \psi_2(0)\phi^*\,, \quad \text{as } y \gg 1\,. \tag{3.28c}$$

Again, the last condition (3.28c) is obtained by matching the inner expansion, through the overlap zone, with the outer expansion. In fact, since for $\eta \sim \mathcal{O}(\epsilon)$, the outer solution admits, using (3.7), (3.10), (3.15) and (3.27), the expansion

$$\begin{aligned}
&\epsilon u_1(x,y) + \epsilon^2 u_2(x,y) + \mathcal{O}(\epsilon^3) = v_0(x,\eta) + \epsilon v_1(x,\eta) + \epsilon^2 v_2(x,\eta) + \mathcal{O}(\epsilon^3)\\
=&v_0(x,0) + \eta\frac{\partial v_0}{\partial \eta}(x,0) + \frac{\eta^2}{2}\frac{\partial^2 v_0}{\partial \eta^2}(x,0) + \epsilon[v_1(x,0) + \eta\frac{\partial v_1}{\partial \eta}(x,0)] + \epsilon^2 v_2(x,0) + \mathcal{O}(\epsilon^3)\\
=&\epsilon\big[\psi_0'(0)\big(y\phi^*(x) - i\phi_s^*(x)\big) + \psi_1(0)\phi^*(x)\big]\\
&+\epsilon^2\Big\{\psi_0''(0)[\frac{y^2}{2}\phi^*(x) - iy\phi_s^*(x) - \frac{1}{2}\phi_{ss}^*(x)] + \psi_1'(0)[y\phi^*(x)) - i\phi_s^*(x)] + \psi_2(0)\phi^*(x)\Big\} + \mathcal{O}(\epsilon^3)\,.
\end{aligned}$$

By matching the terms of order ϵ^2 we readily obtain (3.28c).

It is easy to show (see the previous footnote) that the function

$$(x,y) \mapsto \frac{y^2}{2}\phi^*(x) - iy\phi_s^*(x) - \frac{1}{2}\phi_{ss}^*(x)$$

solves (3.28a). Thus, as we did for the $\mathcal{O}(\epsilon)$ inner balance, we set

$$w_2(x,y) = u_2(x,y) - \psi_0''(0)\Big[\frac{1}{2}y^2\phi^*(x) - iy\phi_s^*(x) - \frac{1}{2}\phi_{ss}^*(x)\Big] - \psi_1'(0)[y\phi^*(x) - i\phi_s^*(x)]\,,$$

and substitute it into (3.28) to obtain

$$(\mathcal{A}_0 - E_0^*)w_2 = 0 \qquad \text{in } \mathbb{R}_+^2\,, \tag{3.29a}$$

$$w_2 = i\psi_1'(0)\phi_s^* + \frac{1}{2}\psi_0''(0)\phi_{ss}^* \quad \text{on } \partial\mathbb{R}_+^2\,, \tag{3.29b}$$

$$w_2 \sim \psi_2(0)\phi^* \qquad \text{as } y \to +\infty\,. \tag{3.29c}$$

We now observe that by (3.23) $\psi_0''(0) = 0$. Hence we can rely on Theorem 2.1 to show that if

$$\psi_2(0) = i\alpha_1\psi_1'(0)\,, \tag{3.30}$$

then there exists w_2 satisfying (3.29).

A uniformly valid quasimode.

One can continue the above process to obtain higher order terms up to the desired accuracy. Once we have obtained the outer and the inner expansions we can combine them into a quasimode which would approximate the eigenfunctions in both the inner and the outer zones. Denote this uniform approximation by U_ϵ. The standard manner, by which the uniform quasimode is constructed (see [14, 20]) is by setting

$$U_\epsilon(x,y) = u(x,y) + v(x,\epsilon y) - u(+\infty)\,,$$

where $u(+\infty)$ stands for the asymptotic behaviour of u as $y \to +\infty$ (it can be exchanged for the asymptotic behaviour of v near $\eta = 0$). For instance, at the $\mathcal{O}(\epsilon)$ balance $u(+\infty)$ is given by (3.16c). Thus,

$$\begin{aligned} U_\epsilon(x,y) =& \psi_0(\eta)\phi^*(x) + \epsilon\big\{ - i\phi_s^*\psi_0'(\eta) + [\psi_1(\eta) - \psi_1(0)]\phi^*(x) + w_1(x,y)\big\} \\ &+ \epsilon^2\Big[- \frac{1}{2}\phi_{ss}^*(x)\psi_0''(\eta) - i\phi_s^*(x)\psi_1'(\eta) + [\psi_2(\eta) - \psi_2(0)]\phi^*(x) + w_2(x,y)\Big]\,, \end{aligned} \tag{3.31}$$

with $\eta = \epsilon y$.

It can be readily verified that

$$U_\epsilon(x,0) = 0\,.$$

We now show that, for a suitably chosen ψ_2, U_ϵ is also an approximate solution for the equation

$$(\mathcal{A}_{0,c} - E_0^* - \epsilon^2\lambda_2)U = 0\,, \tag{3.32}$$

with $\epsilon = c^{\frac{1}{3}}$ and an error of size $o(\epsilon^2)$. More precisely, we choose in (3.31) $\psi_2 \equiv 0$, and accordingly, by (3.30) and (3.29), we can impose

$$\psi_1'(0) = 0\,, \tag{3.33}$$

and get from the uniqueness part in Theorem 2.1 that $w_2 \equiv 0$. Then we have

$$\begin{aligned} U_\epsilon(x,y) =&\psi_0(\eta)\phi^*(x) + \epsilon\big\{ - i\phi_s^*\psi_0'(\eta) + [\psi_1(\eta) - \psi_1(0)]\phi^*(x) + w_1(x,y)\big\} \\ &+ \epsilon^2\Big[- \frac{1}{2}\phi_{ss}^*(x)\psi_0''(\eta) - i\phi_s^*(x)\psi_1'(\eta)\Big]\,. \end{aligned} \tag{3.34}$$

For this choice of U_ϵ we have:

Proposition 3.1. *Let U_ϵ be given by (3.34) where ψ_0 is given by (3.25), $\psi_1 \in \mathcal{S}(\overline{\mathbb{R}_+},\mathbb{C})$ satisfies (3.19) with $\psi_1'(0) = 0$, and w_1 satisfies (3.18). Let further,*

$$\Lambda = E_0^* + \epsilon^2\lambda_2\,, \tag{3.35}$$

where λ_2 is given by (3.24), and let

$$f = (\mathcal{A}_c^+ - \Lambda)U_\epsilon\,, \tag{3.36}$$

with $c = \epsilon^3$.
Then we have the following conclusions.

(i) *$U_\epsilon \in \mathcal{D}(\mathcal{A}_c^+)$, the domain of the operator $\mathcal{A}_c^+$.*
(ii) *For any $p,\ k,\ n \in \mathbb{N}$, there exists $C = C(p,k,n) > 0$ such that,*

$$\Big\| |x|^p(\epsilon y)^k \frac{\partial^n f}{\partial y^n}\Big\|_2 \leq C(p,k,n)\,\epsilon^3\,\|U_\epsilon\|_2\,. \tag{3.37}$$

Proof. Step 1. We first prove (ii). We have

$$\begin{aligned} f =&(\mathcal{A}_{0,c} - \Lambda)U_\epsilon \\ =&(iy\epsilon^4 + \lambda_2\epsilon^3)\big[w_1(x,y) - \psi_1(0)\phi^*(x)\big] \\ &+ \epsilon^3\Big\{ - i\phi_s^*(x)\psi_0^{(3)}(\eta) + \psi_1''(\eta)\phi^*(x) + \lambda_2[-i\phi_s^*(x)\psi_0'(\eta) + \psi_1(\eta)\phi^*(x)] \\ &\quad - i\Big(\frac{x^2}{2} - s^*\Big)[\phi_{ss}^*(x)\psi_0^{(3)}(\eta) + 2\phi_s^*(x)\psi_1''(\eta)]\Big\} \\ &+ \epsilon^4\big[- i\phi_s^*(x)\psi_1^{(3)}(\eta) - i\lambda_2\phi_s^*(x)\psi_1'(\eta)\big]\,, \end{aligned} \tag{3.38}$$

with $\eta = \epsilon y$. To prove (ii) one has to show that (3.37) holds for each term in (3.38). There are two types of terms: the first includes those which depend only on the slow coordinate η, the second includes those which depend on the fast coordinate y. Consider then a term of the first type

$$h_1(x,y) = \epsilon^3\psi_1''(\eta)\phi^*(x)\,. \tag{3.39}$$

Since $\psi_1 \in \mathcal{S}(\overline{\mathbb{R}_+},\mathbb{C})$ and $\phi^* \in S(\overline{\mathbb{R}},\mathbb{C})$, it easily follows that

$$\begin{aligned} \Big\| |x|^p(\epsilon y)^k \frac{\partial^n h_1}{\partial y^n}\Big\|_2^2 = \int_{\mathbb{R}^2_+} |x|^{2p}(\epsilon y)^{2k}\Big|\frac{\partial^n h_1}{\partial y^n}\Big|^2\,dxdy = \\ \epsilon^{5+2n}\int_{\mathbb{R}_+}\eta^{2k}\big|\psi_1^{(n+2)}(\eta)\big|^2\,d\eta\int_{\mathbb{R}}|x|^{2p}|\phi^*(x)|^2\,dx \leq C_{p,k,n}\,\epsilon^{5+2n}\,. \end{aligned} \tag{3.40}$$

All terms of this type can similarly be estimated.

Consider next an example of a term of the second type

$$h_2(x,y) = \epsilon^3[w_1(x,y) - \psi_1(0)\phi^*(x)]\,. \tag{3.41}$$

In view of (2.8), (2.37), and (2.5c) we have that

$$\int_{B(P_0,1)} |h_2(x,y)|^2\, dxdy \leq \frac{C(k,p)}{|x_0|^p y_0^k}\,\epsilon^6\,, \quad \forall k\in\mathbb{N}\,,$$

where $P_0=(x_0,y_0)$ and $B(P_0,1)$ is the ball with center P_0 and radius 1. Standard elliptic estimates, boot-strapping and Sobolev embeddings then show the existence of a constant $C(k,p,n)$ such that,

$$\|h_2\|_{H^n(B(P_0,1/2))} \leq \frac{C(k,p,n)}{|x_0|^p y_0^k}\,\epsilon^3\,, \quad \forall k\in\mathbb{N}\,.$$

Using the above and a countable covering of $\mathbb{R}^2_+$ (see for instance the proof of Remark 2.8) we obtain that

$$\Big\||x|^p(\epsilon y)^k\frac{\partial^n h_2}{\partial y^n}\Big\|_2 \leq C(k,p,n)\,\epsilon^{k+3}\,. \tag{3.42}$$

The above, in conjunction with (3.40), yields

$$\Big\||x|^p(\epsilon y)^k\frac{\partial^n f}{\partial y^n}\Big\|_2 \leq C(k,p,n)\,\epsilon^{5/2}\,. \tag{3.43}$$

By the choice of C_0 in (3.25) we have

$$\|U_\epsilon\|_2 = \frac{1}{\epsilon^{1/2}} + \mathcal{O}(1)\,. \tag{3.44}$$

Combining (3.43) and (3.44) we obtain (3.37).

Step 2. We now prove (i). As, by (3.43) we have that $\mathcal{A}_c^+U_\epsilon\in L^2(\mathbb{R}^2_+,\mathbb{C})$, it remains to show only that $U_\epsilon\in H_0^{1,\text{mag}}(\mathbb{R}^2_+,\mathbb{C})$. For product terms, such as $\psi_0(\eta)\phi^*(x)$, this fact easily follows from the exponential rate of decay of Airy's functions and its derivatives (note that ψ_1 decays exponentially fast as well). For w_1, this fact follows from Remark 2.8. ∎

4. Some Preliminary Estimates

In the following we list some of the properties of

$$f = (\mathcal{A}_{0,c}-\Lambda)U_\epsilon\,, \tag{4.1}$$

where Λ is given by (3.35) and U_ϵ by (3.31). In addition we prove some basic properties of the solution of

$$\begin{cases} (\mathcal{A}_{0,c}-\lambda)u = g & \text{in } \mathbb{R}^2_+\,,\\ u=0 & \text{on } \partial\mathbb{R}^2_+\,, \end{cases} \tag{4.2}$$

where $g\in\mathcal{S}\big(\overline{\mathbb{R}^2_+},\mathbb{C}\big)$. To describe the topology in $\mathcal{S}\big(\overline{\mathbb{R}^2_+},\mathbb{C}\big)$, we use here the increasing (with respect to k) ϵ-dependent family of norms

$$g\longmapsto p_k(g) = \sum_{p+q+r+s\leq k}\big\||x|^p(\epsilon y)^q\partial_x^r\partial_y^s f\big\|_{L^2(\mathbb{R}^2_+)}\,, \tag{4.3}$$

for $k \in \mathbb{N}$, where

$$\epsilon = c^{1/3}.$$

We use this definition in the sequel whenever a norm appears in our calculations, except for the cases where the exact form of the norm should be introduced. In this section we shall denote the norm $\|\cdot\|_{L^2(\mathbb{R}^2_+)}$ by $\|\cdot\|_2$, and denote the inner product $\langle\cdot,\cdot\rangle_{L^2(\mathbb{R}^2_+)}$ by $\langle\cdot,\cdot\rangle_2$.

The following equations are needed in the sequel. Let ϕ be any smooth real-valued function with a compact support in $\overline{\mathbb{R}^2_+}$. We multiply (4.2) by $\phi^2\bar{u}$ and integrate to get

$$\begin{aligned}&\Big\|\Big(\nabla - i\frac{x^2}{2}\hat{\mathbf{i}}_y\Big)(\phi u)\Big\|_2^2 - \|u\nabla\phi\|_2^2 + 2i\mathrm{Im}\Big\langle\Big(\nabla - i\frac{x^2}{2}\hat{\mathbf{i}}_y\Big)(\phi u), u\nabla\phi\Big\rangle_2 + ic\|y^{1/2}\phi u\|_2^2\\ =&\lambda\|\phi u\|_2^2 + \langle\phi g, \phi u\rangle_2\,.\end{aligned} \tag{4.4}$$

Taking the real part of (4.4) we obtain

$$\Big\|\Big(\nabla - i\frac{x^2}{2}\hat{\mathbf{i}}_y\Big)(\phi u)\Big\|_2^2 = \|u\nabla\phi\|_2^2 + (\mathrm{Re}\,\lambda)\|\phi u\|_2^2 + \mathrm{Re}\,\langle\phi g, \phi u\rangle_2\,. \tag{4.5}$$

Lemma 4.1. *Let $C > 0$, $c_0 > 0$ and $k \in \mathbb{N}$. There exists a positive constant $C(k)$ such that, for any $g \in \mathcal{S}\big(\overline{\mathbb{R}^2_+}, \mathbb{C}\big)$, $c \in (0, c_0)$ and $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda \le C$, such that the solution u of (4.2) satisfies*

$$\|x^k u\|_2 \le C(k)\big(\|u\|_2 + p_k(g)\big)\,. \tag{4.6}$$

Proof. Step 1. For $j \ge 1$, let $\eta_j \in C^\infty(\mathbb{R}, [0,1])$ satisfy

$$\eta_j(t) = \begin{cases} 0\,, & \text{if } t < -1\,,\\ 1\,, & \text{if } 0 < t < j\,,\\ 0\,, & \text{if } 2j < t\,,\end{cases} \tag{4.7}$$

and

$$|\eta_j'(t)| \le \frac{C_0}{j} \quad \text{for all } t \in \mathbb{R}\,.$$

Denote further by η_j^+ the restriction of η_j to $\mathbb{R}_+$. Setting $\phi = |x|^{m/2}\eta_j(x)\eta_k^+(y)$ in (4.4), with m a non-negative integer we get from the real part

$$\Big\|\Big(\nabla - i\frac{x^2}{2}\hat{\mathbf{i}}_y\Big)(|x|^{m/2}\eta_j\eta_k^+ u)\Big\|_2^2 = \|(|x|^{m/2}\eta_j\eta_k^+)_x u\|_2^2 + \mathrm{Re}\,\lambda\||x|^{m/2}\eta_j\eta_k^+ u\|_2^2 + \mathrm{Re}\,\langle x^m\eta_j^2 u, g\rangle_2\,.$$

Letting $k \to +\infty$ then yields

$$\Big\|\Big(\nabla - i\frac{x^2}{2}\hat{\mathbf{i}}_y\Big)(|x|^{m/2}\eta_j u)\Big\|_2^2 = \|(|x|^{m/2}\eta_j)_x u\|_2^2 + \mathrm{Re}\,\lambda\||x|^{m/2}\eta_j u\|_2^2 + \mathrm{Re}\,\langle x^m\eta_j^2 u, g\rangle_2\,. \tag{4.8}$$

We now claim that there exists a constant $K(m)$ depending on m such that

$$\Big\||x|^{(m+1)/2}\eta_j u\Big\|_2^2 \le K\{\|(|x|^{m/2}\eta_j)_x u\|_2^2 + \mathrm{Re}\,\lambda\||x|^{m/2}\eta_j u\|_2^2 + \||x|^{m/2}\eta_j u\|_2\,\||x|^{m/2}\eta_j g\|_2\}\,, \tag{4.9}$$

which can be proved by using Theorem 4 in [15] (cf. also (2.24) in [4] and (5.16) in [5]) together with a partition of unity. In fact we can find a constant M_1 such that, for any $\psi \in H_0^{1,mag}(\mathbb{R}^2_+,\mathbb{C})$, we have

$$\int_{\mathbb{R}^2_+} |B_{kl}||\psi|^2 dx \leq M_1 \left(\|\nabla_{\mathbf{A}}\psi\|_2^2 + \|\psi\|_2^2\right) , \tag{4.10}$$

where

$$B_{kl} = i[\partial_{x_k} - iA_k, \partial_{x_l} - iA_l] ,$$

and $[P,Q]$ denotes the commutator of the operators P and Q. For $\mathbf{A} = (0, \frac{x^2}{2})$ we have $B_{12} = x$. Hence, from (4.10) we get the existence of $M_2 > 0$ such that

$$\||x|^{1/2}\psi\|_2^2 \leq M_2 \left(\left\| \left(\nabla - i\frac{x^2}{2}\hat{\mathbf{i}}_y\right)\psi\right\|_2^2 + \|\psi\|_2^2 \right) .$$

Clearly, the above inequality is valid for $\psi = |x|^{m/2}\eta_j u$, where m is a non-negative integer. Consequently,

$$\||x|^{(m+1)/2}\eta_j\, u\|_2^2 \leq M_2 \left(\left\|(\nabla - i\frac{x^2}{2}\hat{\mathbf{i}}_y)(|x|^{m/2}\eta_j\, u)\right\|_2^2 + \||x|^{m/2}\eta_j\, u\|_2^2 \right) .$$

From the above, in conjunction with (4.8), we get (4.9).

Step 2. Based on (4.9) we prove (4.6) invoking inductive arguments. We first consider the case $m = 0$. By (4.9) we have

$$\left\||x|^{1/2}\eta_j\, u\right\|_2^2 \leq K\Big\{\frac{1}{j}\|u\|_2^2 + \operatorname{Re}\lambda\|\eta_j\, u\|_2^2 + \|u\|_2\,\|g\|_2\Big\} .$$

Taking limit as $j \to +\infty$ we obtain

$$\left\|\mathbf{1}_{\mathbb{R}_+}(x)|x|^{1/2}u\right\|_2^2 \leq M_3 \left(\|u\|_2^2 + \|u\|_2\,\|g\|_2\right) , \tag{4.11}$$

where $M_3 = K(0)\max\{|\lambda|, 1\}$, and $\mathbf{1}_{\mathbb{R}_+}(x)$ is the characteristic function of the set $\mathbb{R}_+$. Since $u(-x,y)$ is a solution of (4.2), we apply (4.11) to $u(-x,y)$ and then change the variables $(x,y) \to (-x,y)$ to get

$$\left\|\mathbf{1}_{\mathbb{R}_-}(x)|x|^{1/2}u\right\|_2^2 \leq M_3 \left(\|u\|_2^2 + \|u\|_2\,\|g\|_2\right) ,$$

from which we easily obtain the existence of C_1 such that,

$$\left\||x|^{1/2}u\right\|_2 \leq C_1 \left(\|u\|_2 + \|g\|_2\right) .$$

Suppose now, by induction, that there exists C_l such that,

$$\||x|^{l/2}u\|_2^2 \leq C_l\big(\|u\|_2^2 + p_{[l/2]}(g)\|u\|_2\big) , \tag{4.12}$$

for all $1 \leq l \leq m$, where $[\cdot]$ denotes the integer part of the term in brackets. By (4.9) and (4.12) we get a constant C_{m+1} such that, for all j and g,

$$\left\||x|^{(m+1)/2}\eta_j u\right\|_2^2 \leq C_{m+1} \left(\|u\|_2^2 + p_{[(m+1)/2]}(g)\|u\|_2\right) .$$

Taking limit as $j \to \infty$ and using again the fact that $u(-x,y)$ is a solution of (4.2), we obtain

$$\Big\| |x|^{(m+1)/2} u \Big\|_2^2 \leq C_{m+1} \left(\|u\|_2^2 + p_{[(m+1)/2]}(g) \|u\|_2 \right).$$

Thus, by induction, (4.12) follows for all $l \geq 1$. In particular, if we set $l = 2k$ in (4.12), then (4.6) easily follows with $C(k) = C_{2k}$. ∎

Lemma 4.2. *Let $C > 0$, $c_0 > 0$ and $n \in \mathbb{N}$. There exists $C(n)$ such that, for any $g \in \mathcal{S}(\overline{\mathbb{R}^2_+}, \mathbb{C})$, $\lambda \in \mathbb{C}$ with $|\lambda| \leq C$ and $0 < c < c_0$, the solution u of (4.2) satisfies*

$$\Big\| \frac{\partial^n u}{\partial y^n} \Big\|_2 \leq C(n) \big(\|u\|_2 + p_{2n}(g) \big). \tag{4.13}$$

Proof. Step 1. Taking the limit as $j \to +\infty$ in (4.8), we obtain by (4.6) that, for any m, there exists a constant $K_1(m)$ such that:

$$\Big\| \Big(\nabla - i \frac{x^2}{2} \hat{\mathbf{i}}_y \Big) (|x|^{m/2} u) \Big\|_2^2 \leq K_1(m) \left(\|u\|_2^2 + p_{[(m+1)/2]}(g) \|u\|_2 \right).$$

Using the above together with Cauchy's inequality we obtain, for any m, the existence of a constant $K_2(m)$ such that:

$$\|(|x|^{m/2} u)_x\|_2 + \||x|^{m/2} u_y\|_2 \leq K_2(m) \big(\||x|^{m/2+2} u\|_2 + \|u\|_2 + p_{[(m+1)/2]}(g) \big),$$

and hence, taking $m = 2k$ in this inequality and using (4.6) we get the existence of a constant $C_1(k)$ depending on k such that

$$\|x^k u_x\|_2 + \|x^k u_y\|_2 \leq C_1(k) \left(\|u\|_2 + p_{k+2}(g) \right). \tag{4.14}$$

Step 2. Let $\zeta \in C_0^\infty(\mathbb{R}_+)$ be given by

$$\zeta(t) = \begin{cases} 1 & \text{if } t < 1, \\ 0 & \text{if } 2 < t, \end{cases}$$

and

$$0 \leq \zeta \leq 1, \quad |\zeta'| \leq 2.$$

Let further

$$\zeta_\ell(t) = \zeta(2^\ell t).$$

Clearly, for any ℓ,

$$\zeta_\ell \leq \zeta_{\ell-1}, \quad |\zeta_j'| \leq 2^{\ell+1} \zeta_{\ell-1}. \tag{4.15}$$

In the following, for fixed x_0 and y_0 we write

$$\zeta_{\ell,x_0}(x) = \zeta_\ell(|x - x_0|), \quad \zeta_{\ell,y_0}(y) = \zeta_\ell(|y - y_0|).$$

Let $\phi = \zeta_{\ell,y_0}(y) \zeta_{k,x_0}(x)$ in (4.4) to obtain for the real part

$$\Big\| \Big(\nabla - i \frac{x^2}{2} \hat{\mathbf{i}}_y \Big) (\zeta_{\ell,y_0} \zeta_{k,x_0} u) \Big\|_2^2 = \operatorname{Re} \lambda \|\zeta_{\ell,y_0} \zeta_{k,x_0} u\|_2^2 + \|\nabla(\zeta_{\ell,y_0} \zeta_{k,x_0}) u\|_2^2 + \operatorname{Re} \langle \zeta_{\ell,y_0} \zeta_{k,x_0} u, \zeta_\ell g \rangle_2 .$$

Letting $k \to +\infty$ yields

$$\begin{aligned}\Big\|\Big(\nabla - i\frac{x^2}{2}\hat{\mathbf{i}}_y\Big)(\zeta_{\ell,y_0}u)\Big\|_2^2 =& \mathrm{Re}\,\lambda\|\zeta_{\ell,y_0}u\|_2^2 + \|\zeta'_{\ell,y_0}u\|_2^2 + \mathrm{Re}\,\langle\zeta_{\ell,y_0}u,\zeta_\ell g\rangle_2 \\ \leq& K_3(\ell)\left(\|\zeta_{\ell-1,y_0}u\|_2^2 + \|\zeta_{\ell,y_0}g\|_2\|\zeta_{\ell,y_0}u\|_2\right),\end{aligned} \tag{4.16}$$

where $K_3(\ell)$ depends only on ℓ. Here we have used (4.15). From the imaginary part of (4.4) we obtain, after the limit $k \to \infty$ is taken, that

$$\begin{aligned}c\|y^{1/2}\zeta_{\ell,y_0}u\|_2^2 =& \mathrm{Im}\,\lambda\|\zeta_{\ell,y_0}u\|_2^2 + \mathrm{Im}\,\langle\zeta_{\ell,y_0}u,\zeta_{j,y_0}g\rangle_2 + 2\mathrm{Im}\,\Big\langle\Big(\frac{\partial}{\partial y} - i\frac{x^2}{2}\Big)(\zeta_{\ell,y_0}u),\zeta'_{\ell,y_0}u\Big\rangle_2 \\ \leq& |\mathrm{Im}\,\lambda|\|\zeta_{j,y_0}u\|_2^2 + \|\zeta_{\ell,y_0}g\|_2\|\zeta_{\ell,y_0}u\|_2 + 2\Big\|\Big(\nabla - i\frac{x^2}{2}\hat{\mathbf{i}}_y\Big)(\zeta_{\ell,y_0}u)\Big\|_2\|\zeta'_{\ell,y_0}u\|_2 .\end{aligned} \tag{4.17}$$

With the aid of (4.15) and (4.16) we thus obtain, for any ℓ the existence of $C_2(\ell) \geq 1$ such that,

$$c^{1/2}\|y^{1/2}\zeta_{\ell,y_0}u\|_2 \leq C_2(\ell)\big(\|\zeta_{\ell-1,y_0}u\|_2 + \|\zeta_{\ell-1,y_0}g\|_2\big)\,. \tag{4.18}$$

Step 3. We now establish the H^2 estimates of u. Using (4.2) we obtain that

$$\begin{cases} -\Delta u = -\Big[\dfrac{x^4}{4} - \lambda\Big]u - ix^2u_y + \lambda u - i\,cyu + g\,, & \text{in } \mathbb{R}^2_+\,, \\ u = 0 \qquad \text{on } \partial\mathbb{R}^2_+\,. \end{cases} \tag{4.19}$$

For any $\mathbf{x}_0 = (x_0, y_0)$ with $x_0 \in \mathbb{R}$ and $y_0 > 0$ and $\ell \in \mathbb{N}$, we let

$$\zeta_{\ell,x_0,y_0}(x,y) = \zeta_{\ell,x_0}(x)\,\zeta_{\ell,y_0}(y)\chi_+(y)\,,$$

where $\chi_+(y)$ is the characteristic function of the positive y-axis. Applying the standard elliptic L^2 estimates to (4.19) and taking $\ell = 4$, we can show that, for some constant K_4,

$$\|u\|_{H^2(B(\mathbf{x}_0,2^{-4}))} \leq K_4\left\{\|(x^4+1)\zeta_{4,x_0,y_0}u\|_2 + \|x^2\zeta_{4,x_0,y_0}u_y\|_2 + c\|y\zeta_{4,x_0,y_0}u\|_2 + \|\zeta_{4,x_0,y_0}g\|_2\right\}. \tag{4.20}$$

Then we use a sequence of intervals $A_i = (x_i - 2^{-5}, x_i + 2^{-5})$, $i \in \mathbb{Z}$, to cover the x-axis, and a sequence of intervals $B_j = (y_j - 2^{-5}, y_j + 2^{-5})$, $j \in \mathbb{N}$, to cover the positive y axis. We can choose $\{x_i\}$ such that

$$\sum_{i\in\mathbb{Z}} \frac{1}{1+|x_i|^2} < \infty.$$

We set $x_0 = x_i$ and $y_0 = y_j$ in (4.20) and take sum in i and j to get the existence of K_5 such that:

$$\begin{aligned}\|u\|_{H^2(\mathbb{R}^2_+)} \leq& K_5\Big[\sum_{i\in\mathbb{Z}}\big[\|x^4\zeta_{4,x_i}u\|_2 + \|x^2\zeta_{4,x_i}u_y\|_2 \\ &+ c\sum_{j\in\mathbb{N}}\|y\zeta_{4,y_j}u\|_2 + (\|u\|_2 + \|g\|_2)\Big]\,.\end{aligned} \tag{4.21}$$

Step 3.1. We now claim that, for any integer $k \geq 1$, there exists a constant $C_3(k)$ such that for any i it holds that

$$\|x^4\zeta_{4,x_i}u\|_2 + \|x^2\zeta_{4,x_i}u_y\|_2 \leq \frac{C_3(k)}{1+|x_i|^k}\big(\|u\|_2 + p_{k+4}(g)\big)\,, \quad \forall k \geq 1\,. \tag{4.22}$$

In the following we denote the constant $C(k)$ in (4.6) by $C_0(k)$. It is easy to show that (4.22) is valid with $C_3(k) \geq 21(1+2^k)$ for all i such that $|x_i| \leq 2$. We now assume $|x_i| > 2$. For any $k \geq 1$ we use (4.6) to obtain

$$\begin{aligned} C_0(k+4)(\|u\|_2 + p_{k+4}(g)) &\geq \|x^{k+4}u\|_2 \geq \|x^{k+4}\zeta_{4,x_i}u\|_2 \\ &\geq \Big(\frac{1+|x_i|}{2}\Big)^k \|x^4\eta_{4,x_i}u\|_2 \geq \frac{1+|x_i|^k}{2^k}\|x^4\eta_{4,x_i}u\|_2\,. \end{aligned}$$

Here we have used the fact that, on the support of ζ_{4,x_i} we have $|x-x_i| \leq 2^{-3}$, yielding that $|x| \geq |x_i| - 2^{-3} \geq (|x_i|+1)/2$. Hence

$$\|x^4\eta_{4,x_i}u\|_2 \leq \frac{2^kC_0(k+4)}{1+|x_i|^k}(\|u\|_2 + p_{k+4}(g))\,. \tag{4.23}$$

Similarly, we obtain that

$$\|x^2\eta_{4,x_i}u_y\|_2 \leq \frac{2^kC_1(k+2)}{1+|x_i|^k}(\|u\|_2 + p_{k+4}(g))\,, \tag{4.24}$$

where $C_1(k+2)$ is the constant appearing in (4.14). Combining (4.23) and (4.24) yields (4.22)

Step 3.2. Next, we claim that there exists $C_4 > 0$ such that, for any j such that $y_j > 2$,

$$c\,\|y\zeta_{4,y_j}u\|_2 \leq C_4\left(\|\zeta_{2,y_j}u\|_2 + p_1(\zeta_{2,y_j}g)\right). \tag{4.25}$$

To prove (4.25), we first note that for $\ell \geq 2$, if $\zeta_{\ell,y_j}(y) \neq 0$ then $|y-y_j| \leq 2^{1-\ell}$, and hence $y_j/2 \leq y \leq 2y_j$. Using (4.18) we obtain

$$c^{1/2}(\frac{y_j}{2})^{1/2}\|\zeta_{4,y_j}u\|_2 \leq c\|y^{1/2}\zeta_{4,y_j}u\|_2 \leq C_2(4)\big(\|\zeta_{3,y_j}u\|_2 + \|\zeta_{3,y_j}g\|_2\big)\,.$$

Therefore,

$$\|\zeta_{4,y_j}u\|_2 \leq \frac{2^{1/2}C_2(4)}{c^{1/2}y_j^{1/2}}\big(\|\zeta_{3,y_j}u\|_2 + \|\zeta_{3,y_i}g\|_2\big)\,.$$

For the same reason we have

$$\|\zeta_{3,y_j}u\|_2 \leq \frac{2^{1/2}C_2(3)}{c^{1/2}y_j^{1/2}}\big(\|\zeta_{2,y_j}u\|_2 + \|\zeta_{2,y_j}g\|_2\big)\,.$$

Using the above two inequalities we have just proved, and noting that $C_2(n) \geq 1$, we obtain that

$$\|\zeta_{4,y_j}u\|_2 \leq \frac{K_6}{cy_j}\Big\{\|\zeta_{2,y_j}u\|_2 + \|\zeta_{2,y_j}g\|_2 + \|\epsilon y\zeta_{3,y_j}g\|_2\Big\}.$$

Since $g \in \mathcal{S}\big(\overline{\mathbb{R}^2_+},\mathbb{C}\big)$, it easily follows the existence of a constant C_4 such that, for any j, g and $c > 0$,

$$c\|y\zeta_{4,y_j}u\|_2 \leq 2c\|y_j\zeta_{4,y_j}u\|_2 \leq C_4\big(\|\zeta_{2,y_j}u\|_2 + p_1(\zeta_{2,y_j}g)\big)\,,$$

and this completes the proof of (4.25).

For later reference, we repeat the above procedure twice more to obtain, for some constant C_5,

$$c^2\|y^2\zeta_{4,y_j}u\|_2 \leq C_5\big(\|\zeta_{0,y_j}u\|_2 + p_2(\zeta_{0,y_j}g)\big)\,. \tag{4.26}$$

Step 3.3. We now combine (4.21), (4.22) and (4.25) to obtain, for any $k\geq 2$,

$$\begin{aligned}\|u\|_{H^2(\mathbb{R}^2_+)} \leq & K_5\sum_{i\in\mathbb{Z}}\frac{C_3(k)}{1+|x_i|^k}\big(\|u\|_2 + p_{k+4}(g)\big)\\ & + K_5(C_4+1)\sum_{j\in\mathbb{N}}\big(\|\zeta_{2,y_j}u\|_2 + p_1(\zeta_{2,y_j}g)\big) + K_5(\|u\|_2+\|g\|_2).\end{aligned}$$

Setting $k=2$ in the above inequality yields the existence of C_6 such that:

$$\|u\|_{H^2(\mathbb{R}^2_+)} \leq C_6\left(\|u\|_2 + p_6(g)\right). \tag{4.27}$$

Hence, (4.13) holds for $n=2$.

Step 4. To bound the H^3 norm of u we need to bound the H^1 norm the right-hand-side of (4.19), which is bounded by

$$\begin{aligned}\|\Delta u\|_{H^1(\mathbb{R}^2_+,\mathbb{C})} \leq & \|x^2u_{yy}\|_2 + \|x^2u_{xy}\|_2 + \|(x^2+1)u_x\|_2 + 4\|(x^2+1)u_y\|_2 + 2\|(x^2+1)u\|_2\\ & + c\|(y+1)u\|_2 + c\|yu_x\|_2 + c\|yu_y\|_2\,.\end{aligned} \tag{4.28}$$

Once we manage to obtain a bound for the right-hand-side of the above inequality, we can use the standard regularity theory for the Dirichlet Laplacian on the half-space.

Step 4.1. We first compute the terms $\|x^2u_{yy}\|_2 + \|x^2u_{xy}\|_2$ in (4.28). Let D^2u denote the Hessian matrix of u. Using (4.20) we have, for any $x_0\in\mathbb{R}$ and $y_0>0$

$$\begin{aligned}&(x_0^2+1)\|D^2u\|_{L^2(B(\mathbf{x}_0,2^{-4}))}\\ \leq & C_3(x_0^2+1)\{\|(x^4+1)\zeta_{4,x_0,y_0}u\|_2 + \|x^2\zeta_{4,x_0,y_0}u_y\|_2 + c\|y\zeta_{4,x_0,y_0}u\|_2 + \|\zeta_{4,x_0,y_0}g\|_2\}\,,\end{aligned}$$

which can be written as

$$\begin{aligned}\|(x^2+1)D^2u\|_{L^2(B(\mathbf{x}_0,2^{-4}))} \leq & K_7\{\|(x^6+1)\zeta_{4,x_0,y_0}u\|_2 + \|(x^4+1)\zeta_{4,x_0,y_0}u_y\|_2\\ & + 2[(x_0^2+1)^2 + c^2y_0^2]\|\zeta_{4,x_0,y_0}u\|_2 + \|(x^2+1)\zeta_{4,x_0,y_0}g\|_2\}\,.\end{aligned}$$

As in Step 3 we cover the x-axis by the intervals $\{A_i\}$ and cover the positive y-axis by the intervals $\{B_j\}$. Then, we let $x_0=x_i$ and $y_0=y_j$ in the above inequality, and then sum in i and j to obtain

$$\begin{aligned}\|(x^2+1)D^2u\|_2 \leq & K_8\Big[\sum_{i\in\mathbb{Z}}\Big\{\|(x^6+1)\zeta_{4,x_i}u\|_2 + \|(x^4+1)\zeta_{4,x_i}u_y\|_2 + \|(x^2+1)\zeta_{4,x_i}g\|_2\Big\}\\ & + c^2\sum_{j\in\mathbb{N}}\|y^2\zeta_{4,y_j}u\|_2\Big]\,.\end{aligned} \tag{4.29}$$

We now estimate the terms in the right side of (4.29). For the first two terms we use the same argument as in the proof of (4.22). Thus,

$$\|(x^6+1)\zeta_{4,x_i}u\|_2+\|(x^4+1)\zeta_{4,x_i}u_y\|_2\leq\frac{C_3(k+2)}{1+|x_i|^k}\big(\|u\|_2+p_{k+4}(g)\big)\,,\quad\forall k\geq 1\,. \tag{4.30}$$

For the third term we use (4.26) to obtain

$$c^2\|y^2\zeta_{4,x_i,y_j}u\|_2\leq C_5\big(\|\zeta_{0,y_j}u\|_2+p_2(\zeta_{0,y_j}g)\big)\,. \tag{4.31}$$

Now we use (4.30) and (4.31) to bound the terms on the right-hand-side of (4.29), and take $k=2$ to obtain the existence of K_8 and K_9 such that:

$$\begin{aligned}\|(x^2+1)D^2u\|_2\leq&K_8\Big[\sum_{i\in\mathbb{Z}}\Big\{\frac{1}{1+|x_i|^2}(\|u\|_2+p_4(g))\\&+\|(x^2+1)\zeta_{4,x_i}g\|_2\Big\}+\sum_{j\in\mathbb{N}}\|\zeta_{0,y_j}u\|_2+p_2(\zeta_{0,y_j}g)\Big]\\&\leq K_9\left(\|u\|_2+p_6(g)\right).\end{aligned} \tag{4.32}$$

Step 4.2. We next estimate the term $c\|y\nabla u\|_2$ in (4.28). From (4.25) we know that $yu\in L^2(\mathbb{R}^2_+,\mathbb{C})$. Hence, via a density argument, we can apply the first equality in (4.5) is valid with $\phi=y$, and

$$\Big\|\Big(\nabla-i\frac{x^2}{2}\hat{\mathbf{i}}_y\Big)(yu)\Big\|_2^2=\|u\|_2^2+(\operatorname{Re}\lambda)\|yu\|_2^2+\operatorname{Re}\langle yg,yu\rangle_2\,,$$

From this we have

$$\begin{aligned}c^2\|y\nabla u\|_2^2=&c^2\|\partial_x(yu)\|_2^2+c^2\Big\|\partial_y(yu)-\frac{ix^2}{2}(yu)+(\frac{ix^2y}{2}-1)u\Big\|_2^2\\\leq&2c^2\Big\|\Big(\nabla-i\frac{x^2}{2}\hat{\mathbf{i}}_y\Big)(yu)\Big\|_2^2+2c^2\Big\|(\frac{ix^2y}{2}-1)u\Big\|_2^2\\\leq&8c^2(\|u\|_2^2+\|yg\|_2^2)+c^4\|y^2u\|_2^2+\|x^4u\|_2^2\,.\end{aligned}$$

From the above, with the aid of (4.26) and (4.6), we obtain a constant C_7 such that:

$$c^2\|y\nabla u\|_2^2\leq C_7\left(\|u\|_2^2+p_4(g)\right). \tag{4.33}$$

Step 4.3. The rest of the terms in (4.28) can be easily estimated. From Lemma 4.1 and (4.14) we have

$$\|(x^2+1)u_x\|_2+4\|(x^2+1)u_y\|_2+2\|(x^2+1)u\|_2\leq C_8\left(\|u\|_2+p_4(g)\right),$$

and from (4.25) we derive

$$c\|(y+1)u\|_2\leq C_9\left(\|u\|_2+p_4(g)\right).$$

Since all the terms on the right-hand-side of (4.28) have been estimated we have

$$\|\Delta u\|_{H^1(\mathbb{R}^2_+,\mathbb{C})}\leq C_{10}\left(\|u\|_2+p_6(g)\right).$$

Hence standard elliptic estimates applied to (4.19) yield

$$\|u\|_{H^3(\mathbb{R}^2_+)}\leq C_{11}\left(\|u\|_2+p_6(g)\right). \tag{4.34}$$

Hence (4.13) holds for $n = 3$. Higher order Sobolev norms can similarly be obtained in a recursive manner. ∎

Recall that $c = \epsilon^3$. We next define the following cutoff function

$$\chi_\epsilon(y) = \begin{cases} y \exp\big(-\epsilon^{-\alpha}(1-\epsilon^{\alpha+\gamma})\big) & \text{if } 0 \le y \le 1, \\ \exp\big(-\epsilon^{-\alpha}(1-\epsilon^{\alpha+\gamma}y)\big) & \text{if } 1 < y \le \epsilon^{-(\alpha+\gamma)}, \\ 1 & \text{if } \epsilon^{-(\alpha+\gamma)} \le y, \end{cases} \tag{4.35}$$

where α and γ satisfy $\alpha > 0, 0 < \gamma < 1$ and $\alpha + \gamma \le 1$. Furthermore,

$$\chi_\epsilon'(y) = \begin{cases} \exp\big(-\epsilon^{-\alpha}(1-\epsilon^{\alpha+\gamma})\big) & \text{if } 0 \le y \le 1, \\ \epsilon^\gamma \exp\big(-\epsilon^{-\alpha}(1-\epsilon^{\alpha+\gamma}y)\big) = \epsilon^\gamma \chi_\epsilon & \text{if } 1 < y \le \epsilon^{-(\alpha+\gamma)}, \\ 0 & \text{if } \epsilon^{-(\alpha+\gamma)} < y. \end{cases}$$

Thus, for sufficiently ϵ,

$$\chi_\epsilon'(y) \ge 0, \quad |\chi_\epsilon'| \le \epsilon^\gamma \chi_\epsilon + 2e^{-\epsilon^{-\alpha}}. \tag{4.36}$$

Hence for every $u \in L^2(\mathbb{R}^2_+, \mathbb{C})$ we have

$$\|\chi_\epsilon' u\|_2 \le \epsilon^\gamma \|\chi_\epsilon u\|_2 + 2e^{-\epsilon^{-\alpha}} \|u\|_2 \,. \tag{4.37}$$

The proof of the next lemma relies on the following elementary implication:

$$\text{For } (a,b) \in \mathbb{R}_+ \times \mathbb{R}_+, \text{ if } X^2 \le a^2 + bX \,, \text{ then } X^2 \le 2a^2 + b^2 \,. \tag{4.38}$$

Lemma 4.3. *Let $c_0 > 0$, $M > 0$, $\delta \in (0, 1/2)$, $\alpha \in (0, 2\delta]$ and $n \in \mathbb{N}$. There exists a constant $C_{n,\delta} > 0$ such that, for any $g \in \mathcal{S}(\overline{\mathbb{R}^2_+}, \mathbb{C})$, $c \in (0, c_0)$, $\epsilon = c^{1/3}$, and $\lambda \in \mathbb{C}$ with $|\lambda - E_0^*| \le M\epsilon$, the solution u of (4.2) satisfies*

$$\begin{aligned} \Big\|\chi_\epsilon y \frac{\partial^n u}{\partial y^n}\Big\|_2 \le & C_{n,\delta}\Big[\frac{1}{\epsilon^{2+\delta}}\Big\|\chi_\epsilon \frac{\partial^n u}{\partial y^n}\Big\|_2 + \epsilon^\delta \Big\|\frac{\partial^{n-1} u}{\partial y^{n-1}}\Big\|_2 + \frac{1}{\epsilon^3}\Big\|\chi_\epsilon \frac{\partial^n g}{\partial y^n}\Big\|_2\Big] \\ & + C_{n,\delta} e^{-\frac{1}{6}\epsilon^{-\alpha}} \big(\|u\|_2 + p_{2n}(g)\big) \,. \end{aligned} \tag{4.39}$$

Proof. Step 1. Differentiating (4.2) n times yields

$$(\mathcal{A}_{0,c} - \lambda)\frac{\partial^n u}{\partial y^n} = \frac{\partial^n g}{\partial y^n} + i\, n\, \epsilon^3 \, \frac{\partial^{n-1} u}{\partial y^{n-1}} \quad \text{in } \mathbb{R}^2_+ \,. \tag{4.40}$$

Multiplying (4.40) by $\chi_\epsilon^2(y)\partial^n \bar{u}/\partial y^n$ and integrating by parts yields for the real part

$$\begin{aligned} \Big\|\Big(\nabla - i\frac{x^2}{2}\hat{\mathbf{i}}_y\Big)\Big(\chi_\epsilon \frac{\partial^n u}{\partial y^n}\Big)\Big\|_2^2 = & \operatorname{Re}\lambda \, \Big\|\chi_\epsilon \frac{\partial^n u}{\partial y^n}\Big\|_2^2 \Big\|\chi_\epsilon' \frac{\partial^n u}{\partial y^n}\Big\|_2^2 \\ & + \operatorname{Re}\Big\langle \chi_\epsilon \frac{\partial^n u}{\partial y^n}, \chi_\epsilon\Big(\frac{\partial^n g}{\partial y^n} + in\epsilon^3 \frac{\partial^{n-1} u}{\partial y^{n-1}}\Big)\Big\rangle \,, \end{aligned} \tag{4.41}$$

and for the imaginary part

$$\begin{aligned} \epsilon^3 \Big\|y^{1/2} \chi_\epsilon \frac{\partial^n u}{\partial y^n}\Big\|_2^2 = & \operatorname{Im}\lambda \Big\|\chi_\epsilon \frac{\partial^n u}{\partial y^n}\Big\|_2^2 + \Big\langle \chi_\epsilon \frac{\partial^n u}{\partial y^n}, \chi_\epsilon \frac{\partial^n g}{\partial y^n}\Big\rangle - 2n\epsilon^3 \Big\langle \chi_\epsilon \chi_\epsilon', \Big|\frac{\partial^{n-1} u}{\partial y^{n-1}}\Big|^2\Big\rangle \\ & + 2\operatorname{Im}\Big\langle \Big(\frac{\partial}{\partial y} - i\frac{x^2}{2}\Big)\Big(\chi_\epsilon \frac{\partial^n u}{\partial y^n}\Big), \chi_\epsilon' \frac{\partial^n u}{\partial y^n}\Big\rangle \,. \end{aligned} \tag{4.42}$$

By (4.41), (4.13) and (4.37), we have that, for all $0 < c < c_0$,

$$\begin{aligned}\Big\|\Big(\nabla - i\frac{x^2}{2}\hat{\mathbf{i}}_y\Big)\Big(\chi_\epsilon\frac{\partial^n u}{\partial y^n}\Big)\Big\|_2^2 \leq & C_1\Big\|\chi_\epsilon\frac{\partial^n u}{\partial y^n}\Big\|_2^2 + 2\Big\|\chi_\epsilon\frac{\partial^n g}{\partial y^n}\Big\|_2^2 + 4n\epsilon^6\Big\|\chi_\epsilon\frac{\partial^{n-1} u}{\partial y^{n-1}}\Big\|_2^2 \\ & + C_1 e^{-\epsilon^{-\alpha}}\big(\|u\|_2 + p_{2n}(g)\big)^2 ,\end{aligned} \tag{4.43}$$

where C_1, like all other constants introduced in the following, depends on M, n, c_0, α and γ. In particular we have that

$$\Big\|\Big(\nabla - i\frac{x^2}{2}\hat{\mathbf{i}}_y\Big)\Big(\chi_\epsilon\frac{\partial^n u}{\partial y^n}\Big)\Big\|_2 \leq C_2\left(\|u\|_2 + p_{2n}(g)\right). \tag{4.44}$$

Next, we estimate the terms in the right side of (4.42). Since $\chi_\epsilon' \geq 0$, the third term in the right side of (4.42) is negative. For the fourth term on the right-hand-side of (4.42), we use (4.13), (4.44) and(4.37) to obtain

$$\begin{aligned}&\Big|2\,\mathrm{Im}\Big\langle\Big(\frac{\partial}{\partial y} - i\frac{x^2}{2}\Big)\Big(\chi_\epsilon\frac{\partial^n u}{\partial y^n}\Big), \chi_\epsilon'\frac{\partial^n u}{\partial y^n}\Big\rangle\Big| \\ \leq & 2\epsilon^\gamma\Big\|\Big(\nabla - i\frac{x^2}{2}\hat{\mathbf{i}}_y\Big)\Big(\chi_\epsilon\frac{\partial^n u}{\partial y^n}\Big)\Big\|_2 \cdot \Big\|\chi_\epsilon\frac{\partial^n u}{\partial y^n}\Big\| + C_3\, e^{-\epsilon^{-\alpha}}(\|u\|_2 + p_{2n}(g))^2 .\end{aligned}$$

From the above and (4.42) we obtain

$$\begin{aligned}\epsilon^3\Big\|y^{1/2}\chi_\epsilon\frac{\partial^n u}{\partial y^n}\Big\|_2^2 \leq & M\,\epsilon\Big\|\chi_\epsilon\frac{\partial^n u}{\partial y^n}\Big\|_2^2 + \Big\|\chi_\epsilon\frac{\partial^n u}{\partial y^n}\Big\|_2\Big\|\chi_\epsilon\frac{\partial^n g}{\partial y^n}\Big\|_2 \\ & + 2\epsilon^\gamma\Big\|\Big(\nabla - i\frac{x^2}{2}\hat{\mathbf{i}}_y\Big)\Big(\chi_\epsilon\frac{\partial^n u}{\partial y^n}\Big)\Big\|_2\Big\|\chi_\epsilon\frac{\partial^n u}{\partial y^n}\Big\|_2 + C_3\, e^{-\epsilon^{-\alpha}}\big(\|u\|_2 + p_{2n}(g)\big)^2 .\end{aligned} \tag{4.45}$$

Combining (4.43) with (4.45) yields

$$\begin{aligned}\epsilon^3\Big\|y^{1/2}\chi_\epsilon\frac{\partial^n u}{\partial y^n}\Big\|_2^2 \leq & C_4\epsilon^\gamma\Big\|\chi_\epsilon\frac{\partial^n u}{\partial y^n}\Big\|_2^2 + C_4\,\Big\|\chi_\epsilon\frac{\partial^n u}{\partial y^n}\Big\|_2\Big\|\chi_\epsilon\frac{\partial^n g}{\partial y^n}\Big\|_2 \\ & + C_4\epsilon^3\Big\|\chi_\epsilon\frac{\partial^n u}{\partial y^n}\Big\|_2\Big\|\chi_\epsilon\frac{\partial^{n-1} u}{\partial y^{n-1}}\Big\|_2 + C_4 C(n) e^{-\epsilon^{-\alpha}}\big(\|u\|_2 + p_{2n}(g)\big)^2 .\end{aligned}$$

Using the above in conjunction with Cauchy's inequality yields

$$\Big\|\chi_\epsilon\frac{\partial^n u}{\partial y^n}\Big\|_2\Big\|\chi_\epsilon\frac{\partial^{n-1} u}{\partial y^{n-1}}\Big\|_2 \leq \frac{\epsilon^\gamma}{2}\Big\|\chi_\epsilon\frac{\partial^n u}{\partial y^n}\Big\|_2^2 + \frac{1}{2\epsilon^\gamma}\Big\|\chi_\epsilon\frac{\partial^{n-1} u}{\partial y^{n-1}}\Big\|_2^2 .$$

Consequently,

$$\begin{aligned}\Big\|y^{1/2}\chi_\epsilon\frac{\partial^n u}{\partial y^n}\Big\|_2^2 \leq & C_5\Big[\frac{1}{\epsilon^{3-\gamma}}\Big\|\chi_\epsilon\frac{\partial^n u}{\partial y^n}\Big\|_2^2 + \epsilon^{3-\gamma}\Big\|\chi_\epsilon\frac{\partial^{n-1} u}{\partial y^{n-1}}\Big\|_2^2 \\ & + \frac{1}{\epsilon^3}\Big\|\chi_\epsilon\frac{\partial^n u}{\partial y^n}\Big\|_2\Big\|\chi_\epsilon\frac{\partial^n g}{\partial y^n}\Big\|_2 + \frac{1}{\epsilon}e^{-\epsilon^{-\alpha}}\big(\|u\|_2 + p_n(g)\big)^2\Big] .\end{aligned} \tag{4.46}$$

Step 2. Multiplying (4.40) by $y\chi_\epsilon^2(y)\partial^n\bar{u}/\partial y^n$ and integrating by parts yields for the imaginary part

$$\epsilon^3\Big\|y\chi_\epsilon\frac{\partial^n u}{\partial y^n}\Big\|_2^2 = \operatorname{Im}\lambda\Big\|\chi_\epsilon y^{1/2}\frac{\partial^n u}{\partial y^n}\Big\|_2^2 + \Big\langle\chi_\epsilon y\frac{\partial^n u}{\partial y^n},\chi_\epsilon\frac{\partial^n g}{\partial y^n}\Big\rangle$$
$$- n\epsilon^3\Big\langle(\chi_\epsilon^2+2y\chi_\epsilon\chi_\epsilon'),\Big|\frac{\partial^{n-1}u}{\partial y^{n-1}}\Big|^2\Big\rangle + \operatorname{Im}\Big\langle\Big(\frac{\partial}{\partial y}-i\frac{x^2}{2}\Big)\Big(\chi_\epsilon\frac{\partial^n u}{\partial y^n}\Big),(\chi_\epsilon+2\chi_\epsilon' y)\frac{\partial^n u}{\partial y^n}\Big\rangle\,. \tag{4.47}$$

Using again the fact $\chi_\epsilon'(y)\geq 0$, the third term in the right side of (4.47) is negative, and hence

$$\begin{aligned}\epsilon^3\Big\|y\chi_\epsilon\frac{\partial^n u}{\partial y^n}\Big\|_2^2 \leq & M\epsilon\Big\|y^{1/2}\chi_\epsilon\frac{\partial^n u}{\partial y^n}\Big\|_2^2 + \Big\|y\chi_\epsilon\frac{\partial^n u}{\partial y^n}\Big\|_2\Big\|\chi_\epsilon\frac{\partial^n g}{\partial y^n}\Big\|_2 \\ & + \Big\|\Big(\nabla - i\frac{x^2}{2}\hat{\mathbf{i}}_y\Big)\Big(\chi_\epsilon\frac{\partial^n u}{\partial y^n}\Big)\Big\|_2\Big\|(\chi_\epsilon+2\chi_\epsilon' y)\frac{\partial^n u}{\partial y^n}\Big\|_2\,.\end{aligned} \tag{4.48}$$

Note that, when $\epsilon\leq 1$,

$$\begin{aligned} y\chi_\epsilon'(y) = & \begin{cases} y\chi_\epsilon(y) & \text{if } 0\leq y\leq 1\,, \\ y\epsilon^\gamma\chi_\epsilon \leq \epsilon^{-\alpha}\chi_\epsilon(y) & \text{if } 1<y\leq\epsilon^{-(\alpha+\gamma)}\,, \\ 0 & \text{if } \epsilon^{-(\alpha+\gamma)}<y\,, \end{cases} \\ \leq & \quad \epsilon^{-\alpha}\chi_\epsilon(y)\,. \end{aligned}$$

Hence,

$$\Big\|(\chi_\epsilon+2\chi_\epsilon' y)\frac{\partial^n u}{\partial y^n}\Big\|_2 \leq 3\epsilon^{-\alpha}\Big\|\chi_\epsilon\frac{\partial^n u}{\partial y^n}\Big\|_2\,. \tag{4.49}$$

Substituting (4.49) into (4.48) yields, with the aid of (4.38), that

$$\begin{aligned}\Big\|y\chi_\epsilon\frac{\partial^n u}{\partial y^n}\Big\|_2^2 \leq & \frac{2M}{\epsilon^2}\Big\|y^{1/2}\chi_\epsilon\frac{\partial^n u}{\partial y^n}\Big\|_2^2 + \frac{1}{\epsilon^6}\Big\|\chi_\epsilon\frac{\partial^n g}{\partial y^n}\Big\|_2^2 \\ & + \frac{6}{\epsilon^{3+\alpha}}\Big\|\Big(\nabla - i\frac{x^2}{2}\hat{\mathbf{i}}_y\Big)\Big(\chi_\epsilon\frac{\partial^n u}{\partial y^n}\Big)\Big\|_2\Big\|\chi_\epsilon\frac{\partial^n u}{\partial y^n}\Big\|_2\,.\end{aligned} \tag{4.50}$$

We next substitute (4.46) and (4.43) into (4.50) to obtain

$$\begin{aligned}\Big\|y\chi_\epsilon\frac{\partial^n u}{\partial y^n}\Big\|_2^2 \leq C_6\Big[& \frac{1}{\epsilon^{5-\gamma}}\Big\|\chi_\epsilon\frac{\partial^n u}{\partial y^n}\Big\|_2^2 + \epsilon^{1-\gamma}\Big\|\chi_\epsilon\frac{\partial^{n-1}u}{\partial y^{n-1}}\Big\|_2^2 + \frac{1}{\epsilon^6}\Big\|\chi_\epsilon\frac{\partial^n g}{\partial y^n}\Big\|_2^2 \\ & \epsilon^{-\alpha}\Big\|\chi_\epsilon\frac{\partial^{n-1}u}{\partial y^{n-1}}\Big\|_2\Big\|\chi_\epsilon\frac{\partial^n u}{\partial y^n}\Big\|_2 + \frac{1}{\epsilon^5}\Big\|\chi_\epsilon\frac{\partial^n u}{\partial y^n}\Big\|_2\Big\|\chi_\epsilon\frac{\partial^n g}{\partial y^n}\Big\|_2 \\ & + \frac{1}{\epsilon^{3+\alpha}}e^{-\frac{1}{2}\epsilon^{-\alpha}}\big(\|u\|_2+p_{2n}(g)\big)\Big\|\chi_\epsilon\frac{\partial^n u}{\partial y^n}\Big\|_2 + \frac{1}{\epsilon^3}e^{-\epsilon^{-\alpha}}\big(\|u\|_2+p_{2n}(g)\big)^2\Big]\,.\end{aligned} \tag{4.51}$$

Making use of Cauchy's inequality yields

$$2\epsilon^{-\alpha}\Big\|\chi_\epsilon\frac{\partial^{n-1}u}{\partial y^{n-1}}\Big\|_2\Big\|\chi_\epsilon\frac{\partial^n u}{\partial y^n}\Big\|_2 \leq \frac{1}{\epsilon^{5-\gamma}}\Big\|\chi_\epsilon\frac{\partial^n u}{\partial y^n}\Big\|_2^2 + \epsilon^{5-\gamma-2\alpha}\Big\|\chi_\epsilon\frac{\partial^{n-1}u}{\partial y^{n-1}}\Big\|_2^2\,, \tag{4.52a}$$

$$\Big\|\chi_\epsilon\frac{\partial^n u}{\partial y^n}\Big\|_2\Big\|\chi_\epsilon\frac{\partial^n g}{\partial y^n}\Big\|_2 \leq \epsilon^\gamma\,\Big\|\chi_\epsilon\frac{\partial^n u}{\partial y^n}\Big\|_2^2 + \epsilon^{-\gamma}\,\Big\|_2\Big\|\chi_\epsilon\frac{\partial^n g}{\partial y^n}\Big\|_2^2\,. \tag{4.52b}$$

Furthermore, it follows from (4.13) that

$$\frac{e^{-\frac{1}{2}\epsilon^{-\alpha}}}{\epsilon^{3+\alpha}}\left(\|u\|_2+p_{2n}(g)\right)\Big\|\chi_\epsilon\frac{\partial^n u}{\partial y^n}\Big\|_2 \leq \frac{e^{-\frac{1}{2}\epsilon^{-\alpha}}}{\epsilon^{3+\alpha}}\left(\|u\|_2+p_{2n}(g)\right)^2. \tag{4.52c}$$

Substituting (4.52a,b,c) back to (4.51) we get

$$\Big\|y\chi_\epsilon\frac{\partial^n u}{\partial y^n}\Big\|_2^2 \leq C_7\Bigg[\frac{1}{\epsilon^{5-\gamma}}\Big\|\chi_\epsilon\frac{\partial^n u}{\partial y^n}\Big\|_2^2+\epsilon^{1-\gamma}\Big\|\chi_\epsilon\frac{\partial^{n-1}u}{\partial y^{n-1}}\Big\|_2^2 + \frac{1}{\epsilon^6}\Big\|\chi_\epsilon\frac{\partial^n g}{\partial y^n}\Big\|_2^2+\frac{e^{-\frac{1}{2}\epsilon^{-\alpha}}}{\epsilon^3}\left(\|u\|_2+p_{2n}(g)\right)^2\Bigg]. \tag{4.53}$$

Let $\delta = 1-\gamma/2$ (which readily yields $\alpha < 2\delta$). Substituting into (4.53) we obtain, for sufficiently small ϵ,

$$\Big\|y\chi_\epsilon\frac{\partial^n u}{\partial y^n}\Big\|_2^2 \leq C_8\Bigg[\frac{1}{\epsilon^{4+2\delta}}\Big\|\chi_\epsilon\frac{\partial^n u}{\partial y^n}\Big\|_2^2+\epsilon^{2\delta}\Big\|\chi_\epsilon\frac{\partial^{n-1}u}{\partial y^{n-1}}\Big\|_2^2+\frac{1}{\epsilon^6}\Big\|\chi_\epsilon\frac{\partial^n g}{\partial y^n}\Big\|_2^2+e^{-\frac{1}{3}\epsilon^{-\alpha}}\left(\|u\|_2+p_{2n}(g)\right)^2\Bigg].$$

Finally taking square root of the both sides we get (4.39) with $C_{n,\delta}=(C_8)^{\frac{1}{2}}$. ∎

We conclude this section by the following obvious estimate

Lemma 4.4. *Let f be given by (3.38), and χ_ϵ by (4.35). For every $n\in\mathbb{N}$ there exists $C_n>0$, such that for all sufficiently small ϵ we have*

$$\Big\|\chi_\epsilon\frac{\partial^n f}{\partial y^n}\Big\|_2 \leq C_n\,\epsilon^{n+5/2}\,. \tag{4.54}$$

Proof. As in the proof of Proposition 3.1 we prove (4.54) by estimating separately the derivatives of the two different types of terms in the expression of f mentioned there.

Let h_1 be a term of the first type given by (3.39). Then,

$$\Big\|\chi_\epsilon\frac{\partial^n h_1}{\partial y^n}\Big\|_2 \leq \Big\|\frac{\partial^n h_1}{\partial y^n}\Big\|_2 = \epsilon^{3+n}\|\psi_1^{(n+2)}\|_2 \leq C_1(n)\epsilon^{n+5/2}\,, \tag{4.55}$$

where the last equality is due to the fact that $\psi_1\in\mathcal{S}(\overline{\mathbb{R}_+},\mathbb{C})$.

Let h_2 be a term of the second type given by (3.41). Then,

$$\begin{aligned}\Big\|\chi_\epsilon\frac{\partial^n h_2}{\partial y^n}\Big\|_2 &= \epsilon^3\,\Big\|\chi_\epsilon\frac{\partial^n w_1}{\partial y^n}\Big\|_2\\ &\leq \epsilon^3\Big\|\frac{\partial^n w_1}{\partial y^n}\Big\|_{L^2(\mathbb{R}\times(\frac{1}{2}\epsilon^{-(\alpha+\gamma)},\infty))}+e^{-\frac{1}{2}\epsilon^{-\alpha}}\,\Big\|\frac{\partial^n w_1}{\partial y^n}\Big\|_{L^2(\mathbb{R}\times(0,\frac{1}{2}\epsilon^{-(\alpha+\gamma)}))}.\end{aligned}$$

Applying standard H^n elliptic estimates to (3.18) on the domain $\mathbb{R}\times(\frac{1}{4}\epsilon^{-(\alpha+\gamma)},+\infty)$ in conjunction with (2.5c), it is easy to show that there exists a constant K, and for every $k\geq 1$ $C_2(k)>0$, such that for all sufficiently small ϵ we have

$$\Big\|\frac{\partial^n w_1}{\partial y^n}\Big\|_{L^2(\mathbb{R}\times(\frac{1}{2}\epsilon^{-(\alpha+\gamma)},+\infty)}\leq K\,\|w_1\|_{L^2(\mathbb{R}\times(\frac{1}{4}\epsilon^{-(\alpha+\gamma)},+\infty)}\leq C_2(k)\,\epsilon^{k(\alpha+\gamma)}\,.$$

The same H^n estimates can also be applied to (3.18) on the domain $\mathbb{R}\times(0,\frac{3}{4}\epsilon^{-(\alpha+\gamma)})$, yielding

$$\Big\|\frac{\partial^n w_1}{\partial y^n}\Big\|_{L^2(\mathbb{R}\times(0,\frac{1}{2}\epsilon^{-(\alpha+\gamma)}))}\leq K\|w_1\|_{L^2(\mathbb{R}\times(0,\frac{3}{4}\epsilon^{-(\alpha+\gamma)}))}\leq C_3\,\epsilon^{-(\alpha+\gamma)}\,,$$

and hence

$$\Big\|\chi_\epsilon\frac{\partial^n h_2}{\partial y^n}\Big\|_2\leq C_2(k)\,\epsilon^{3+k(\alpha+\gamma)}+C_3\,\epsilon^{-(\alpha+\gamma)}e^{-\frac{1}{2}\epsilon^{-\alpha}}\leq C_4(k)\,\epsilon^{3+k(\alpha+\gamma)}\,.$$

Combining the above with (4.55) yields (4.54). ∎

5. "Outer" Estimates

To estimate the resolvent for λ in the vicinity of (3.35) we need to prove first some auxiliary estimates. Note that by (3.31), to leading order, the eigenfunction corresponding to an eigenvalue near Λ is expected to be proportional to ϕ^*. Thus, we define the following projections from $L^2(\mathbb{R}^2_+,\mathbb{C})$ into $L^2(\mathbb{R}^2_+,\mathbb{C})$

$$P_\|=\langle\cdot,\phi^*\rangle_x\,\phi^*\,,\qquad P_\perp=I-P_\|\,,$$

where $\langle\cdot,\cdot\rangle_x$ denotes the inner product in $L^2(\mathbb{R},\mathbb{C})$ with respect to the variable x. Set then, for any $v\in L^2(\mathbb{R}^2_+,\mathbb{C})$

$$v_\|=P_\|v,\qquad v_\perp=P_\perp v\,.$$

Then

$$v(x,y)=\phi^*(x)\psi[v](y)+v_\perp(x,y)\,,\qquad\psi[v](y)=\langle v(\cdot,y),\phi^*(\cdot)\rangle_x\,,\quad y\in\mathbb{R}_+\,.$$

A useful observation to make is that

$$\frac{\partial^n}{\partial y^n}v(x,y)=\phi^*(x)\psi[v]^{(n)}(y)+\frac{\partial^n}{\partial y^n}v_\perp(x,y)\,,$$

and hence

$$\frac{\partial^n}{\partial y^n}P_\perp v(x,y)=P_\perp\frac{\partial v}{\partial y^n}(x,y)\,. \tag{5.1}$$

The next lemma is necessary to show that the image of functions of small L^2 norm under the resolvent $(\mathcal{A}_c^+-\lambda)^{-1}$ lies almost entirely in $P_\|L^2(\mathbb{R}^2_+,\mathbb{C})$.

Lemma 5.1. *There exists positive constant C_1 such that, for any $u\in H_0^{1,\mathrm{mag}}(\mathbb{R}^2_+,\mathbb{C})$ and any $\gamma>0$ satisfying*

$$\Big\|\Big(\nabla-i\Big[\frac{x^2}{2}-s^*\Big]\hat{\mathbf{i}}_y\Big)u\Big\|_2^2-E_0^*\|u\|_2^2\leq\gamma^2\,, \tag{5.2}$$

it holds that

$$\|u_\perp\|_2\leq C_1\gamma^{1/2}\|u\|_2^{1/2}\,. \tag{5.3}$$

Proof. Assume first $u \in C_c^\infty(\mathbb{R}^2_+,\mathbb{C})$. Then $u_\perp = P_\perp u \in C_0^\infty(\mathbb{R}^2_+,\mathbb{C})$. Recall the notation

$$\|u\|_2 \equiv \|u\|_{L^2(\mathbb{R}^2_+)} = \|u\|_{L^2(\mathbb{R}^2)} = \|\hat{u}\|_{L^2(\mathbb{R}^2)} \equiv \|\hat{u}\|_2\,.$$

Step 1. Let as before $\hat{w}$ denote the partial Fourier transform of the extension to $\mathbb{R}^2$ of w with respect to y, defined by (2.13). We claim that

$$\int_{\mathbb{R}} |\hat{u}(x,s)|^2\,dx = \int_{\mathbb{R}} |\hat{u}_\parallel(x,s)|^2 dx + \int_{\mathbb{R}} |\hat{u}_\perp(x,s)|^2\,dx \quad \text{for all } s\in\mathbb{R}\,. \tag{5.4}$$

In fact, from the definition of $P_\parallel$ and $P_\perp$ we have

$$u_\parallel(x,y) = \langle u(\cdot,y),\phi^*(\cdot)\rangle_x \phi^*(x), \quad u_\perp(x,y) = u(x,y) - \langle u(\cdot,y),\phi^*(\cdot)\rangle_x\phi^*(x)\,.$$

Taking the partial Fourier transform in y yields

$$\begin{aligned}\hat{u}_\parallel(x,s) =&\mathcal{F}_y[\langle u(\cdot,y),\phi^*(\cdot)\rangle_x\phi^*(x)] = \langle \mathcal{F}_y[u(\cdot,y)],\phi^*(\cdot)\rangle_x\phi^*(x)\\ =&\langle \hat{u}(\cdot,s),\phi^*(\cdot)\rangle_x\phi^*(x) = b(s)\phi^*(x)\,,\\ \hat{u}_\perp(x,s) =&\mathcal{F}_y[u - \langle u,\phi^*\rangle_x\phi^*(x)] = \hat{u}(x,s) - \langle\hat{u}(\cdot,s),\phi^*(\cdot)\rangle_x\phi^*(x)\\ =&\hat{u}(x,s) - b(s)\phi^*(x)\,.\end{aligned}$$

Hence,

$$\hat{u}(x,s) = \hat{u}_\parallel(x,s) + \hat{u}_\perp(x,s)\,,$$

where

$$b(s) = \langle \hat{u}(\cdot,s),\phi^*(\cdot)\rangle_x\,.$$

Note that for any $s\in\mathbb{R}$

$$\begin{aligned}&\int_{\mathbb{R}} \hat{u}_\parallel(x,s)\cdot\overline{\hat{u}_\perp(x,s)}\,dx = \int_{\mathbb{R}} b(s)\phi^*(x)\cdot\overline{\hat{u}(x,s)-b(s)\phi^*(x)}\,dx\\ =&b(s)\int_{\mathbb{R}}\phi^*(x)[\bar{\hat{u}}(x,s) - \overline{b(s)\phi^*(x)}]\,dx\\ =&b(s)\int_{\mathbb{R}}\phi^*(x)\bar{\hat{u}}(x,s)dx - |b(s)|^2\int_{\mathbb{R}}|\phi^*(x)|^2\,dx = 0\,.\end{aligned}$$

Thus, $\hat{u}_\parallel(\cdot,s)$ and $\hat{u}_\perp(\cdot,s)$ are orthogonal to each other in $L^2(\mathbb{R}_x,\mathbb{C})$ for any fixed s, thereby verifying (5.4).

Step 2. Let

$$S_\delta = \{s\in\mathbb{R}\,:\, E_0(s) - E_0^* > \delta\}\,.$$

The precise value of $0<\delta<1$ will be assigned later. We now show that

$$\|\hat{u}_\perp\|_{L^2(\mathbb{R}\times S_\delta)} \leq \frac{\gamma}{\delta^{1/2}}\,. \tag{5.5}$$

To prove (5.5) we rewrite (5.2) in terms of the partial Fourier transform of u in y:

$$\int_{\mathbb{R}^2}\Big(|\partial_x\hat{u}|^2 + \Big|\Big(\frac{x^2}{2} - s^* - s\Big)\hat{u}\Big|^2\Big)\,dxds \leq \gamma^2 + E_0^*\int_{\mathbb{R}^2}|\hat{u}|^2 dxds\,. \tag{5.6}$$

Consequently,

$$\int_{\mathbb{R}^2}[E_0(s) - E_0^*]|\hat{u}|^2\,dxds \leq \gamma^2\,,$$

and hence,

$$\|\hat{u}\|^2_{L^2(\mathbb{R}\times S_\delta)} \leq \frac{\gamma^2}{\delta}\,. \tag{5.7}$$

On the other hand, integrating (5.4) with respect to s on S_δ yields

$$\|\hat{u}\|^2_{L^2(\mathbb{R}\times S_\delta)} = \|\hat{u}_\|\|^2_{L^2(\mathbb{R}\times S_\delta)} + \|\hat{u}_\perp\|^2_{L^2(\mathbb{R}\times S_\delta)}\,.$$

From the above and (5.7), we easily verify (5.5).

Step 3: We next show that there exists $C_0 > 0$, such that for any $\psi \in L^2(\mathbb{R},\mathbb{C})$ with $\|\psi\|_{L^2(\mathbb{R})} = 1$, we have

$$\Big|\int_{\mathbb{R}\times(\mathbb{R}\setminus S_\delta)} \hat{u}_\perp(x,s)\psi(s)\phi_0(x,s)\,dxds\Big| \leq C_0\,\delta^{1/2}\,\|u_\perp\|_2\,, \tag{5.8}$$

where $\phi_0(\cdot,s)$ is the normalized eigenfunction of $\mathcal{M}_s$ associated with the lowest eigenvalue $E_0(s)$, [cf. (2.2)].

To prove (5.8), we first note that

$$\int_{\mathbb{R}\times(\mathbb{R}\setminus S_\delta)} \hat{u}_\perp(x,s)\psi(s)\phi_0(x,s)\,dxds = \int_{\mathbb{R}\times(\mathbb{R}\setminus S_\delta)} \hat{u}_\perp(x,s)\psi(s)\big(\phi_0(x,s)-\phi^*(x)\big)\,dxds.$$

Hence,

$$\begin{aligned}&\Big|\int_{\mathbb{R}\times(\mathbb{R}\setminus S_\delta)} \hat{u}_\perp(x,s)\psi(s)\phi_0(x,s)\,dxds\Big| \\ \leq&\|\hat{u}_\perp\|_{L^2(\mathbb{R}\times(\mathbb{R}\setminus S_\delta))}\,\big\|\psi(s)\big(\phi_0(x,s)-\phi^*(x)\big)\big\|_{L^2(\mathbb{R}\times(\mathbb{R}\setminus S_\delta))}\,.\end{aligned} \tag{5.9}$$

For any $s > s^*$ we have

$$\int_{\mathbb{R}} |\phi_0(x,s)-\phi^*(x)|^2\,dx \leq |s-s^*|^2 \sup_{\theta\in(s^*,s)} \int_{\mathbb{R}} \Big|\frac{\partial\phi_0}{\partial s}(x,\theta)\Big|^2\,dx\,.$$

A similar inequality holds for $s < s^*$.

Set

$$C_0(\delta) = \sup_{s\in\mathbb{R}\setminus S_\delta} \int_{\mathbb{R}} \Big|\frac{\partial\phi_0}{\partial s}(x,s)\Big|^2\,dx\,.$$

As $0 < \delta < 1$, we have that $S_1 \subset S_\delta$. Hence, if we can show that $C_0(1) < \infty$, it would follow that

$$C_0(\delta) \leq C_0(1) < \infty\,. \tag{5.10}$$

The boundedness of $C_0(1)$ results immediately from the fact that, by Kato's perturbation theory $s \mapsto \phi_0(\cdot,s)$, is a C^1 function with values in $L^2(\mathbb{R})$.

As

$$\begin{aligned}&\big\|\psi(s)\big(\phi_0(x,s)-\phi^*(x)\big)\big\|_{L^2(\mathbb{R}\times(\mathbb{R}\setminus S_\delta))}\|^2 \\ \leq&C_0(1)\int_{\mathbb{R}\setminus S_\delta} |\psi(s)|^2|s-s^*|^2\,ds \leq \tilde{C}_0\,\delta \int_{\mathbb{R}\setminus S_\delta} |\psi(s)|^2\,ds = C_0\,\delta\,.\end{aligned}$$

The validity of (5.8) then easily follows from (5.9).

Step 4. Let

$$\hat{u}^s_\| = \langle \hat{u}(\cdot,s),\phi_0(\cdot,s)\rangle_x \phi_0(x,s)\,, \tag{5.11}$$

and let

$$\hat{u} = \hat{u}_{\|}^{s} + \hat{u}_{\perp}^{s}\,. \tag{5.12}$$

We next estimate $\|\hat{u}_{\perp}^{s}\|_2$.

As in Step 1 we can show that

$$\langle \hat{u}_{\|}^{s}, \hat{u}_{\perp}^{s}\rangle_x = 0 \quad \text{for all } s \in \mathbb{R}\,. \tag{5.13}$$

On the other hand, since for any fixed $s \in \mathbb{R}$, the function

$$x \to \hat{u}_{\|}^{s}(x, s+s^*)$$

is the eigenfunction of $\mathcal{M}_{s+s^*}$ associated with the first eigenvalue $E_0(s+s^*)$, we have that

$$\begin{aligned}
&\mathrm{Re}\int_{\mathbb{R}} \Big\{\partial_x \hat{u}_{\|}^{s}(x,s+s^*)\overline{\partial_x \hat{u}_{\perp}^{s}(x,s+s^*)} + \big(\frac{x^2}{2} - s - s^*\big)^2 \hat{u}_{\|}^{s}(x,s+s^*)\overline{\hat{u}_{\perp}^{s}(x,s+s^*)}\Big\}dx \\
=&\mathrm{Re}\int_{\mathbb{R}} \hat{u}_{\|}^{s}(x,s+s^*)\overline{\hat{u}_{\perp}^{s}(x,s+s^*)}dx = 0\,.
\end{aligned}$$

To obtain the last equality we used (5.13). Consequently, we have that

$$\begin{aligned}
&\int_{\mathbb{R}^2} |\partial_x \hat{u}|^2 + \Big|\big(\frac{x^2}{2} - s^* - s\big)\hat{u}\Big|^2\,dxds \\
=&\int_{\mathbb{R}} E_0(s+s^*)\|\hat{u}_{\|}^{s}(\cdot,s)\|_{L^2(\mathbb{R})}^2\,ds + \int_{\mathbb{R}^2}\left(|\partial_x \hat{u}_{\perp}^{s}|^2 + \Big|\big(\frac{x^2}{2} - s^* - s\big)\hat{u}_{\perp}^{s}\Big|^2\right)\,dxds\,.
\end{aligned} \tag{5.14}$$

From (5.13) and using the variational characterization of the second eigenvalue $E_1(s+s^*)$, we see that the right side of (5.14) is no less than

$$E_0^*\|\hat{u}_{\|}^{s}\|_2^2 + E_1^*\|\hat{u}_{\perp}^{s}\|_2^2\,.$$

Then we plug (5.14) back to the left side of (5.6) and get

$$E_0^*\|\hat{u}_{\|}^{s}\|_2^2 + E_1^*\|\hat{u}_{\perp}^{s}\|_2^2 - E_0^*\|\hat{u}\|_2^2 \leq \gamma^2\,. \tag{5.15}$$

It follows from (5.15) that

$$\|\hat{u}_{\perp}^{s}\|_2^2 \leq \frac{\gamma^2}{E_1^* - E_0^*} \leq C_2^2\,\gamma^2\,. \tag{5.16}$$

Step 5. We now prove (5.3). Assuming first that $a := \|\hat{u}_{\|}^{s}\|_{L^2(\mathbb{R})} > 0$, we let in (5.8)

$$\psi(s) = \frac{1}{a}\langle \hat{u}(\cdot,s), \phi_0(\cdot,s)\rangle_x\,,$$

and noting that $a \leq \|\hat{u}\|_2$, we obtain

$$\Big|\int_{\mathbb{R}\times(\mathbb{R}\setminus S_\delta)} \overline{\hat{u}}_{\perp}(x,s)\,\hat{u}_{\|}^{s}(x,s)\,dxds\Big| \leq a\,C_0\,\delta^{1/2}\,\|\hat{u}_{\perp}\|_2 \leq C_0\,\delta^{1/2}\,\|\hat{u}_{\perp}\|_2\|u\|_2\,. \tag{5.17}$$

It can be verified that the above inequality holds for $a = 0$ as well.

From (5.13) we see that, for any fixed $s \in \mathbb{R}$,

$$\int_{\mathbb{R}} |\hat{u}_\perp(x,s)|^2 dx = \int_{\mathbb{R}} \bar{\hat{u}}_\perp(x,s) \cdot \hat{u}_\perp(x,s)\, dx$$
$$= \int_{\mathbb{R}} \bar{\hat{u}}_\perp(x,s) \cdot [\hat{u}(x,s) - \hat{u}_\parallel(x,s)]\, dx = \int_{\mathbb{R}} \bar{\hat{u}}_\perp(x,s) \cdot \hat{u}(x,s)\, dx$$
$$= \int_{\mathbb{R}} \bar{\hat{u}}_\perp(x,s) \cdot [\hat{u}^s_\parallel(x,s) + \hat{u}^s_\perp(x,s)]\, dx\,.$$

Integrating the above equality in s on $\mathbb{R} \setminus S_\delta$, and using (5.17) and (5.16), we obtain

$$\|\hat{u}_\perp\|^2_{L^2(\mathbb{R}\times(\mathbb{R}\setminus S_\delta))} = \int_{\mathbb{R}\times\mathbb{R}\setminus S_\delta)} \bar{\hat{u}}_\perp(x,s) \cdot [\hat{u}^s_\parallel(x,s) + \hat{u}^s_\perp(x,s)]\, dxds$$
$$\leq \|\hat{u}_\perp\|_{L^2(\mathbb{R}\times(\mathbb{R}\setminus S_\delta)}\|\hat{u}^s_\parallel\|_{L^2(\mathbb{R}\times\mathbb{R}\setminus S_\delta)} + \|\hat{u}_\perp\|_{L^2(\mathbb{R}\times(\mathbb{R}\setminus S_\delta))}\|\hat{u}^s_\perp\|_{L^2(\mathbb{R}\times\mathbb{R}\setminus S_\delta)}$$
$$\leq C_0 \left(\delta^{1/2}\|\hat{u}_\perp\|_2\|u\|_2 + \gamma\|\hat{u}_\perp\|_2\right).$$

From the above and (5.5) we get

$$\|\hat{u}_\perp\|^2_2 = \|\hat{u}_\perp\|^2_{L^2(\mathbb{R}\times S_\delta)} + \|\hat{u}_\perp\|^2_{L^2(\mathbb{R}\times(\mathbb{R}\setminus S_\delta))} \leq C\Big(\frac{\gamma^2}{\delta} + \delta^{1/2}\|\hat{u}_\perp\|_2\|u\|_2 + \gamma\|\hat{u}_\perp\|_2\Big)\,,$$

from which we easily obtain, using Cauchy's inequality, that

$$\|\hat{u}_\perp\|_2 \leq C_1 \left(\frac{\gamma}{\delta^{1/2}} + \delta^{1/2}\,\|u\|_2\right).$$

We now distinguish between two different cases: $\|u\|_2 \geq \gamma$ and $\|u\|_2 < \gamma$. In the former case we set $\delta = \gamma/\|u\|_2 \leq 1$, and (5.3) easily follows. In the latter case we have

$$\|\hat{u}_\perp\|_2 \leq \|u\|_2 \leq \gamma^{1/2}\,\|u\|_2^{1/2} \leq C_1\,\gamma^{1/2}\,\|u\|_2^{1/2}\,,$$

which is precisely (5.3).

Step 6. As (5.3) is proved only for $u \in C^\infty_c(\mathbb{R}^2_+, \mathbb{C})$ we extend it to any $u \in H^{1,\mathrm{mag}}_0$ by a standard density argument. ∎

An immediate, but useful, corollary is given by:

Corollary 5.2. *Let $u \in H^{1,\mathrm{mag}}_0(\mathbb{R}^2_+, \mathbb{C})$ satisfy (5.2) and $P_\parallel u = 0$. Then*

$$\|u\|_2 \leq C\gamma\,. \tag{5.18}$$

The next corollary is both useful and simple to prove.

Corollary 5.3. *Let $u \in H^{2,\mathrm{mag}}(\mathbb{R}^2_+, \mathbb{C})$ satisfy*

$$(\mathcal{A}_{0,c} - \lambda)u = g\,, \tag{5.19}$$

where $g \in L^2(\mathbb{R}^2_+, \mathbb{C})$.

(i) *Let $\Upsilon \in C^1(\mathbb{R}_+, [0,1]) \cap W^{1,\infty}(\mathbb{R}_+, [0,1])$ satisfy $\Upsilon(0) = 0$. Then*

$$\|\Upsilon u_\perp\|_2 \leq C_1 \Big\{|\mathrm{Re}\,\lambda - E^*_0|^{1/4}\|\Upsilon u\|_2 + \|\Upsilon g\|_2^{1/4}\|\Upsilon u\|_2^{3/4} + \|\Upsilon' u\|_2^{1/2}\|\Upsilon u\|_2^{1/2}\Big\}\,. \tag{5.20}$$

(ii) *Furthermore, if $P_{\|}u=0$ then there exists a positive constant C_2 such that, if $|\mathrm{Re}\,\lambda - E_0^*| \leq 1/(4C_1^4)$, then*

$$\|\Upsilon u\|_2 \leq C_2 \Big\{ \|\Upsilon g_\perp\|_2 + \|\Upsilon' u\|_2 \Big\} . \tag{5.21}$$

Proof. Step 1. We first prove (i). Suppose first that $\Upsilon \in C_c^1(\mathbb{R}_+,[0,1])$. Since $u \in H^{1,\mathrm{mag}}(\mathbb{R}^2_+,\mathbb{C})$ we have that $\Upsilon u \in H_0^{1,\mathrm{mag}}(\mathbb{R}^2_+,\mathbb{C})$. Multiplying (5.19) by $\Upsilon^2\bar{u}$ and integrating over $\mathbb{R}^2_+$ we obtain

$$\Big\| \Big(\nabla - i\Big[\frac{x^2}{2} - s^*\Big]\hat{\mathbf{i}}_y\Big)(\Upsilon u)\Big\|_2^2 - E_0^*\|\Upsilon u\|_2^2 = (\mathrm{Re}\,\lambda - E_0^*)\|\Upsilon u\|_2^2 + \|\Upsilon' u\|_2^2 + \mathrm{Re}\,\langle \Upsilon u, \Upsilon g\rangle\,.$$

Denote

$$\gamma = \Big(|\mathrm{Re}\,\lambda - E_0^*|\|\Upsilon u\|_2^2 + \|\Upsilon' u\|_2^2 + |\mathrm{Re}\,\langle \Upsilon u, \Upsilon g\rangle| \Big)^{1/2}.$$

Note that $(\Upsilon u)_\perp = \Upsilon u_\perp$. Applying Lemma 5.1 to Υu we obtain

$$\|\Upsilon u_\perp\|_2 \leq C_1\,\gamma^{1/2}\,\|\Upsilon u\|_2^{1/2} . \tag{5.22}$$

Using Cauchy's inequality to bound $|\mathrm{Re}\,\langle \Upsilon u, \Upsilon g\rangle|$ yields

$$\begin{aligned}\gamma^{1/2}\|\Upsilon u\|_2^{1/2} =& (\gamma\|\Upsilon u\|_2)^{1/2} \leq \Big(|\mathrm{Re}\,\lambda - E_0^*|\|\Upsilon u\|_2^4 + \|\Upsilon' u\|_2^2\|\Upsilon u\|_2^2 + \|\Upsilon u\|_2^3\|\Upsilon g\|_2 \Big)^{1/4} \\ \leq& |\mathrm{Re}\,\lambda - E_0^*|^{1/4}\|\Upsilon u\|_2 + \|\Upsilon' u\|_2^{1/2}\|\Upsilon u\|_2^{1/2} + \|\Upsilon u\|_2^{3/4}\|\Upsilon g\|_2^{1/4} .\end{aligned}$$

Substituting the above into (5.22) we obtain (5.20).

Step 2. We next prove (ii). Now we assume $P_{\|}u=0$. Then

$$P_{\|}(\Upsilon u) = \Upsilon P_{\|} u = 0 .$$

Thus, by applying Corollary 5.2 to Υu with γ given above, we obtain

$$\|\Upsilon u\|_2^2 \leq C_1^4\gamma^2 = C_1^4\Big(|\mathrm{Re}\,\lambda - E_0^*|\|\Upsilon u\|_2^2 + \|\Upsilon' u\|_2^2 + |\mathrm{Re}\,\langle \Upsilon u, \Upsilon g\rangle| \Big) . \tag{5.23}$$

Since ϕ^* is a real-valued function, so $P_{\|}$ and $P_\perp$ are self-adjoint operators. Thus

$$\langle \Upsilon u, \Upsilon g\rangle = \langle P_\perp(\Upsilon u), \Upsilon g\rangle = \langle \Upsilon u, P_\perp(\Upsilon g)\rangle = \langle \Upsilon u, \Upsilon g_\perp\rangle .$$

Hence

$$|\mathrm{Re}\,\langle \Upsilon u, \Upsilon g\rangle| \leq |\langle \Upsilon u, \Upsilon g_\perp\rangle| \leq \|\Upsilon u\|_2\|\Upsilon g_\perp\|_2 \leq \frac{1}{4C_1^4}\,\|\Upsilon u\|_2^2 + C_1^4\,\|\Upsilon g_\perp\|_2^2 .$$

Next, suppose that

$$|\mathrm{Re}\,\lambda - E_0^*| \leq \frac{1}{4C_1^4} .$$

Substituting the above back into (5.23) yields

$$\begin{aligned}\|\Upsilon u\|_2^2 \leq& C_1^4\left(\frac{1}{4C_1^4}\,\|\Upsilon u\|_2^2 + \|\Upsilon' u\|_2^2 + \frac{1}{4C_1^4}\,\|\Upsilon u\|_2^2 + C_1^4\,\|\Upsilon g_\perp\|_2^2 \right) \\ =& \frac{1}{2}\,\|\Upsilon u\|_2^2 + C_1^4\left(\|\Upsilon' u\|_2^2 + C_1^4\,\|\Upsilon g_\perp\|_2^2 \right),\end{aligned}$$

from which (5.21) is readily verified.

Step 3: A simple density argument extends the corollary to any $\Upsilon \in C^1(\mathbb{R}_+,[0,1]) \cap W^{1,\infty}(\mathbb{R}_+,[0,1])$ satisfying $\Upsilon(0)=0$. ∎

Once the above auxiliary results have been established, we can begin our attempt to estimate the norm $\|(A_c^+-\lambda)^{-1}\|$ in a close vicinity of (3.35). Let then

$$w=(\mathcal{A}_c^+-\lambda)^{-1}f\,, \tag{5.24}$$

where

$$f=(\mathcal{A}_{0,c}-\lambda)U_\epsilon\,, \tag{5.25}$$

and U_ϵ is given by (3.31). In the following we prove that

$$\|w\|_2 \leq \frac{1}{2\epsilon^{1/2}}\,. \tag{5.26}$$

Once (5.26) has been verified, the eigenvalue estimate would easily follow as we later demonstrate.

We prove (5.26) by negation, that is, we suppose for a contradiction that

Assumption 5.4.

$$\|w\|_2 \geq \frac{1}{2\epsilon^{1/2}}\,. \tag{5.27}$$

We note that the above assumption is merely a technical measure. One can prove (5.26) directly, but that would considerably complicate many of the estimates in the sequel.

It easily follows from (3.37), (3.43) and (3.44) that

$$p_k(f)\leq C(k)\,\epsilon^{5/2}\,, \tag{5.28}$$

where $p_k(f)$ is defined in (4.3). From (5.27) and (5.28) we then get

$$p_k(f)\leq C(k)\,\epsilon^3\,\|w\|_2\,. \tag{5.29}$$

Let

$$w_\|=P_\|w=\phi^*(x)\psi_0(y)\,,\quad w_\perp=P_\perp w\,. \tag{5.30}$$

Then

$$w=\phi^*(x)\psi_0(y)+w_\perp\,,\quad \psi_0(y)=\langle w(\cdot,y),\phi^*(\cdot)\rangle_x\,,\quad y\in\mathbb{R}_+\,. \tag{5.31}$$

Recall that (see (3.10))

$$\psi_0(0)=0\,. \tag{5.32}$$

In the following we derive an estimate for ψ_0. As the outer solution constructed in Section 3 depends solely on a slow coordinate $\eta=\epsilon y$, we expect the derivative of the outer solution with respect to y to have smaller norms. The next lemma establishes this fact for $\psi_0'(\eta)$.

Lemma 5.5. *Suppose that* (5.27) *holds. Let* ϵ_0 *and* C_0 *be both positive. Let* $\lambda=\lambda(\epsilon)$ *satisfy for* $0<\epsilon<\epsilon_0$

$$|\mathrm{Re}\,\lambda-E_0^*|\leq C_0\,\epsilon^2\,. \tag{5.33}$$

Then, there exist $C>0$ *and* $0<\epsilon_1\leq\epsilon_0$*, such that for* $\epsilon\in(0,\epsilon_1)$*,*

$$\|\psi_0'\|_{L^2(\mathbb{R})}\leq C\,\epsilon^{1/2}\,\|\psi_0\|_{L^2(\mathbb{R})}\,. \tag{5.34}$$

We note that the above estimate is not optimal: one expects a factor of ϵ instead of $\epsilon^{1/2}$ in (5.34). We shall later prove a better estimate using (5.34) to this end.

Proof. Step 1. Recall the notation $\mathcal{A}_0 \overset{def}{=} \mathcal{A}_{0,0}$ (see the paragraph after (1.10)) in Section 1. Clearly,

$$\operatorname{Re}\langle \phi^*\psi_0, (\mathcal{A}_{0,c}-\lambda)w\rangle = \operatorname{Re}\langle \phi^*\psi_0, (\mathcal{A}_0-\lambda)w\rangle = \operatorname{Re}\langle (\mathcal{A}_0-\lambda)\phi^*\psi_0, w\rangle\,.$$

Consequently

$$(E_0^*-\operatorname{Re}\lambda)\|\psi_0\|^2_{L^2(\mathbb{R})} - \operatorname{Re}\langle \psi_0'', \psi_0\rangle_y + \operatorname{Re}\Big\langle 2i\Big(\frac{x^2}{2}-s^*\Big)\phi^*\psi_0', w\Big\rangle = \operatorname{Re}\langle \phi^*\psi_0, f\rangle\,.$$

In view of (5.32)), and since $\phi^* \perp (x^2/2-s^*)\phi^*$ we have that

$$\|\psi_0'\|^2_{L^2(\mathbb{R})} = (\operatorname{Re}\lambda - E_0^*)\|\psi_0\|^2_{L^2(\mathbb{R})} + \operatorname{Re}\Big\langle -2i\Big(\frac{x^2}{2}-s^*\Big)\phi^*\psi_0', w_\perp\Big\rangle + \operatorname{Re}\langle \phi^*\psi_0, f\rangle\,. \quad (5.35)$$

Step 2. From (5.24), (5.33), and (5.29) we get

$$\begin{aligned}\Big\|\Big(\nabla - i\Big[\frac{x^2}{2}-s^*\Big]\hat{\mathbf{i}}_y\Big)w\Big\|_2^2 - E_0^*\|w\|_2^2 &= (\operatorname{Re}\lambda - E_0^*)\|w\|_2^2 + \operatorname{Re}\langle f, w\rangle_2\\ \le C_0\,\epsilon^2\,\|w\|_2^2 + \|f\|_2\|w\|_2 &\le (C_0\,\epsilon^2 + C_1\,\epsilon^3)\|w\|_2^2 \le C_2\,\epsilon^2\|w\|_2^2\,.\end{aligned}$$

Then we apply Lemma 5.1 to w with $\gamma = \epsilon\sqrt{C_2}\,\|w\|_2$ to obtain that

$$\|w_\perp\|_2 \le C\,\epsilon^{1/2}\,\|w\|_2\,. \quad (5.36)$$

As

$$\|w\|_2 \le \|\psi_0\|_2 + \|w_\perp\|_2\,,$$

we have, for sufficiently small ϵ, that

$$\|w\|_2 \le 2\,\|\psi_0\|_2\,. \quad (5.37)$$

Consequently, by (5.29) and (5.37) we have, for every k, that

$$p_k(f) \le C_k\,\epsilon^3\,\|\psi_0\|\,, \quad (5.38)$$

and by (5.36)

$$\|w_\perp\|_2 \le C\,\epsilon^{1/2}\,\|\psi_0\|_2\,. \quad (5.39)$$

Thus, by (5.35) we obtain that

$$\|\psi_0'\|_2^2 \le C\left(\epsilon^2\,\|\psi_0\|_2^2 + \|f\|_2\|\psi_0\|_2\right) + C\,\epsilon^{1/2}\,\|\psi_0'\|_2\,\|\psi_0\|_2\,.$$

With the aid of (5.38), we readily obtain (5.34). ∎

We have thus shown that the norm of ψ_0' is small compared to that of ψ_0, although it is evaluated over both the "inner" and the "outer" regions described in Section 3. The norm is small despite the fact that $\partial w/\partial y$ is expected to be $\mathcal{O}(1)$ inside the inner region and not necessarily $\mathcal{O}(\epsilon)$ as in the outer region. This smallness of $\|\psi_0'\|_2$ can be attributed to the fact that the leading term in the inner expansion of the quasimode U_ϵ in Section 3 is of order $\mathcal{O}(\epsilon)$.

For higher order derivatives, however, one should not expect that $\|\psi_0^{(k)}\|_2$ would be much smaller than $\|\psi_0^{(k-1)}\|_2$. Nevertheless, the expected slow variation of ψ_0 far away from the boundary, should yield small norms calculated over the outer region only. We therefore make use of the cutoff function χ_ϵ defined in (4.35) to establish recursive estimates for the "outer norms" of higher order derivatives.

Remark 5.6. *Assuming* (5.27)*, it easily follows from* (4.54) *that*

$$\Big\|\chi_\epsilon\frac{\partial^n f}{\partial y^n}\Big\|_2 \leq C_n\,\epsilon^{n+3}\,\|\psi_0\|_2\,. \tag{5.40}$$

Lemma 5.7. *Under the conditions of Lemma 5.5, (in particular assuming* (5.27)*) there exist, for every $n\in\mathbb{N}$, $C_n>0$ and $\epsilon_n>0$, such that, for $\epsilon\in(0,\epsilon_n)$,*

$$\begin{aligned}\|\chi_\epsilon\psi_0^{(n+1)}\|_{L^2(\mathbb{R}_+)} \leq & C_n\Big[\epsilon^\gamma\,\|\chi_\epsilon\psi_0^{(n)}\|_{L^2(\mathbb{R}_+)}+\epsilon^2\,\|\chi_\epsilon\psi_0^{(n-1)}\|_{L^2(\mathbb{R}_+)}+\Big\|\chi_\epsilon\frac{\partial^n w_\perp}{\partial y^n}\Big\|_2\\ &+\epsilon^{n+2}\,\|\psi_0\|_{L^2(\mathbb{R}_+)}\Big]\,,\end{aligned} \tag{5.41a}$$

$$\begin{aligned}\Big\|\chi_\epsilon\frac{\partial^n w_\perp}{\partial y^n}\Big\|_2 \leq & C_n\Big[\epsilon^{1/2}\,\|\chi_\epsilon\psi_0^{(n)}\|_{L^2(\mathbb{R}_+)}+\epsilon^{3/2}\Big(\Big\|\chi_\epsilon\frac{\partial^{n-1}w_\perp}{\partial y^{n-1}}\Big\|_2+\|\chi_\epsilon\psi_0^{(n-1)}\|_{L^2(\mathbb{R}_+)}\Big)\\ &+\epsilon^{n+3/2}\,\|\psi_0\|_{L^2(\mathbb{R}_+)}\Big]\,.\end{aligned} \tag{5.41b}$$

Proof. Step 1. We prove first (5.41b). Clearly, by the definition of w we have that

$$(\mathcal{A}_{0,c}-\lambda)\frac{\partial^n w}{\partial y^n}=\frac{\partial^n f}{\partial y^n}+in\epsilon^3\frac{\partial^{n-1}w}{\partial y^{n-1}}\,. \tag{5.42}$$

Furthermore, as

$$\frac{\partial^n w}{\partial y^n}=\phi^*(x)\psi_0^{(n)}(y)+\frac{\partial^n w_\perp}{\partial y^n}\,, \tag{5.43}$$

we have that

$$P_\perp(\chi_\epsilon\frac{\partial^n w}{\partial y^n})=\chi_\epsilon\frac{\partial^n w_\perp}{\partial y^n}\,.$$

Hence we can apply (5.20) to (5.42), with Υ replaced by χ_ϵ and u by $\partial^n w/\partial y^n$, and then use (5.33) to obtain that

$$\begin{aligned}\Big\|\chi_\epsilon\frac{\partial^n w_\perp}{\partial y^n}\Big\|_2 \leq & C\Big[\epsilon^{1/2}\Big\|\chi_\epsilon\frac{\partial^n w}{\partial y^n}\Big\|_2+\Big(\Big\|\chi_\epsilon\frac{\partial^n f}{\partial y^n}\Big\|_2^{1/4}+\epsilon^{3/4}\Big\|\chi_\epsilon\frac{\partial^{n-1}w}{\partial y^{n-1}}\Big\|_2^{1/4}\Big)\Big\|\chi_\epsilon\frac{\partial^n w}{\partial y^n}\Big\|_2^{3/4}\\ &+\Big\|\chi_\epsilon'\frac{\partial^n w}{\partial y^n}\Big\|_2^{1/2}\Big\|\chi_\epsilon\frac{\partial^n w}{\partial y^n}\Big\|_2^{1/2}\Big]\,.\end{aligned}$$

With the aid of (4.37), (5.43), (4.13), (5.38), (5.40), and (5.37), we obtain that

$$\begin{aligned}\Big\|\chi_\epsilon\frac{\partial^n w_\perp}{\partial y^n}\Big\|_2 \leq & C_n\Big[\epsilon^{1/2}\Big\|\chi_\epsilon\frac{\partial^n w}{\partial y^n}\Big\|_2+\epsilon^{3/4}\Big\|\chi_\epsilon\frac{\partial^{n-1}w}{\partial y^{n-1}}\Big\|_2^{1/4}\Big\|\chi_\epsilon\frac{\partial^n w}{\partial y^n}\Big\|_2^{3/4}\\ &+\epsilon^\gamma\Big\|\chi_\epsilon\frac{\partial^n w}{\partial y^n}\Big\|_2+\epsilon^{(3+n)/4}\|\psi_0\|_2^{1/4}\Big\|\chi_\epsilon\frac{\partial^n w}{\partial y^n}\Big\|_2^{3/4}+e^{-\epsilon^{-\alpha}}\,\|\psi_0\|_{L^2(\mathbb{R}_+)}\Big]\,.\end{aligned}$$

Using Young's inequality we have that

$$\epsilon^{(3+n)/4}\|\psi_0\|_2^{1/4}\Big\|\chi_\epsilon\frac{\partial^n w}{\partial y^n}\Big\|_2^{3/4} \leq \frac{3\epsilon^{1/2}}{4}\Big\|\chi_\epsilon\frac{\partial^n w}{\partial y^n}\Big\|_2 + \frac{\epsilon^{n+3/2}}{4}\|\psi_0\|_2\,,$$

and that

$$\epsilon^{3/4}\Big\|\chi_\epsilon\frac{\partial^{n-1} w}{\partial y^{n-1}}\Big\|_2^{1/4}\Big\|\chi_\epsilon\frac{\partial^n w}{\partial y^n}\Big\|_2^{3/4} \leq \frac{3\epsilon^{1/2}}{4}\Big\|\chi_\epsilon\frac{\partial^n w}{\partial y^n}\Big\|_2 + \frac{\epsilon^{3/2}}{4}\Big\|\chi_\epsilon\frac{\partial^{n-1} w}{\partial y^{n-1}}\Big\|_2 .$$

Hence,

$$\Big\|\chi_\epsilon\frac{\partial^n w_\perp}{\partial y^n}\Big\|_2 \leq C_n\left[\epsilon^{1/2}\Big\|\chi_\epsilon\frac{\partial^n w}{\partial y^n}\Big\|_2 + \epsilon^{3/2}\Big\|\chi_\epsilon\frac{\partial^{n-1} w}{\partial y^{n-1}}\Big\|_2 + \epsilon^{n+3/2}\|\psi_0\|_{L^2(\mathbb{R}_+)}\right].$$

The above inequality in conjunction with (5.43) readily yields (5.41b) for sufficiently small ϵ.

Step 2. To prove (5.41a), we multiply (5.42) by $\chi_\epsilon^2\phi^*\bar{\psi}_0^{(n)}$ and integrate by parts to obtain

$$\begin{aligned}
&\big\|\big(\chi_\epsilon\psi_0^{(n)}\big)'\big\|_{L^2(\mathbb{R}_+)}^2 - \big\|\chi_\epsilon'\psi_0^{(n)}\big\|_{L^2(\mathbb{R}_+)}^2 \\
=&\big(E_0^* - \operatorname{Re}\lambda\big)\big\|\chi_\epsilon\psi_0^{(n)}\big\|_{L^2(\mathbb{R}_+)}^2 + \operatorname{Re}\Big\langle 2i\Big(\frac{x^2}{2}-s^*\Big)\phi^*\big(\chi_\epsilon^2\psi_0^{(n)}\big)', \frac{\partial^n w_\perp}{\partial y^n}\Big\rangle \\
&-\epsilon^3\operatorname{Re}\big\langle i\chi_\epsilon^2\psi_0^{(n)},\psi_0^{(n-1)}\big\rangle + \operatorname{Re}\Big\langle\chi_\epsilon^2\phi^*\psi_0^{(n)},\frac{\partial^n f}{\partial y^n}\Big\rangle .
\end{aligned} \tag{5.44}$$

To estimate the second term in the right side of (5.44), we represent it as follows:

$$\begin{aligned}
&\operatorname{Re}\Big\langle 2i\Big(\frac{x^2}{2}-s^*\Big)\phi^*(x)\big(\chi_\epsilon^2\psi_0^{(n)}\big)', \frac{\partial^n w_\perp}{\partial y^n}\Big\rangle \\
=&\operatorname{Re}\Big\langle 2i\Big(\frac{x^2}{2}-s^*\Big)\phi^*(x)\big[\big(\chi_\epsilon\psi_0^{(n)}\big)'\chi_\epsilon + \chi_\epsilon\psi_0^{(n)}\chi_\epsilon'\big], \frac{\partial^n w_\perp}{\partial y^n}\Big\rangle \\
=&\operatorname{Re}\Big\langle 2i\Big(\frac{x^2}{2}-s^*\Big)\phi^*(x)\big(\chi_\epsilon\psi_0^{(n)}\big)', \chi_\epsilon\frac{\partial^n w_\perp}{\partial y^n}\Big\rangle + \operatorname{Re}\Big\langle 2i\Big(\frac{x^2}{2}-s^*\Big)\phi^*(x)\chi_\epsilon\psi_0^{(n)}, \chi_\epsilon'\frac{\partial^n w_\perp}{\partial y^n}\Big\rangle.
\end{aligned}$$

Using the fact that $x^2\phi^*\in L^2(\mathbb{R})$ together with Cauchy's inequality and (4.37) yields

$$\begin{aligned}
&\Big|\operatorname{Re}\Big\langle 2i\Big(\frac{x^2}{2}-s^*\Big)\phi^*(x)\big(\chi_\epsilon^2\psi_0^{(n)}\big)', \frac{\partial^n w_\perp}{\partial y^n}\Big\rangle\Big| \\
\leq& C\,\big\|\big(\chi_\epsilon\psi_0^{(n)}\big)'\big\|_2\Big\|\chi_\epsilon\frac{\partial^n w_\perp}{\partial y^n}\Big\|_2 + C\,\big\|\chi_\epsilon\psi_0^{(n)}\big\|_2\Big\|\chi_\epsilon'\frac{\partial^n w_\perp}{\partial y^n}\Big\|_2 \\
\leq& C\,\big\|\big(\chi_\epsilon\psi_0^{(n)}\big)'\big\|_2\Big\|\chi_\epsilon\frac{\partial^n w_\perp}{\partial y^n}\Big\|_2 + C\,\big\|\chi_\epsilon\psi_0^{(n)}\big\|_2\Big\{\epsilon^\gamma\Big\|\chi_\epsilon\frac{\partial^n w_\perp}{\partial y^n}\Big\|_2 + e^{-\epsilon^{-\alpha}}\Big\|\frac{\partial^n w_\perp}{\partial y^n}\Big\|_2\Big\}.
\end{aligned}$$

For every $\delta>0$ we have

$$\big\|\big(\chi_\epsilon\psi_0^{(n)}\big)'\big\|_{L^2(\mathbb{R}_+)}^2 - \big\|\chi_\epsilon'\psi_0^{(n)}\big\|_{L^2(\mathbb{R}_+)}^2 \geq (1-\delta)\big\|\chi_\epsilon\psi_0^{(n+1)}\big\|_{L^2(\mathbb{R}_+)}^2 - \frac{1}{\delta}\big\|\chi_\epsilon'\psi_0^{(n)}\big\|_{L^2(\mathbb{R}_+)}^2 .$$

From the above, (5.44), and (5.33), we get

$$\begin{aligned}\big\|\chi_\epsilon\psi_0^{(n+1)}\big\|_{L^2(\mathbb{R}_+)}^2 \leq C_n \Bigg[&\epsilon^{2\gamma}\big\|\chi_\epsilon\psi_0^{(n)}\big\|_{L^2(\mathbb{R}_+)}^2 + \big\|\chi_\epsilon\psi_0^{(n+1)}\big\|_{L^2(\mathbb{R}_+)}\Big\|\chi_\epsilon\frac{\partial^n w_\perp}{\partial y^n}\Big\|_2 \\ &+ \epsilon^\gamma \big\|\chi_\epsilon\psi_0^{(n)}\big\|_{L^2(\mathbb{R}_+)}\Big\|\frac{\partial^n w_\perp}{\partial y^n}\Big\|_2 + \epsilon^3 \big\|\chi_\epsilon\psi_0^{(n)}\big\|_{L^2(\mathbb{R}_+)}\big\|\chi_\epsilon\psi_0^{(n-1)}\big\|_{L^2(\mathbb{R}_+)} \\ &+ \big\|\chi_\epsilon\psi_0^{(n)}\big\|_{L^2(\mathbb{R}_+)}\Big\|\chi_\epsilon\frac{\partial^n f}{\partial y^n}\Big\|_2 + e^{-\epsilon^{-\alpha}}\Big(\|\psi_0^{(n)}\|_{L^2(\mathbb{R}_+)}^2 + \|\psi_0^{(n)}\|_{L^2(\mathbb{R}_+)}\Big\|_2\frac{\partial^n w_\perp}{\partial y^n}\Big\|_2\Big)\Bigg]\,. \end{aligned} \tag{5.45}$$

Next, we use (5.43), then apply (4.13) with $u = w$ and $g = f$, and then use (5.37) and (5.38) to bound the L^2 norms of w and of f. We obtain

$$\begin{aligned}\Big\|\frac{\partial^n w_\perp}{\partial y^n}\Big\|_2^2 + \|\psi_0^{(n)}\|_{L^2(\mathbb{R}_+)}^2 = \Big\|\frac{\partial^n w}{\partial y^n}\Big\|_2^2 \leq C(n)\big(\|w\|_2^2 + p_{2n}(f)^2\big) \\ \leq C(n)\big(4\|\psi_0\|_2^2 + \tilde{C}_n\,\epsilon^6\,\|\psi_0\|_2^2\big) \leq C_n\,\|\psi_0\|_{L^2(\mathbb{R}_+)}^2\,.\end{aligned}$$

Hence, (5.41a) readily follows from (5.38) and (5.45). ∎

The next corollary employs (5.41) with $n = 1$ and $n = 2$.

Corollary 5.8. *Under the conditions of Lemma 5.5, there exist $C > 0$ and $\epsilon_0 > 0$, such that, for all $\epsilon \in (0, \epsilon_0)$,*

$$\|\chi_\epsilon\psi_0''\|_{L^2(\mathbb{R}_+)} \leq C\,\epsilon\,\|\psi_0\|_{L^2(\mathbb{R}_+)}\,, \tag{5.46a}$$

$$\|\chi_\epsilon\psi_0^{(3)}\|_{L^2(\mathbb{R}_+)} \leq C\,\epsilon^{3/2}\,\|\psi_0\|_{L^2(\mathbb{R}_+)}\,. \tag{5.46b}$$

Proof. We first use (5.41b) with $n = 1$ to obtain that

$$\Big\|\chi_\epsilon\frac{\partial w_\perp}{\partial y}\Big\|_2 \leq C\Big[\epsilon^{1/2}\,\|\chi_\epsilon\psi_0'\|_{L^2(\mathbb{R}_+)} + \epsilon^{3/2}\Big(\Big\|\chi_\epsilon w_\perp\Big\|_2 + \|\chi_\epsilon\psi_0\|_{L^2(\mathbb{R}_+)}\Big) + \epsilon^{5/2}\,\|\psi_0\|_{L^2(\mathbb{R}_+)}\Big]\,. \tag{5.47}$$

Then, substituting (5.39) and (5.34) into (5.47), we obtain that

$$\Big\|\chi_\epsilon\frac{\partial w_\perp}{\partial y}\Big\|_2 \leq C\,\epsilon\,\|\psi_0\|_{L^2(\mathbb{R}_+)}\,. \tag{5.48}$$

Substituting the above into (5.41a) with $n = 1$, and then using (5.34) again, yields (5.46a) (recall that $\gamma > 1/2$).

The proof of (5.46b) follows in exactly the same manner. ∎

The formal expansion in Section 3 suggests that (5.34) and (5.46) are not optimal. To obtain better estimates on the derivatives of ψ_0 we have to obtain a more accurate approximation of $w_\perp$ than (5.39). Set then

$$w_\perp(x,y) = -i\phi_s^*(x)\psi_0'(y) + w_\perp^1(x,y)\,. \tag{5.49}$$

We can now prove the following

Lemma 5.9. *Under the conditions of Lemma 5.5, there exist $C>0$ and $\epsilon_0>0$, such that, for $\epsilon\in(0,\epsilon_0)$,*

$$\|\chi_\epsilon\psi_0'\|_{L^2(\mathbb{R}_+)} \leq C\,\epsilon^{\gamma}\,\|\psi_0\|_{L^2(\mathbb{R}_+)}\,, \tag{5.50a}$$

$$\|\chi_\epsilon\psi_0''\|_{L^2(\mathbb{R}_+)} \leq C\,\epsilon^{\gamma+1/2}\,\|\psi_0\|_{L^2(\mathbb{R}_+)}\,, \tag{5.50b}$$

$$\|\chi_\epsilon w_\perp^1\|_2 \leq C\,\epsilon^{\gamma+1/2}\,\|\psi_0\|_{L^2(\mathbb{R}_+)}\,. \tag{5.50c}$$

Proof. Step 1. A straightforward calculation yields

$$\begin{aligned}(\mathcal{A}_{0,c}-\lambda)w_\perp^1 =& -i(E_0^*-\lambda-i\epsilon^3y)\phi_s^*\psi_0'+f+\phi^*[\psi_0''-(i\epsilon^3y+E_0^*-\lambda)\psi_0]\\ &-2\Big(\frac{x^2}{2}-s^*\Big)\phi_s^*\psi_0''-i\psi_0^{(3)}\phi_s^*\,.\end{aligned} \tag{5.51}$$

Applying (5.21) to $w_\perp^1$ we obtain, for sufficiently small ϵ,

$$\begin{aligned}\|\chi_\epsilon w_\perp^1\|_2 \leq& C\,\big\{\|\chi_\epsilon' w_\perp^1\|_2+\epsilon^2\,\|\chi_\epsilon\psi_0'\|_{L^2(\mathbb{R}_+)}+\epsilon^3\,\|\chi_\epsilon y\psi_0'\|_{L^2(\mathbb{R}_+)}+\|\chi_\epsilon f_\perp\|_2\\ &+\|\chi_\epsilon\psi_0''\|_{L^2(\mathbb{R}_+)}+\|\chi_\epsilon\psi_0^{(3)}\|_{L^2(\mathbb{R}_+)}\big\}\,.\end{aligned} \tag{5.52}$$

Here we have used the fact that both $\phi^*(x)$ and $\phi_s^*(x)$ are in $\mathcal{S}(\mathbb{R}_+)$.

Step 2. From (5.31) we get

$$\chi_\epsilon(y)y\frac{\partial w}{\partial y}(\cdot,y)=\phi^*(\cdot)\chi_\epsilon(y)y\psi_0'(y)+\chi_\epsilon(y)y\frac{\partial w_\perp}{\partial y}(\cdot,y),\quad y\in\mathbb{R}_+,$$

which is an orthogonal $L^2(\mathbb{R},\mathbb{C})$ decomposition in x. Hence,

$$\int_{\mathbb{R}}|\chi_\epsilon(y)y\frac{\partial w}{\partial y}(x,y)|^2dx=\int_{\mathbb{R}}|\phi^*(x)\chi_\epsilon(y)y\psi_0'(y)|^2dx+\int_{\mathbb{R}}|\chi_\epsilon(y)y\frac{\partial w_\perp}{\partial y}(x,y)|^2dx.$$

Next we integrate the above equation with respect to y to obtain

$$\Big\|\chi_\epsilon y\frac{\partial w}{\partial y}\Big\|_2^2=\|\chi_\epsilon y\psi_0'\|_{L^2(\mathbb{R}_+)}^2+\Big\|\chi_\epsilon y\frac{\partial w_\perp}{\partial y}\Big\|_2^2\geq\|\chi_\epsilon y\psi_0'\|_{L^2(\mathbb{R}_+)}^2.$$

From the above and (4.39) with $n=1$ we get

$$\begin{aligned}\|\chi_\epsilon y\psi_0'\|_{L^2(\mathbb{R}_+)}\leq\Big\|\chi_\epsilon y\frac{\partial w}{\partial y}\Big\|_2\leq& C_\delta\,\Big[\frac{1}{\epsilon^{2+\delta}}\Big(\|\chi_\epsilon\psi_0'\|_{L^2(\mathbb{R}_+)}+\Big\|\chi_\epsilon\frac{\partial w_\perp}{\partial y}\Big\|_2\Big)\\ &+\epsilon^\delta\,\|w\|_2+\frac{1}{\epsilon^3}\Big\|\chi_\epsilon\frac{\partial f}{\partial y}\Big\|_2\Big]+C\,e^{-\frac13\epsilon^{-\alpha}}\big(\|w\|_2+p_2(f)\big).\end{aligned}$$

Next we use (5.48) to estimate $\|\chi_\epsilon\frac{\partial w_\perp}{\partial y}\|_2$, (4.54) with $n=1$ to estimate $\frac{1}{\epsilon^3}\|\chi_\epsilon\frac{\partial f}{\partial y}\|_2$, and (5.29) to control $p_2(f)$. We then obtain from the above inequality that

$$\|\chi_\epsilon y\psi_0'\|_{L^2(\mathbb{R}_+)}\leq\frac{C_\delta}{\epsilon^{2+\delta}}\Big(\|\chi_\epsilon\psi_0'\|_{L^2(\mathbb{R}_+)}+\epsilon\,\|\psi_0\|_{L^2(\mathbb{R}_+)}\Big)+C\,e^{-\frac13\epsilon^{-\alpha}}\,\|w\|_2\,. \tag{5.53}$$

Substituting (5.53) together with (4.37), (5.34), (5.46), (3.43), and the fact that

$$\|w_\perp^1\|_2\leq\|w_\perp\|_2+C\,\|\psi_0'\|_{L^2(\mathbb{R}_+)}\leq C\,\|\psi_0\|_{L^2(\mathbb{R}_+)}\,,$$

into (5.52) we obtain that

$$\|\chi_\epsilon w_\perp^1\|_2\leq C\,\epsilon\,\|\psi_0\|_{L^2(\mathbb{R}_+)}\,. \tag{5.54}$$

Step 3. We now multiply (5.42) with $n=0$ by $\chi_\epsilon^2\phi^*\bar{\psi_0}$ and integrate by parts to obtain

$$\begin{aligned}\|\chi_\epsilon\psi_0'\|_{L^2(\mathbb{R}_+)}^2 =&\|\chi_\epsilon'\psi_0\|_{L^2(\mathbb{R}_+)}^2 + (\operatorname{Re}\lambda - E_0^*)\|\chi_\epsilon\psi_0\|_{L^2(\mathbb{R}_+)}^2 \\ &+ \operatorname{Re}\Big\langle -2i\Big(\frac{x^2}{2}-s^*\Big)\chi_\epsilon^2\phi^*\psi_0', w_\perp\Big\rangle + \operatorname{Re}\langle\chi_\epsilon^2\phi^*\psi_0, f\rangle\,.\end{aligned}$$

Substituting (5.49) into the above yields

$$\begin{aligned}\beta\|\chi_\epsilon\psi_0'\|_{L^2(\mathbb{R}_+)}^2 =&\|\chi_\epsilon'\psi_0\|_{L^2(\mathbb{R}_+)}^2 + (\operatorname{Re}\lambda - E_0^*)\|\chi_\epsilon\psi_0\|_{L^2(\mathbb{R}_+)}^2 \\ &+ \operatorname{Re}\Big\langle -2i\Big(\frac{x^2}{2}-s^*\Big)\chi_\epsilon^2\phi^*\psi_0', w_\perp^1\Big\rangle + \operatorname{Re}\langle\chi_\epsilon^2\phi^*\psi_0, f\rangle\,,\end{aligned}$$

in which $\beta = E_0''(s^*)$. With the aid of (4.37), (5.54), (5.38), and (3.43) we obtain

$$\|\chi_\epsilon\psi_0'\|_{L^2(\mathbb{R}_+)}^2 \leq C\left(\epsilon^{2\gamma}\|\psi_0\|_{L^2(\mathbb{R}_+)}^2 + \epsilon\,\|\chi_\epsilon\psi_0'\|_{L^2(\mathbb{R}_+)}\|\psi_0\|_{L^2(\mathbb{R}_+)}\right),$$

from which (5.50a) readily follows.

Step 4. To prove (5.50b) we return back to (5.47) and substitute into it (5.50a) instead of (5.34) to obtain

$$\Big\|\chi_\epsilon\frac{\partial w_\perp}{\partial y}\Big\|_2 \leq C\,\epsilon^{\gamma+1/2}\,\|\psi_0\|_{L^2(\mathbb{R}_+)}\,.$$

Substituting the above into (5.41a) with $n=1$ yields (5.50b).

Finally, we revisit (5.52), this time armed with (5.50b) in hand. The proof of (5.50c) then readily follows. ∎

The following estimates follow immediately from (5.41) and (5.50a,b).

Corollary 5.10. *Under the conditions of Lemma 5.5, there exist $C>0$ and $\epsilon_0>0$, such that for $\epsilon\in(0,\epsilon_0)$,*

$$\|\chi_\epsilon\psi_0^{(3)}\|_{L^2(\mathbb{R}_+)} \leq C\,\epsilon^{1+\gamma}\,\|\psi_0\|_{L^2(\mathbb{R}_+)}, \tag{5.55a}$$

$$\|\chi_\epsilon\psi_0^{(4)}\|_{L^2(\mathbb{R}_+)} \leq C\,\epsilon^{\gamma+3/2}\,\|\psi_0\|_{L^2(\mathbb{R}_+)}\,. \tag{5.55b}$$

The estimates (5.50b) and (5.55) are still non-optimal as we expect $\psi_0^{(k)}$ to be of order ϵ^k.

Lemma 5.11. *Under the conditions of Lemma 5.5, there exist $C>0$ and $\epsilon_0>0$, such that, for $\epsilon\in(0,\epsilon_0)$,*

$$\Big\|\chi_\epsilon\psi_0''\Big\|_{L^2(\mathbb{R}_+)} \leq C\,\epsilon^{2\gamma}\,\|\psi_0\|_{L^2(\mathbb{R}_+)}\,. \tag{5.56}$$

Proof. Step 1. By (5.51) we have that

$$\begin{aligned}(\mathcal{A}_{0,c}-\lambda)\frac{\partial w_\perp^1}{\partial y} =& -i(E_0^*-\lambda-i\epsilon^3y)\phi_s^*\psi_0'' + f_y + \phi^*[\psi_0^{(3)} - (i\epsilon^3y+E_0^*-\lambda)\psi_0'] \\ &-2\Big(\frac{x^2}{2}-s^*\Big)\phi_s^*\psi_0^{(3)} - i\psi_0^{(4)}\phi_s^*\,.\end{aligned}$$

We now use (5.21) to obtain that

$$\begin{aligned}\Big\|\chi_\epsilon\frac{\partial w_\perp^1}{\partial y}\Big\|_2 \leq& C\Big[\Big\|\chi_\epsilon'\frac{\partial w_\perp^1}{\partial y}\Big\|_2+\epsilon^2\|\chi_\epsilon\psi_0''\|_{L^2(\mathbb{R}_+)}+\epsilon^3\|\chi_\epsilon y\psi_0''\|_{L^2(\mathbb{R}_+)}+\epsilon^3\|\chi_\epsilon\psi_0'\|_{L^2(\mathbb{R}_+)}\\ &+\Big\|\chi_\epsilon\frac{\partial f_\perp}{\partial y}\Big\|_2+\|\chi_\epsilon\psi_0^{(3)}\|_{L^2(\mathbb{R}_+)}+\|\chi_\epsilon\psi_0^{(4)}\|_{L^2(\mathbb{R}_+)}\Big]\,.\end{aligned} \tag{5.57}$$

Here we have used (5.1) to obtain $P_\perp f_y=\frac{\partial}{\partial y}f_\perp$.

As

$$\chi_\epsilon(y)y\frac{\partial^2 w}{\partial y^2}(\cdot,y)=\phi^*(\cdot)\chi_\epsilon(y)y\psi_0''(y)+\chi_\epsilon(y)y\frac{\partial^2 w_\perp}{\partial y^2}(\cdot,y),\quad y\in\mathbb{R}_+,$$

(see step 2 in the proof of Lemma 5.9), we have

$$\|\chi_\epsilon y\psi_0''\|_{L^2(\mathbb{R}_+)}\leq\Big\|\chi_\epsilon y\frac{\partial^2 w}{\partial y^2}\Big\|_2.$$

Employing (4.39) with $n=2$ and $u=w$ yields

$$\begin{aligned}\Big\|\chi_\epsilon y\frac{\partial^2 w}{\partial y^2}\Big\|_2 \leq& C_\delta\left[\frac{1}{\epsilon^{2+\delta}}\Big\|\chi_\epsilon\frac{\partial^2 w}{\partial y^2}\Big\|_2+\epsilon^\delta\left\|\frac{\partial w}{\partial y}\right\|_2+\frac{1}{\epsilon^3}\left\|\chi_\epsilon\frac{\partial^2 f}{\partial y^2}\right\|_2\right]+C_\delta\,e^{-\frac13\epsilon^{-\alpha}}\left(\|w\|_2+p_4(f)\right)\\ \leq& C_\delta\left[\frac{1}{\epsilon^{2+\delta}}\Big(\|\chi_\epsilon\psi_0''\|_2+\Big\|\chi_\epsilon\frac{\partial^2 w_\perp}{\partial y^2}\Big\|_2\Big)+\epsilon^\delta\Big\|\frac{\partial w}{\partial y}\Big\|_2+\frac{1}{\epsilon^3}\left\|\chi_\epsilon\frac{\partial^2 f}{\partial y^2}\right\|_2\right]\\ &+C_\delta\,e^{-\frac13\epsilon^{-\alpha}}\left(\|w\|_2+p_4(f)\right).\end{aligned}$$

Therefore, we find that

$$\begin{aligned}\|\chi_\epsilon y\psi_0''\|_{L^2(\mathbb{R}_+)}\leq&\Big\|\chi_\epsilon y\frac{\partial^2 w}{\partial y^2}\Big\|_2\leq C_\delta\Big[\frac{1}{\epsilon^{2+\delta}}\Big(\|\chi_\epsilon\psi_0''\|_{L^2(\mathbb{R}_+)}+\Big\|\chi_\epsilon\frac{\partial^2 w_\perp}{\partial y^2}\Big\|_2\Big)\\ &+\epsilon^\delta\Big(\|\chi_\epsilon\psi_0'\|_2+\Big\|\chi_\epsilon\frac{\partial w_\perp}{\partial y}\Big\|_2\Big)+\frac{1}{\epsilon^3}\Big\|\chi_\epsilon\frac{\partial^2 f}{\partial y^2}\Big\|_2\Big]+C\,e^{-\frac13\epsilon^{-\alpha}}\left(\|w\|_2+p_4(f)\right).\end{aligned} \tag{5.58}$$

By (5.41b) with $n=2$ we have

$$\begin{aligned}\Big\|\chi_\epsilon\frac{\partial^2 w_\perp}{\partial y^2}\Big\|_2 \leq& C\left[\epsilon^{1/2}\|\chi_\epsilon\psi_0''\|_{L^2(\mathbb{R}_+)}+\epsilon^{3/2}\left(\Big\|\chi_\epsilon\frac{\partial w_\perp}{\partial y}\Big\|_2+\|\chi_\epsilon\psi_0'\|_{L^2(\mathbb{R}_+)}\right)\|\chi_\epsilon\psi_0''\|_{L^2(\mathbb{R}_+)}^{3/4}\right.\\ &\left.+\epsilon^{7/2}\,\|\psi_0\|_{L^2(\mathbb{R}_+)}\right],\end{aligned}$$

which together with (5.48), (5.29), and (5.50) yields

$$\Big\|\chi_\epsilon\frac{\partial^2 w_\perp}{\partial y^2}\Big\|_2\leq C\left[\epsilon^{1+\gamma}\,\|\chi_\epsilon\psi_0\|_{L^2(\mathbb{R}_+)}+\epsilon^{7/2}\,\|\psi_0\|_{L^2(\mathbb{R}_+)}\right].$$

Substituting the above together with (5.48), (5.50), and (5.29) into (5.58) leads to

$$\|\chi_\epsilon y\psi_0''\|_{L^2(\mathbb{R}_+)}\leq\frac{C_\delta}{\epsilon^\delta}\,\|\psi_0\|_{L^2(\mathbb{R}_+)}\,.$$

Substituting the above inequality, (4.37), (3.37) (4.39), (5.55) and (5.50) into (5.57), we obtain that

$$\Big\|\chi_\epsilon\frac{\partial w_\perp^1}{\partial y}\Big\|_2 \leq C\,\epsilon^{2\gamma}\,\|\psi_0\|_{L^2(\mathbb{R}_+)}\,. \tag{5.59}$$

Step 2. Next, we substitute (5.49) into (5.44) with $n=1$. We obtain

$$\begin{aligned}&\big\|\big(\chi_\epsilon\psi_0'\big)'\big\|_{L^2(\mathbb{R}_+)}^2 - \big\|\chi_\epsilon'\psi_0'\big\|_{L^2(\mathbb{R}_+)}^2\\ &=\big(E_0^*-\operatorname{Re}\lambda\big)\big\|\chi_\epsilon\psi_0'\big\|_{L^2(\mathbb{R}_+)}^2+\operatorname{Re}\Big\langle 2i\Big(\frac{x^2}{2}-s^*\Big)\phi^*\big(\chi_\epsilon^2\psi_0'\big)',\frac{\partial w_\perp^1}{\partial y}\Big\rangle\\ &\quad-\epsilon^3\operatorname{Re}\big\langle i\chi_\epsilon^2\psi_0',\psi_0\big\rangle+(1-\beta)\operatorname{Re}\big\langle\big(\chi_\epsilon^2\psi_0'\big)',\psi_0''\big\rangle+\operatorname{Re}\Big\langle\chi_\epsilon^2\phi^*\psi_0',\frac{\partial f}{\partial y}\Big\rangle\,,\end{aligned}$$

where $\beta=\frac{1}{2}E_0''(s^*)$. With the aid of (4.37), Cauchy's inequality, and (3.43), we then obtain

$$\begin{aligned}\big\|\chi_\epsilon\psi_0''\big\|_{L^2(\mathbb{R}_+)}^2 \leq & C\,\Big[\epsilon^2\,\|\chi_\epsilon\psi_0'\|_{L^2(\mathbb{R}_+)}^2+\|\chi_\epsilon\psi_0''\|_{L^2(\mathbb{R}_+)}\Big\|\chi_\epsilon\frac{\partial w_\perp^1}{\partial y}\Big\|_2\\ &+\epsilon^\gamma\,\big\|\chi_\epsilon\psi_0'\big\|_{L^2(\mathbb{R}_+)}\Big\|\chi_\epsilon\frac{\partial w_\perp^1}{\partial y}\Big\|_2+\epsilon^3\,\|\chi_\epsilon\psi_0'\|_{L^2(\mathbb{R}_+)}\|\chi_\epsilon\psi_0\|_{L^2(\mathbb{R}_+)}\\ &+\epsilon^\gamma\,\|\chi_\epsilon\psi_0''\|_{L^2(\mathbb{R}_+)}\|\chi_\epsilon\psi_0'\|_{L^2(\mathbb{R}_+)}+C\,e^{-\frac{1}{2}\epsilon^{-\alpha}}\,\|\psi_0\|_{L^2(\mathbb{R}_+)}\Big]\,.\end{aligned}$$

The above inequality in conjunction with (5.59), (5.50a,b), (3.37) and (5.38) yields (5.56). ∎

The following improvement of (5.55a) follows immediately from (5.41), (5.50a), and (5.56).

Corollary 5.12. *Under the conditions of Lemma 5.5, there exist $C>0$ and $\epsilon_0>0$ such that, for $\epsilon\in(0,\epsilon_0)$,*

$$\|\chi_\epsilon\psi_0^{(3)}\|_{L^2(\mathbb{R})}\leq C\,\epsilon^{2\gamma+1/2}\,\|\psi_0\|_{L^2(\mathbb{R})}\,. \tag{5.60}$$

We complete the outer estimates by showing that away from the boundary, for $y^{-1}=\mathcal{O}(\epsilon)$, one of the solutions of (3.23) can serve as a good approximation for ψ_0.

Proposition 5.13. *Under the conditions of Lemma 5.5, there exist $C>0$ and $\epsilon_0>0$ such that, for $\epsilon\in(0,\epsilon_0)$,*

$$\big\|\chi_\epsilon\big(-\beta\psi_0''+[i\epsilon^3y+E_0^*-\lambda]\psi_0\big)\big\|_2\leq C\,\epsilon^{2\gamma+1/2}\,\|\psi_0\|_{L^2(\mathbb{R}_+)}, \tag{5.61}$$

where $\beta=\frac{1}{2}E_0''(s^*)$.

Proof. Taking the inner product in $L^2(\mathbb{R})$ of (5.51) with $\phi^*(x)$ yields

$$-\beta\psi_0''+[i\epsilon^3y+E_0^*-\lambda]\psi_0=-2i\Big\langle\Big(\frac{x^2}{2}-s^*\Big)\phi^*,\frac{\partial w_\perp^1}{\partial y}\Big\rangle_x+\langle f,\phi^*\rangle\,.$$

Consequently,

$$\big\|\chi_\epsilon\big(-\beta\psi_0''+[i\epsilon^3y+E_0^*-\lambda]\psi_0\big)\big\|_{L^2(\mathbb{R}_+)}\leq C\Big\|\chi_\epsilon\frac{\partial w_\perp^1}{\partial y}\Big\|_2+\|f\|_2\,. \tag{5.62}$$

As (5.59) is not sufficiently accurate to obtain (5.61), we return to (5.57), and substitute (5.60) instead of (5.55a) into it, along with (4.37), (5.38), (5.55b), and (5.50a). It follows that

$$\Big\|\chi_\epsilon\frac{\partial w_\perp^1}{\partial y}\Big\|_2 \leq C\epsilon^{2\gamma+1/2}\|\psi_0\|_{L^2(\mathbb{R}_+)}\,.$$

substituting the above into (5.62) immediately yields (5.61). ∎

Remark 5.14. *Let* $\mu = (E_0^* - \lambda)/\epsilon^2$ *. By applying the transformation* $\eta = \epsilon y$ *in* (5.61) *we obtain for*

$$\Psi(\eta) = \psi_0(\frac{\eta}{\epsilon}), \quad \zeta(\eta) = \chi_\epsilon(\frac{\eta}{\epsilon})$$

the following

$$\big\|\zeta\big(- \beta\Psi''(\eta) + [i\eta - \mu]\Psi(\eta)\big)\big\|_{L^2(\mathbb{R}_+)} \leq C_\delta\,\epsilon^{1/2-\delta}\,\|\Psi\|_{L^2(\mathbb{R}_+)} \tag{5.63}$$

for any $\delta > 0$*. This manifests the validity of* (3.23) *in the outer region.*

6. "Inner" Estimates

In Section 3 we obtained the equation governing the behaviour of ψ_0 in the outer region (namely (3.23)). Obviously, ψ_0 decays as $y \to +\infty$. Nevertheless, a boundary condition at the lower "edge" of the outer needs yet to be established. In this regard, it is simply not enough to rely on the fact that $\psi_0(0) = w(x, 0) = 0$, since the boundary condition has to be prescribed at some $y_0 \sim \epsilon^{-1+\delta}$, wherein $0 < \delta < 1/2$. The next lemma allows us to obtain a bound on $\psi_0(y_0)$.

Lemma 6.1. *Let* w *satisfy*

$$\begin{cases} (\mathcal{A}_{0,c} - \lambda)w = f & \text{in } \mathbb{R}^2_+, \\ w = 0 & \text{on } \partial\mathbb{R}^2_+, \end{cases} \tag{6.1}$$

with $f \in S(\mathbb{R}^2_+, \mathbb{C})$ *and* $\lambda \in \rho(\mathcal{A}_+)$*. Then, for every* λ *satisfying*

$$|\mathrm{Re}\,\lambda - E_0^*| \leq 1\,, \tag{6.2}$$

there exists $k(\delta) \in \mathbb{N}$*,*

$$\Big\|\frac{\partial w}{\partial y}\Big\|_2^2 \leq C_\delta\Big(|\mathrm{Re}\,\lambda - E_0^*|^{1-\delta}(\|w\|_2^2 + (p_k(f))^2) + |\mathrm{Re}\,\lambda - E_0^*|^{-\delta}\|w\|_2\|f\|_2\Big), \quad \forall\delta > 0\,. \tag{6.3}$$

Proof. The proof follows the same steps of the proof of (2.14f) in Section 2. Multiplying (6.1) by $\bar{w}$ and integrating by part we obtain

$$\Big\|\Big(\nabla - i\Big[\frac{x^2}{2} - s^*\Big]\hat{\mathbf{i}}_y\Big)w\Big\|_2^2 - \mathrm{Re}\,\lambda\|w\|_2^2 = \mathrm{Re}\,\langle w, f\rangle\,.$$

Let $\hat{w}$ denote, once again, the partial Fourier transform with respect to y of the extension of w to $H^1(\mathbb{R}^2, \mathbb{C})$ defined in (2.12) and (2.13) We next employ the decomposition (2.17)

$$\begin{cases} \hat{w}(x, s) = \hat{w}_{\|,s}(x, s) + \hat{w}_{\perp,s}(x, s), \\ \hat{w}_{\|,s}(x, s) = \hat{b}(s^* + s)\phi_0(x, s^* + s), \\ \hat{b}(s^* + s) = \mathbf{1}_{[s^*-1,s^*+1]}(s^* + s)\langle\hat{w}(\cdot, s), \phi_0(\cdot, s^* + s)\rangle_{L^2(\mathbb{R})}. \end{cases} \tag{6.4}$$

Then, we conclude in exactly the same manner as in the derivation (2.23) in § 2 that

$$\|s\hat{b}(s^*+\cdot)\|_{L^2(\mathbb{R})}^2 + \|\hat{w}_{\perp,s}\|_{L^2(\mathbb{R}^2)}^2 \leq C\big(|\mathrm{Re}\,\lambda - E_0^*|\|w\|_2^2 + \|w\|_2\|f\|_2\big)\,. \tag{6.5}$$

Furthermore, using the same argument leading to (2.25) we obtain

$$\Big\|\frac{\partial w_{\perp,s}}{\partial y} - i\Big(\frac{x^2}{2} - s^*\Big)w_{\perp,s}\Big\|_{L^2(\mathbb{R}^2)}^2 \leq |\mathrm{Re}\,\lambda - E_0^*|\|w\|_2^2 + \big|\langle w,f\rangle\big| + E_0^*\|w_{\perp,s}\|_{L^2(\mathbb{R}^2)}^2\,.$$

Combining the above with (6.5) yields

$$\Big\|\frac{\partial w_{\perp,s}}{\partial y}\Big\|_{L^2(\mathbb{R}^2)}^2 \leq C\big(|\mathrm{Re}\,\lambda - E_0^*|\|w\|_2^2 + \|w\|_2\|f\|_2\big) + \Big\|\Big(\frac{x^2}{2} - s^*\Big)w_{\perp,s}\Big\|_{L^2(\mathbb{R}^2)}^2\,. \tag{6.6}$$

In view of (6.4) we have that

$$\begin{aligned}\Big\|\Big(\frac{x^2}{2} - s^*\Big)w_{\perp,s}\Big\|_{L^2(\mathbb{R}^2_+\setminus[-L,L]\times\mathbb{R}_+)} \leq &\Big\|\Big(\frac{x^2}{2} - s^*\Big)w\Big\|_{L^2(\mathbb{R}^2_+\setminus[-L,L]\times\mathbb{R}_+)} \\ &+ \Big\|\Big(\frac{x^2}{2} - s^*\Big)w\Big\|_{L^2(\mathbb{R}^2_+\setminus[-L,L]\times\mathbb{R}_+)}\end{aligned}$$

By (4.6) for all $L>1$ we have

$$\Big\|\Big(\frac{x^2}{2} - s^*\Big)w\Big\|_{L^2(\mathbb{R}^2_+\setminus[-L,L]\times\mathbb{R}_+)} \leq \frac{C_k}{L^k}\left(\|w\|_2 + p_k(f)\right).$$

Similarly, from the well-known properties of the eigenfunctions of the anharmonic oscillator [11] we learn that

$$\Big\|\Big(\frac{x^2}{2} - s^*\Big)w_{\|,s}\Big\|_{L^2(\mathbb{R}^2\setminus[-L,L]\times\mathbb{R})} \leq \frac{C_k}{L^k}\,\|w_{\|,s}\|_{L^2(\mathbb{R}^2)} \leq \frac{C_k}{L^k}\,\|w\|_2\,.$$

Consequently, for all $L>1$ we have

$$\Big\|\Big(\frac{x^2}{2} - s^*\Big)w_{\perp,s}\Big\|_{L^2(\mathbb{R}^2)} \leq C\,L^2\,\|w_{\perp,s}\|_2 + \frac{C_k}{L^k}\left(\|w\|_2 + p_k(f)\right).$$

For a given $k\geq 1$, we choose

$$L = |\mathrm{Re}\,\lambda - E_0^*|^{-1/(k+4)}\,,$$

to obtain with the aid of (6.5) that

$$\Big\|\Big(\frac{x^2}{2} - s^*\Big)w_{\perp,s}\Big\|_{L^2(\mathbb{R}^2)}^2 \leq C_k\,|\mathrm{Re}\,\lambda - E_0^*|^{k/(k+4)}(\|w\|_2^2 + (p_k(f))^2) + C\,|\mathrm{Re}\,\lambda - E_0^*|^{-4/(k+4)}\,\|f\|_2\,\|w\|_2\,.$$

Substituting into (6.6) the lemma readily follows for $\delta = k/(k+4)$. ∎

Relying on (6.3) we obtain the following estimate:

Lemma 6.2. *Assume that*

$$|\mathrm{Re}\,\lambda - E_0^*| \leq C\epsilon^2. \tag{6.7}$$

Let f satisfy (3.37)*, and suppose that w satisfies* (5.27)*. Let ψ_0 be defined in* (5.30)*. Let $\eta = \epsilon y$ and let $\Psi(\eta) = \psi_0(\eta/\epsilon)$. Then, for all $\delta>0$, there exists $C_\delta>0$ such that*

$$|\Psi(\eta)| \leq C_\delta\,\epsilon^{-\delta}\,(2\eta)^{1/2}\|\Psi\|_{L^2(\eta,+\infty)}, \quad \forall \eta \in (0, \epsilon^\delta\, C_\delta^{-1/2})\,. \tag{6.8}$$

Proof. As $w(x,0)\equiv 0$, we have, for any $R>0$,

$$\int_{\mathbb{R}} |w(x,y)|^2\,dx = \int_{\mathbb{R}} |w(x,y)-w(x,0)|^2\,dx = \int_{\mathbb{R}} \Big|\int_0^y \frac{\partial w}{\partial y_1}\,dy_1\Big|^2\,dx$$
$$\leq \int_{\mathbb{R}} y\int_0^y \Big|\frac{\partial w}{\partial y_1}\Big|^2\,dy_1dx \leq y\,\Big\|\frac{\partial w}{\partial y}\Big\|_2^2 .$$

By (5.30), (6.3), (3.37), and (5.27), we then have that, for all $\delta>0$, there exists some $k(\delta)\in\mathbb{N}$ such that

$$|\psi_0(y)|^2 \leq \int_{\mathbb{R}} |w(x,y)|^2\,dx \leq y\,C_\delta\left(|\mathrm{Re}\,\lambda - E_0^*|^{1-\delta}\|w\|_2^2 + |\mathrm{Re}\,\lambda - E_0^*|^{-\delta}\|w\|_2\|f\|_2\right).$$

It readily follows from (6.7), (5.29), and (5.37), that

$$|\psi_0(y)|^2 \leq C_\delta\,y\,\epsilon^{2(1-\delta)}\,\|\psi_0\|_2^2 .$$

Applying the transformation $\eta=\epsilon y$ we obtain that

$$|\Psi(\eta)|^2 \leq C_\delta\,\eta\,\epsilon^{-2\delta}\,\|\Psi\|_2^2 .$$

To complete the proof we use the above inequality to obtain by integration that

$$\int_0^\eta |\Psi(\zeta)|^2\,d\zeta \leq C_\delta\,\frac{\eta^2}{2}\,\epsilon^{-2\delta}\,\|\Psi\|_2^2 . \tag{6.9}$$

Hence, for η satisfying

$$0<\eta<\epsilon^\delta\,C_\delta^{-1/2},$$

we obtain that

$$\|\Psi\|_2^2 \leq 2\int_\eta^{+\infty} |\Psi(\zeta)|^2\,d\zeta ,$$

from which (6.8) readily follows. ∎

7. Proof of Theorem 1.2

Let $\mathcal{L}: D(\mathcal{L})\to L^2(\mathbb{R}_+)$ be defined as

$$\mathcal{L} = -\frac{d^2}{d\eta^2} + i\eta ,$$

where

$$D(\mathcal{L}) = \left\{u\in H_0^1(\mathbb{R}_+,\mathbb{C})\,:\,\mathcal{L}u\in L^2(\mathbb{R}_+)\right\}.$$

It is well-known [3, 13] that $\sigma(\mathcal{L})$ consists of a countable set of eigenvalues $\{\mu_n\}_{n=1}^\infty$ with

$$\mu_n = e^{-i2\pi/3}\alpha_n ,$$

where $\{\alpha_n\}_{n=1}^\infty\subset\mathbb{R}$ denotes the decreasing sequence of the zeroes of Airy's function on the negative real axis. Set

$$r = \frac{|\alpha_2-\alpha_1|}{2} .$$

To prove Theorem 1.2 we need first the following proposition.

Proposition 7.1. *There exists $C > 0$ such that*

$$\|(\mathcal{L}-\mu)^{-1}\| \leq \frac{C}{|\mu-\mu_1|}\,, \quad \forall \mu \in B(\mu_1, r)\,, \tag{7.1}$$

where μ_1 is the leftmost eigenvalue of the operator $\mathcal{L}$.

Proof. The Riesz-Schauder theory of compact operators allows us to represent the resolvent of $\mathcal{L}$ in the form

$$(\mathcal{L}-\mu)^{-1} = \frac{1}{\mu-\mu_1}\Pi_{\mu_1} + T_\mu\,,$$

where T_μ is bounded in $B(\mu_1, r)$ (cf. Eq. (16.1) in [2]) and Π_{μ_1} is the projector (defined by the Dunford integral) associated with μ_1. The proposition readily follows. ∎

We can now prove the following estimate for the resolvent:

Proposition 7.2. *Let, for $\epsilon > 0$, Λ be given by (3.35) and f given by (3.36). Then, for every $\theta \in (0, 1/6)$ there exist $C_\theta > 0$ and $\epsilon_\theta > 0$ such that, for all λ satisfying*

$$|\lambda - \Lambda| = C_\theta\, \epsilon^{13/6-\theta}\,,$$

we have, for $\epsilon \in (0, \epsilon_\theta)$,

$$\|(\mathcal{A}_c^+ - \lambda)^{-1} f\| \leq \frac{1}{2\epsilon^{1/2}}\,. \tag{7.2}$$

Proof. We recall that to obtain the estimates in Sections 5 and 6, we assumed (5.27) by negation. Therefore, if we reach a contradiction, (7.2) would immediately follow. For $\Psi(\eta) = \psi_0(\eta/\epsilon)$ defined in Remark 5.14 let

$$g(\eta) = -\beta\Psi''(\eta) + [i\eta - \mu]\Psi(\eta)\,,$$

where $\beta = \frac{1}{2}E_0''(s^*)$ and $\mu = (\lambda - E_0^*)/\epsilon^2$. By (5.63) for every $\delta > 0$, we have that

$$\|g\|_{L^2(\epsilon^\delta,\infty)} \leq C_\delta\, \epsilon^{1/2-\delta}\, \|\Psi\|_{L^2(\epsilon^\delta,\infty)}\,.$$

Given any $\theta \in (0, 1/6)$, take $\delta = \theta$ and $\eta_0 = \epsilon^{1/3}$. By (6.8), for all $\epsilon > 0$ sufficiently small (such that $\epsilon^{1/2} < \epsilon^\theta / C_\theta^{1/2}$) we have that

$$|\Psi(\eta_0)| \leq \tilde{C}_\theta\, \epsilon^{1/6-\theta}\, \|\Psi\|_{L^2(\eta_0,\infty)}\,. \tag{7.3}$$

Applying the transformation $\eta \to \beta^{-1/3}(\eta - \eta_0)$ to the equation for ψ_0, we obtain the following problem for Ψ

$$\begin{cases} (\mathcal{L}-\nu)\Psi = g & \text{in } \mathbb{R}_+\,, \\ \Psi(0) = \Psi_1, \end{cases}$$

where $\Psi_1 = \Psi(\eta_0)$. In the above

$$\|g\|_{L^2(\mathbb{R}_+)} \leq C\, \epsilon^{1/6}\, \|\Psi\|_{L^2(\mathbb{R}_+)}\,, \quad |\Psi_1| \leq \tilde{C}_\theta\, \epsilon^{1/6-\theta}\, \|\Psi\|_{L^2(\mathbb{R}_+)}\,, \quad \nu = \beta^{-1/3}(\mu - i\epsilon^{1/6})\,. \tag{7.4}$$

Next we write

$$\Psi(\eta) = \Phi(\eta) + \Psi_1 \frac{A_i(e^{i\pi/6}[\eta + i\nu])}{A_i(e^{i2\pi/3}\nu)}\,, \tag{7.5}$$

where A_i denotes Airy's function [1] and

$$\Phi = (\mathcal{L} - \nu)^{-1} g.$$

By (7.1) we have that

$$\|\Phi\|_{L^2(\mathbb{R}_+)} \leq \frac{C}{|\nu - \mu_1|} \, \|g\|_{L^2(\mathbb{R}_+)} \leq C \, \frac{\epsilon^{1/6}}{|\nu - \mu_1|} \,. \tag{7.6}$$

Furthermore, as

$$|A_i(e^{i2\pi/3}\nu)| = \big|A_i'(\alpha_1)(e^{i2\pi/3}\nu - \alpha_1) + o(|\nu - \mu_1|)\big| \leq C|\nu - \mu_1| \,,$$

a straightforward computation yields

$$\frac{\int_0^\infty |A_i(e^{i\pi/6}[\eta + i\nu])|^2}{|A_i(e^{i2\pi/3}\nu)|^2} \leq \frac{C}{|\nu - \mu_1|^2} \,.$$

Combining (7.5), (7.4), and (7.6) yields

$$\|\Psi\|_{L^2(\mathbb{R}_+)} \leq C(\theta) \frac{\epsilon^{1/6-\theta}}{|\nu - \mu_1|} \|\Psi\|_{L^2(\mathbb{R}_+)} \,.$$

Recall that

$$|\lambda - \Lambda| = \beta^{2/3} |\nu - \mu_1| \,.$$

Thus, for $\epsilon > 0$ small and

$$|\nu - \mu_1| \geq \frac{2}{C(\theta)} \,,$$

it follows that $\Psi \equiv 0$ which clearly contradicts (5.27). ∎

Proof of Theorem 1.2.

We now use the same technique as in [5]. Let U_ϵ be given by (3.34) and f by (3.36). Clearly,

$$(\mathcal{A}_c^+ - \lambda)^{-1} U_\epsilon = \frac{1}{\Lambda - \lambda} \big[U_\epsilon - (\mathcal{A}_c^+ - \lambda)^{-1} f \big] \,.$$

Hence, for all $0 < \theta < 1/6$ we have

$$\oint_{\partial B(\Lambda, C_\theta \epsilon^{13/6-\theta})} (\mathcal{A}_c^+ - \lambda)^{-1} U_\epsilon \, d\lambda = \oint_{\partial B(\Lambda, C_\theta \epsilon^{13/6-\theta})} \frac{1}{\lambda - \Lambda} (\mathcal{A}_c^+ - \lambda)^{-1} f \, d\lambda + 2\pi i U_\epsilon \,.$$

By (7.2) and (3.44) we then have

$$\begin{aligned} &\frac{1}{2\pi} \Big\| \oint_{\partial B(\Lambda, C_\theta \epsilon^{13/6-\theta})} (\mathcal{A}_c^+ - \lambda)^{-1} U_\epsilon \Big\|_2 \\ \geq &\frac{1}{\epsilon^{1/2}} - C - \Big\| \oint_{\partial B(\Lambda, C_\theta \epsilon^{13/6-\theta})} \frac{1}{\lambda - \Lambda} (\mathcal{A}_c^+ - \lambda)^{-1} f \, d\lambda \Big\|_2 \\ \geq &\frac{1}{2\epsilon^{1/2}} - C - C' \,. \end{aligned}$$

For sufficiently small ϵ we thus obtain

$$\oint_{\partial B(\Lambda, C_\theta \epsilon^{13/6-\theta})} (\mathcal{A}_c^+ - \lambda)^{-1} \, d\lambda \not\equiv 0 \,.$$

It follows that $(\mathcal{A}_c^+ - \lambda)^{-1}$ is not holomorphic in $B(\Lambda, C_\theta \epsilon^{13/6-\theta})$ and hence, an eigenvalue of $\mathcal{A}_c^+$ must exist there. ∎

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