Non-linear surface superconductivity in the large κ limit

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Abstract

The Ginzburg-Landau model for superconductivity is considered in two dimensions. We show, for smooth bounded domains, that the superconductivity order parameter decays exponentially fast away from the boundary as the Ginzburg-Landau parameter κ tends to infinity. We prove this result for applied magenetic fields satisfying $h_{ex}-\kappa \gg 1/\kappa$, and therefore, improve a recent result of Pan [16].

1 Introduction

Consider a planar superconducting body which is placed in a sufficiently low temperature (below the critical one) under the action of an external magnetic field . Its energy is given by the Ginzburg-Landau energy functional which can be represented in the following dimensionless form [6]

$$E = \int_\Omega \left(-|\Psi|^2 + \frac{|\Psi|^4}{2} + |h - h_{ex}|^2 + \left| \frac{i}{\kappa} \nabla \Psi + A\Psi \right|^2 \right) dxdy \tag{1.1}$$

in which Ψ is the (complex) superconducting order parameter, such that $|\Psi|$ varies from $|\Psi| = 0$ (when the material is at a normal state) to $|\Psi| = 1$ (for the purely superconducting state). The magnetic vector potential is denoted by A (the magnetic field is, then, given by $h = \nabla \times A$), h_{ex} is the constant applied magnetic field, and κ is the Ginzburg-Landau parameter which is a material property. Superconductors for which $\kappa < 1/\sqrt{2}$ are termed type I superconductors, and those for which $\kappa > 1/\sqrt{2}$ are termed type II. The domain Ω is the domain of superconductor whose Gibbs free energy is given by E. Note that E is invariant to the gauge transformation

$$\Psi \to e^{i\kappa\eta}\psi \qquad ; \qquad A \to A + \nabla\eta \, . \tag{1.2}$$

It is known both from experiments [15] and rigorous analysis [10] that for a sufficiently strong magnetic field the normal state ($\psi \equiv 0$, $h = h_{ex}$) would prevail. If the field is then decreased, there is a critical field, depending on the sample's geometry, where the material would enter the superconducting state. For samples with boundaries, this field is known as the onset field and has been termed H_{C_3}.

The simplest case in which the bifurcation from the normal state to the superconducting one was calculated is the case of a half-plane [18]. The analysis in this case is one dimensional: the linearized Ginzburg-Landau equations, which are the most common model for macroscopic superconductivity, were solved on $\mathbb{R}_+$. Even in this simple case the onset field is substantially larger than the bifurcation field on $\mathbb{R}$ [9]. The situation is no different in two dimensions: it was proved in [14] and [7] that the bifurcating mode in $\mathbb{R}^2_+$ is one-dimensional and that the value of H_{C_3} is exactly the same as in the one-dimensional case. Similarly, the bifurcation from the normal state in $\mathbb{R}^2$ takes place when the applied magnetic field is identical with the bifurcation field for $\mathbb{R}$, which has been termed H_{C_2}.

In addition to the difference in the values of the applied field, it was found by Saint-James and de-Gennes [18] that superconductivity is concentrated at the onset near the boundary for a half-plane, i.e. ψ decays exponentially fast away from the boundary. This phenomenon, which appears only in the presence of boundaries have been termed, therefore, surface superconductivity. It was later proved for general two-dimensional domains with smooth boundaries [13, 7], that as the domain's scale tends to infinity the onset field tends to de-Gennes' value, and that. If the boundaries include wedges the onset field will be larger than de-Gennes' value [4, 11, 19, 12].

Surface superconductivity reflects another difference between the problems in $\mathbb{R}^2_+$ and $\mathbb{R}^2$, where the bifurcation takes place in the form of periodic solutions [1, 5, 2] known as Abrikosov's lattices. The transition, as the applied magnetic field decreases, from surface superconductivity to the experimentally observed [8] Abrikosov's lattices is not yet well understood. Rubinstein [17] conjectured that superconductivity remains limited to a neighborhood of the boundary until about H_{C_2} when a new solution which is similar in the bulk to Abrikosov lattice appears.

Two recent contributions [16, 3] study the behaviour of the gobal minimizer of the energy functional (1.1) for external fields satisfying $\kappa = H_{C_2} < h_{ex} < H_{C_3}$. In [16] the limit $\kappa \to \infty$ is considered: it is demonstrated that ψ decays, in L^2 sense, exponentially fast away form the boundary. The results are valid whenever $h_{ex} - \kappa \gg 1$ as $\kappa \to \infty$. In addition the energy of the global minimizer is shown to be evenly distributed along the boundary. In [3] the large domain limit is considered: it is demonstrated for the global minimizer that both

ψ and h tend, in C^α sense, exponentially fast away from the boundary, to the normal state. The result are valid whenever $h_{ex} - \kappa \sim O(1)$ as the domain's size tends to infinity.

In the present contribution we focus on the limit $\kappa \to \infty$. We prove that the global minimizer tends to the normal state exponentially fast away from the boundary as long as $h_{ex} - \kappa \gg 1/\kappa$, which extends the validity of the results in [16]. Furthermore, we show that the magnetic field tends to a constant not only away from the boundaries but also near the boundary for this limit case.

The Euler-Lagrange equations associated with the energy functional defined in (1.1), or the steady state Ginzburg-Landau equations, are given by

$$\left(\frac{i}{\kappa}\nabla + A\right)^2 \psi = \psi\left(1 - |\psi|^2\right) \quad , \tag{1.3a}$$

$$-\nabla \times \nabla \times A = \frac{i}{2\kappa}\left(\psi^* \nabla \psi - \psi \nabla \psi^*\right) + |\psi|^2 A \quad , \tag{1.3b}$$

and the natural boundary conditions by

$$\left(\frac{i}{\kappa}\nabla + A\right)\psi \cdot \hat{n} = 0 \quad ; \quad h = h_{ex} \; . \tag{1.4a,b}$$

We consider two-dimensional settings where we can write $h = (0, 0, h(x, y))$ and $h_{ex} = (0, 0, h_{ex})$. In the next section we consider the global minimizer of (1.1) in smooth bounded domains as $\kappa \to \infty$. We show that for sufficiently large κ, the global minimizer of (1.1) which must solve (1.3) together with (1.4), tends exponentially fast away from the boundaries to a normal state as long as $h_{ex} - \kappa \gg 1/\kappa$. Furthermore we show that

$$\|h - h_{ex}\|_{L^\infty[\Omega]} \leq \frac{C}{\kappa^{1/2}} \min\left(\sqrt{h_{ex} - \kappa}, \frac{1}{\sqrt{h_{ex} - \kappa}}\right) \; . \tag{1.5}$$

To prove the above results we use a differential inequaity which was proved in [3]. Let

$$u = h - \kappa + \frac{1}{2\kappa}\rho^2. \tag{1.6}$$

Then

$$\nabla^2 u - \rho^2 u = \kappa \left|\hat{J}\right|^2 + \left(\kappa - \frac{1}{2\kappa}\right)\rho^4. \tag{1.7}$$

The precise definition of $\hat{J}$ will not concern us. We shall be interested only in its following property

$$\left|\hat{J}\right|^2 \rho^2 = |\nabla u|^2 . \tag{1.8}$$

which is proved in [3]. Finally, in section 3 we briefly discuss a few key points which are not mentioned in section 2.

2 Exponential rate of decay

We prove here the following theorem:

Theorem 2.1 *Let* $\lambda = \sqrt{\kappa(h_{ex}(\kappa) - \kappa)}$*, and let* $(\psi, A) = (\psi(\lambda,\kappa), A(\lambda,\kappa))$ *denote a solution of* (1.3) *and* (1.4)*. Then,* $\exists \lambda_0 > 0$*,* $\kappa_0 > 0$*,* $\beta > 0$*,* $\delta > 0$ *and* $\tilde{h}_\lambda$ *such that for every* $\kappa > \kappa_0$ *and* $\lambda > \lambda_0 \log^{1/2}\kappa$ *we have*

$$|D^\alpha \psi| \le C_\alpha \kappa^\alpha e^{-\beta\lambda d(x,\partial\Omega)} \qquad \textit{for all } \alpha \ge 0 \textit{ and } x \in \Omega \tag{2.1a}$$

$$\left|h - \tilde{h}_\lambda)\right| \le C \max\left(e^{-\beta\lambda\delta}, \frac{1}{\lambda} e^{-\beta\lambda d(x_0,\partial\Omega)}\right) \tag{2.1b}$$

$$\left|D^\alpha (h - \tilde{h}_\lambda)\right| \le C_\alpha \kappa^{\alpha-1} e^{-\beta\lambda d(x,\partial\Omega)} \qquad \textit{for all } \alpha \ge 0 \textit{ and } x \in \Omega \tag{2.1c}$$

$$\left|\tilde{h}_\lambda - h_{ex}\right| \le C\frac{\log\kappa}{\kappa} \tag{2.1d}$$

To prove the theorem we need first a number of auxiliary results. The first of them includes the following well-known estimates:

Lemma 2.2 *Let* $h_{ex} \geq \kappa$*. Then,*

$$\|\rho\|_{L^\infty(\Omega)} < 1 \tag{2.2a}$$

$$\|h - h_{ex}\|_{C^1(\bar{\Omega})} \leq C \tag{2.2b}$$

$$\| \left(\frac{i}{\kappa}\nabla + A \right) \psi\|_{L^\infty(\bar{\Omega})} \leq C \tag{2.2c}$$

Proof: The proof of (2.2a) is well known and follows immediately from (1.3a) and the real part of the boundary condition (1.4a). The proof of (2.2b) and (2.2c) can be found in [16].

Lemma 2.3 *Let* $h_{ex} \geq \kappa$*. Then, any solution of* (1.3) *and* (1.4) *satisfies, for sufficiently large* κ

$$\int_\Omega \rho^4 \leq \frac{C}{\kappa} \tag{2.3a}$$

$$\int_\Omega |h - h_{ex}|^2 \leq C \frac{\log^2 \kappa}{\kappa^2} \tag{2.3b}$$

where C *is independent of* κ*.*

Proof: We first prove (2.3a). To this end, we integrate (1.7) over Ω. In view of (2.2b) we have,

$$\int_\Omega \rho^2 u + \left(\kappa - \frac{1}{2\kappa} \right) \int_\Omega \rho^4 \leq \int_{\partial\Omega} \frac{\partial u}{\partial n} \leq C \tag{2.4}$$

Hence, apllying (2.2b) once again, we have

$$\kappa \int_\Omega \rho^4 \leq C + \int_\Omega \rho^2 (h - h_{ex}) \leq C \left(1 + \left[\int_\Omega \rho^4 \right]^{1/2} \right) \tag{2.5}$$

from which (2.3a) is readily verified.

To prove (2.3b) we integrate (1.3a) multiplied by $\rho^2\bar{\psi}$ and integrate over Ω. We obtain,

$$\int_\Omega \rho^2 \left|\left(\frac{i}{\kappa}\nabla + A\right)\Psi\right|^2 + \frac{1}{\kappa^2}\int_\Omega \rho^2|\nabla\rho|^2 = \int_\Omega \rho^4(1-\rho^2).$$

By (1.3b) we have

$$\int_\Omega |\nabla h|^2 \leq \int_\Omega \rho^2 \left|\left(\frac{i}{\kappa}\nabla + A\right)\Psi\right|^2 \leq \int_\Omega \rho^4.$$

We now apply Poincare inequality and (1.4b) to obtain

$$\int_\Omega |h-h_{ex}|^2 \leq \int_\Omega \rho^4 \leq \frac{C}{\kappa}. \tag{2.6}$$

In a similar manner to [7, 3] we now define a local coordinate system near $\partial\Omega$. Let η denote the distance from the boundary, s the arclength along the boundary, with some point $x_0 \in \partial\Omega$ corresponding to $s=0$, and $\kappa_1(s)$ the curvature of $\partial\Omega$, which must be uniformly bounded in $[-L/2, L/2]$. This local coordinate system is well defined in the rectangle

$$S = \{(s,\eta)| -L/2 < s < L/2,\ 0 < \eta < \eta_0\} \tag{2.7}$$

where L denotes the arclength of $\partial\Omega$, and η_0 is chosen such that $\inf_{s\in[-L/2,L/2]} 1-\kappa_1(s)\eta_0 > 0$.

Denote by Ω'_α the domain enclosed in $\eta=\alpha$. Integrating (1.7) on Ω'_α yields

$$\kappa\int_{\Omega'_\alpha}\left|\frac{\nabla u}{\rho}\right|^2 + \int_{\Omega'_\alpha}\rho^2 u \leq \int_{\partial\Omega'_\alpha}\frac{\partial u}{\partial n} \leq C\left[\int_{\partial\Omega'_\alpha}\left|\frac{\nabla u}{\rho}\right|^2\right]^{1/2} \tag{2.8}$$

However, by (2.6) we have

$$-\int_{\Omega'_\alpha}\rho^2 u \leq \int_{\Omega'_\alpha}\rho^2\left(h-h_{ex}\right) \leq \frac{C}{\kappa}. \tag{2.9}$$

Furthermore, for every $0 < \delta < \eta_0$, there exists $0 < \alpha < \delta$ such that

$$\int_{\partial\Omega'_\alpha} \left|\frac{\nabla u}{\rho}\right|^2 \leq \frac{C}{\delta} \int_{\Omega'_\alpha} \left|\frac{\nabla u}{\rho}\right|^2 .$$

Cobining the above with (2.8) and (2.9) yields

$$\kappa \int_{\Omega'_\alpha} \left|\frac{\nabla u}{\rho}\right|^2 \leq fracC\delta^{1/2} \int \left[\int_{\Omega'_\alpha} \left|\frac{\nabla u}{\rho}\right|^2\right]^{1/2} ,$$

from which we obtain

$$\int_{\Omega'_\delta} |\nabla u|^2 \leq \int_{\Omega'_\alpha} \left|\frac{\nabla u}{\rho}\right|^2 \leq \frac{C}{\kappa^2(\delta + 1/\kappa)}. \tag{2.10}$$

We can now use Schwarz Inequalty, and the local coordinate system defined in (2.7), to obtain

$$\int_{-L/2}^{L/2} |u(s,0) - u(s,\delta|^2 ds \leq \int_{\Omega\setminus\Omega_\delta} |\nabla u|^2 \left(\eta + \frac{1}{\kappa}\right) ds d\eta \int_0^\delta \frac{d\eta}{(\eta + 1/\kappa)} \leq$$
$$\leq C \log(1 + \kappa\delta) \left[\int_0^\delta d\eta' \int_{\Omega_{\eta'}\setminus\Omega_\delta} |\nabla u|^2 \, ds d\eta + \frac{1}{\kappa^2}\right]$$

Combining the above with (2.10) yields

$$\int_{-L/2}^{L/2} |u(s,0) - u(s,\delta|^2 ds \leq C \frac{\log^2(1 + \kappa\delta)}{\kappa^2}.$$

Consequently,

$$\int_{\partial\Omega_\delta} |u - (h_{ex} - \kappa)|^2 \leq 2 \int_{-L/2}^{L/2} |u(s,0) - u(s,\delta|^2 ds + 2 \int_{\partial\Omega} |u - (h_{ex} - \kappa)|^2 \leq C \frac{\log^2(1 + \kappa\delta)}{\kappa^2}. \tag{2.11}$$

However,

$$\int_{\Omega_\delta} |u - (h_{ex} - \kappa)|^2 \leq C \int_{\partial\Omega_\delta} |u - (h_{ex} - \kappa)|^2 + C \int_{\Omega_\delta} |\nabla u|^2,$$

and hence, with the aid of (2.11), we obtain

$$\int_{\Omega_\delta} |u-(h_{ex}-\kappa)|^2 \le C\frac{\log^2(1+\kappa\delta)}{\kappa^2}.$$

Choosing $\delta \sim O(1)$ we prove (2.3b).

□

Lemma 2.4 *Let $h_{ex} > \kappa$ and $\lambda = \sqrt{\kappa(h_{ex}-\kappa)}$. Let $\{x\}_{\lambda \ge \lambda_0}$ denote a family of points in Ω. Let $s_\lambda = d(x_\lambda, \partial\Omega)$ Then,*

$$F(x_\lambda, s) \le \frac{C(\epsilon)}{(\lambda s_\lambda)^{4-\epsilon}} \; \forall s \le \frac{1}{2}s_\lambda \tag{2.12a}$$

where $F(x,r)$ is given by

$$F(x,r) = \int_{B(0,r)} \lambda^2 w^+ \left|\tilde{J}\right|^2 + \rho^2 \left(w^+\right)^2 + \left|\nabla w^+\right|^2 \tag{2.12b}$$

$$\tilde{J} = \frac{1}{\lambda}\hat{J} \tag{2.12c}$$

and

$$w = \frac{u}{h_{ex}-\kappa} \tag{2.12d}$$

Proof: By (1.7) w satisfies

$$\nabla^2 w - \rho^2 w = \lambda^2 \left|\tilde{J}\right|^2 + \left(\kappa^2 - \frac{1}{2}\right)\frac{1}{2\lambda^2}\rho^4. \tag{2.13}$$

Integrating over $B(x_\lambda, r)$ the product of (2.13) by w^+ we obtain

$$\int_{\partial B(x_\lambda, r)} w^+ \frac{\partial w^+}{\partial r} \ge F(x_\lambda, r) \tag{2.14}$$

Multiplying (2.14) by $1/r$ and integrating between s and s_λ yields, in view of (2.2),

$$\int_s^{s_\lambda} \frac{F(x_\lambda, r)}{r} dr \leq \frac{1}{2} \int_0^{2\pi} \left[\left(w^+(s_\lambda, \theta)\right)^2 - \left(w^+(s, \theta)\right)^2 \right] d\theta \leq C \tag{2.15}$$

In the following we use C to denote a constant which is independent of both λ and x_λ. As F is monotonically increasing in r,

$$\exists \frac{1}{2} < \beta_0 < 1 : \quad F(x_\lambda, \beta_0 s_\lambda) < C. \tag{2.16}$$

It is easy to show that $1/2 < \beta < \beta_0$ exists such that

$$\int_{\partial B_\beta} \lambda^2 w^+ \left|\tilde{J}\right|^2 + \rho^2 \left(w^+\right)^2 \leq \frac{C}{s_\lambda} \int_{B_{\beta_0}} \lambda^2 w^+ \left|\tilde{J}\right|^2 + \rho^2 \left(w^+\right)^2 \tag{2.17}$$

where $B_\beta \stackrel{def}{=} B(x, \beta s)$. Let $\xi_1, \xi_2 \in \partial B_\beta$. Then,

$$\left| \left(w^+\right)^{5/2} (\xi_1) - \left(w^+\right)^{5/2} (\xi_2) \right| \leq C \int_{\partial B_\beta} \left(w^+\right)^{3/2} |\nabla w|$$

By (1.8) $|\nabla w| = \rho \left|\tilde{J}\right|$. Hence,

$$\begin{aligned} \left| \left(w^+\right)^{5/2} (\xi_1) - \left(w^+\right)^{5/2} (\xi_2) \right| &\leq C \left[\int_{\partial B_\beta} w^+ \left|\tilde{J}\right|^2 \right]^{1/2} \left[\int_{\partial B_\beta} \rho^2 \left(w^+\right)^2 \right]^{1/2} \leq \\ &\leq \frac{C}{\lambda} \int_{\partial B_\beta} \lambda^2 w^+ \left|\tilde{J}\right|^2 + \rho^2 \left(w^+\right)^2 \leq \frac{C}{\lambda s_\lambda} F(x_\lambda, \beta_0 s_\lambda) . \end{aligned} \tag{2.18}$$

Let $0 < s < \beta s_\lambda$, and let (r, θ) denote a polar coordinate system centered around x. Then,

$$\int_0^{2\pi} \int_s^{\beta s_\lambda} \left(w^+\right)^{3/2} \frac{\partial w}{\partial r} dr d\theta \leq C \left[\int_A w^+ \left|\tilde{J}\right|^2 \right]^{1/2} \left[\int_A \rho^2 \left(w^+\right)^2 \frac{1}{r^2} \right]^{1/2} \tag{2.19}$$

where $A \stackrel{def}{=} B_\beta \setminus B(x, s)$. Hence,

$$\int_0^{2\pi} \left(w^+\right)^{5/2} \Big|_s^{\beta s_\lambda} \leq \frac{C}{\lambda s} F(x_\lambda, \beta_0 s_\lambda).$$

Utilizing (2.18) together with the inequality

$$\left|x^5 - y^5\right| \geq \left|x^4 - y^4\right|^{5/4} \tag{2.20}$$

and Hölder inequality we obtain

$$\begin{aligned}
&\int_0^{2\pi} \left(w^+\right)^{5/2}\Big|_s^{\beta s_\lambda} d\theta + \frac{2\pi C}{\lambda s_\lambda} F(x_\lambda, \beta_0 s_\lambda) = \\
&\qquad = \int_0^{2\pi} \left\{ \left[\left(w^{+\ 5/2}\left(\beta s_\lambda\right) + \frac{C}{\lambda s_\lambda} F(x_\lambda, \beta_0 s_\lambda) \right)^{1/5} \right]^5 - w^{+\ 5/2}\left(s\right) \right\} d\theta \geq \\
&\qquad \geq \int_0^{2\pi} \left\{ \left(w^{+\ 5/2}\left(\beta s_\lambda\right) + \frac{C}{\lambda s_\lambda} F(x_\lambda, \beta_0 s_\lambda) \right)^{4/5} - w^{+\ 2}\left(s\right) \right\}^{5/4} d\theta \geq \\
&\qquad\qquad \geq C \left\{ \int_0^{2\pi} \left| \left(w^{+\ 5/2}\left(\beta s_\lambda\right) + \frac{C}{\lambda s_\lambda} F(x_\lambda, \beta_0 s_\lambda) \right)^{4/5} - w^{+\ 2}\left(s\right) \right| d\theta \right\}^{5/4} .
\end{aligned}$$

In view of (2.15)

$$\int_0^{2\pi} \left(w^+\right)^2\Big|_s^{\beta s_\lambda} d\theta \geq 0 \, .$$

Consequently,

$$\left\{ \int_0^{2\pi} \left| \left(w^{+\ 5/2}\left(\beta s_\lambda\right) + \frac{C}{\lambda s_\lambda} F(x_\lambda, \beta_0 s_\lambda) \right)^{4/5} - w^{+\ 2}\left(s\right) \right| d\theta \right\}^{5/4} \geq \left\{ \int_0^{2\pi} \left(w^+\right)^2\Big|_s^{\beta s_\lambda} \right\}^{5/4}$$

Combining the above inequalities yields

$$\int_0^{2\pi} \left(w^+\right)^2\Big|_s^{\beta s_\lambda} d\theta \leq C \left(\frac{F(x_\lambda, \beta_0 s_\lambda)}{\lambda s} \right)^{4/5}$$

and by (2.15) we have

$$\int_s^{\beta s_\lambda} \frac{F(x_\lambda, r)}{r} dr \leq C \left(\frac{F(x_\lambda, \beta_0 s_\lambda)}{\lambda s} \right)^{4/5}$$

Thus, since F is monotone increasing

$$\exists \frac{1}{2} < \beta_1 < \beta_0 F(x_\lambda, \beta_1 s_\lambda) \leq C_1 \left(\frac{F(x_\lambda, \beta_0 s_\lambda)}{\lambda s_\lambda} \right)^{4/5} \, . \tag{2.21}$$

It is possible to repeat the above procedure recursively (cf. [3]) to prove the existence of a monotone decreasing sequence $\{\beta_n\}_{n=1}^{\infty}$, which is strictly bounded from below by $1/2$, such that

$$F(x_\lambda, \beta_n s_\lambda) \leq C_n \left(\frac{F(x_\lambda, \beta_{n-1} s_\lambda)}{\lambda s_\lambda} \right)^{4/5} \quad \forall n \geq 1. \tag{2.22}$$

Utilizing the above inequality together with (2.16) proves the lemma.

$\square$

Lemma 2.4 allows us to obtain uniform convergence in Ω of w^+ to a constant, except for a boundary layer of $O(1/\lambda)$ size (as $\lambda \to \infty$).

Lemma 2.5 *For any family of points* $\{x_\lambda\}_{\lambda > \lambda_0}$

$$\exists \tilde{w}_\lambda : \left| w^+(x_\lambda) - \tilde{w}_\lambda \right| \leq \frac{C}{\lambda^{1/2} d(x_\lambda, \partial\Omega)^{1/2}}$$

The lemma can be proved by applying the same arguments as in the proof of lemma 3.4 in [3].

We now find the value of the constant $\tilde{w}_\lambda$ by using the energy estimates in lemma 2.3.

Lemma 2.6 *Let* $h_{ex} > \kappa$. *Then,*

$$|\tilde{w}_\lambda - 1| \leq C \left(\frac{1}{\lambda^{1/2}} + \frac{\log \kappa}{\lambda^2} \right) \tag{2.23}$$

Proof: Let $x \in \Omega$ such that $\partial B(x, r) \subset int(\Omega)$, where r is independent of λ. By lemma 2.3 we have

$$\|1 - w\|_{L^2[B(x,r)]} \leq \|1 - w\|_{L^2[\Omega]} \leq \frac{2}{h_{ex} - \kappa} \|h_{ex} - h\|_{L^2[\Omega]} + \frac{1}{\kappa(h_{ex} - \kappa)} \left\|\rho^4\right\|_{L^2[\Omega]} \leq C \frac{\log \kappa}{\lambda^2}$$

However, by the previous lemma $|w^+ - \tilde{w}_\lambda| \leq C/\lambda^{1/2}$ in $B(x,r)$, and hence, since

$$\|\tilde{w}_\lambda - 1\|_{L^2[B(x,r)]} \leq \left\|1 - w^+\right\|_{L^2[B(x,r)]} + \left\|\tilde{w}_\lambda - w^+\right\|_{L^2[B(x,r)]} \leq C\left(\frac{1}{\lambda^{1/2}} + \frac{\log\kappa}{\lambda^2}\right),$$

from which the lemma immediately follows.

$\square$

We can now obtain better estimates for the rate of decay of $|w^+ - \tilde{w}_\lambda|$ away from the boundaries as $\lambda \to \infty$.

Lemma 2.7 *Let $h_{ex} > \kappa$ and $\{x_\lambda\}_{\lambda \geq \lambda_0}$ denote a family of points such that $x_\lambda \in \Omega$. Let $\lambda s_\lambda = \lambda d(x_\lambda, \partial\Omega) \xrightarrow[\lambda\to\infty]{} \infty$. Then,*

$$\forall n \in \mathbb{N} \quad \exists \frac{1}{2} < \beta_n < 1,\ C_n > 0:\ F(x_\lambda, \beta_n s_\lambda) \leq \frac{C_n}{\lambda^n s_\lambda^n} \tag{2.24a}$$

where F is defined in (2.12)

$$\exists \tilde{w}_\lambda: \quad \left|w^+(x_\lambda) - \tilde{w}_\lambda\right| \leq \frac{C_n}{\lambda^n s_\lambda^n} \tag{2.24b}$$

$$|\tilde{w}_\lambda - 1| \leq C\frac{\log\kappa}{\lambda^2} \tag{2.24c}$$

The proof of (2.24a) and (2.24b) is obtained by following the same line of arguments as in the proof of lemma 3.6 in [3]. To prove (2.24c) we use (2.24b) and apply arguments of the proof of lemma 2.6 once again.

$\square$

Lemma 2.8 *Let $h_{ex} > \kappa$ and $\{x_\lambda\}_{\lambda \geq \lambda_0}$ denote a family of points such that $x_\lambda \in \Omega$. Let $s_\lambda = d(x_\lambda, \partial\Omega) \xrightarrow[\lambda\to\infty]{} \infty$. Then,*

$$\forall n \in \mathbb{N} \quad \exists C_n > 0: \int_{\partial B(x_\lambda, s_\lambda/2)} \rho^2 \leq \frac{C_n}{\lambda^n s_\lambda^n}. \tag{2.25}$$

Proof: By (2.24a)

$$\exists \frac{1}{2} < \beta_n < 1 : \quad \int_{B(x_\lambda,\beta_n s_\lambda)} \rho^2 \left(w^+\right)^2 \leq \frac{C_n}{\lambda^n s_\lambda^n}$$

Writing

$$\|\rho\|_{L^2[B(x_\lambda,\beta_n s_\lambda)]} \leq \left\|\rho w^+\right\|_{L^2[B(x_\lambda,\beta_n s_\lambda)]} + \\ + \left\|\rho\left(w^+ - \tilde{w}_\lambda\right)\right\|_{L^2[B(x_\lambda,\beta_n s_\lambda)]} + \left\|\rho\left(1-\tilde{w}_\lambda\right)\right\|_{L^2[B(x_\lambda,\beta_n s_\lambda)]} ,$$

we obtain, in view of (2.23) and (2.24b),

$$\|\rho\|_{L^2[B(x_\lambda,\beta_n s_\lambda)]} \leq \frac{C_n}{(\lambda s_\lambda)^{n/2}} . \tag{2.26}$$

Consequently, for sufficiently large λ, we may conclude the existence of $\frac{1}{2} < \bar{\beta}_n < \beta_n$ for which

$$\int_{\partial B(x_\lambda,\bar{\beta}_n s_\lambda)} \rho^2 \leq \frac{C_n}{\lambda^n s_\lambda^n}. \tag{2.27}$$

$\square$

Proof of Theorem 1: We prove the theorem by invoking blow up arguments. We first prove that $\exists\lambda_0$ and $\beta > 0$ such that

$$\|\psi\|_{L^2[B(x,\delta)]} \leq C\delta e^{-\beta\lambda d(x,\partial\Omega)} \quad \forall\lambda > \lambda_0,\ 0 < \delta < \frac{1}{\lambda},\ \forall x \in \Omega :\ d(x,\partial\Omega) \geq \frac{1}{\lambda} \tag{2.28}$$

Let

$$\Omega(\lambda,k,s) = \left\{x \in \Omega \middle|\ d(x,\partial\Omega) \geq k\frac{s}{\lambda}\right\}.$$

We prove (2.28) by showing that

$$\exists\lambda_0, s_0 : \sup_{x\in\Omega(\lambda,k,s)} \|\psi(\kappa,\lambda)\|_{L^2[B(x,\delta)]} \leq \frac{1}{2} \sup_{x\in\Omega(\lambda,k+1,s)} \|\psi(\kappa,\lambda)\|_{L^2[B(x,\delta)]} \quad \forall s > s_0\ \lambda > \lambda_0,\ k \in \mathbb{N},\ 0 < \delta < \frac{1}{\lambda} \tag{2.29}$$

Suppose, for a contradiction, that (2.29) does not hold. Then, sequences $\{\lambda_j\}_{j=1}^\infty$,$\{\kappa_j\}_{j=1}^\infty$, $\{s_j\}_{j=1}^\infty$, $\{k_j\}_{j=1}^\infty$ and $\{\delta_j\}_{j=1}^\infty$ exist such that $\lambda_j \uparrow \infty$, $\kappa_j \uparrow \infty$, $s_j \uparrow \infty$, $k_j \in \mathbb{N}$, $0 < \delta_j < 1/\lambda_j$, and

$$\sup_{x \in \Omega(\lambda_j, k_j+1, s_j)} \|\psi(\kappa_j, \lambda_j)\|_{L^2[B(x,\delta_j)]} \geq \frac{1}{2} \left\| \psi(\kappa_j, \lambda_j) \sup_{x \in \Omega(\lambda_j, k_j, s_j)} \right\|_{L^2[B(x,\delta_j)]} \stackrel{def}{=} \frac{1}{2} m_j \tag{2.30}$$

Let

$$\tilde{\psi}_j \stackrel{def}{=} \frac{\psi(\kappa_j, \lambda_j)}{m_j}.$$

By (2.29) there exists $x_j \in \Omega(\lambda_j, k_j+1, s_j)$ such that $\|\tilde{\psi}_j\|_{L^2[B(x_j,\delta_j)]} \geq \frac{1}{2}$. Furthermore, since $B(x_j, 1) \in \Omega(\lambda_j, k_j, s_j)$ we have

$$\frac{1}{2} \leq \left\| \tilde{\psi}_j \right\|_{L^2[B(x_j,\delta_j)]} \leq 1.$$

Define

$$f_j = \tilde{\psi}_j \left(x_j + \frac{x}{\lambda_j} \right) e^{iA_j(x_j)\cdot x/\lambda_j},$$

where $A_j = A(\kappa_j, \lambda_j)$. Let $\tilde{w}_\lambda$ be the same as in (2.24c) and let

$$\tilde{h}_\lambda = \tilde{w}_\lambda (h_{ex} - \kappa) + \kappa.$$

Clearly,

$$\left\| h - \tilde{h}_\lambda \right\|_{L^\infty[\Omega(\lambda,k,s)]} \leq \frac{C_n}{\kappa s^n}$$

It is easy to show that

$$\left(\frac{i\lambda_j}{\kappa_j} \nabla + \frac{\tilde{h}_j}{\lambda_j} \tilde{B}_j \right)^2 f_j = f_j \left(1 - m_j^2 \, |f_j|^2 \right) \quad x \in B(0, s_j) \tag{2.31a}$$

wherein

$$B_j(x) = [A_j(x_j + x) - A_j(x_j)]\,\frac{1}{\tilde{h}_j}, \tag{2.31b}$$

and $\tilde{h}_j = \tilde{h}_{\lambda_j}$. we now define a cut-off function

$$\eta_r = \begin{cases} 1 & \text{in } B(0,r) \\ \\ 0 & \text{in } \mathbb{R}^2/B(0,2r) \end{cases}$$

$$|\nabla \eta_r| \leq C \quad \text{in } \mathbb{R}^2$$

Multiplying (2.31a) by η_r^2, and integrating over $B(0,2r)$ we obtain, for all $r \leq \frac{s_j}{2}$ (cf. [14]), that

$$\int_{B(0,2r)} \left| \left(\frac{i\lambda_j}{\kappa_j} \nabla + \frac{\tilde{h}_j}{\lambda_j} B_j \right) (\eta_r f_j) \right|^2 = \int_{B(0,2r)} \eta_r^2 f_j^2 \left(1 - m_j^2 \, |f_j|^2 \right) + \frac{\lambda_j^2}{\kappa_j^2} \, |\nabla \eta_r|^2 \, f_j^2 \tag{2.32}$$

Let $\hat{A} : \mathbb{R}^2 \to \mathbb{R}^2$ denote any vector field satisfying $\nabla \times \hat{A} = \hat{i}_z$ and $\hat{A}(0) = 0$. Then,

$$\begin{aligned} \int_{B(0,2r)} \left| \left(\frac{i\lambda_j}{\kappa_j} \nabla + \frac{\tilde{h}_j}{\lambda_j} B_j \right) (\eta_r f_j) \right|^2 = \int_{B(0,2r)} \left| \left(\frac{i\lambda_j}{\kappa_j} \nabla + \frac{\tilde{h}_j}{\lambda_j} \hat{A} \right) (\eta_r f_j) \right|^2 + \\ + \int_{B(0,2r)} \frac{\tilde{h}_j}{\kappa_j} (B_j - \hat{A}) \eta_r^2 \left[i \left(\bar{f}_j \nabla f_j - f_j \nabla \bar{f}_j \right) + 2 \, |f_j|^2 \, \frac{\tilde{h}_j \kappa_j}{\lambda_j^2} B_j \right] - \\ - \int_{B(0,2r)} \left(\frac{\tilde{h}_j}{\lambda_j} \right)^2 \left| B_j - \hat{A} \right|^2 \eta_r^2 |f_j|^2 \, . \end{aligned}$$

Clearly,

$$\eta_r^2 \left[i \frac{\lambda_j}{\kappa_j} \left(\bar{f}_j \nabla f_j - f_j \nabla \bar{f}_j \right) + 2 \, |f_j|^2 \, \frac{\tilde{h}_j}{\lambda_j} B_j \right] = 2\Re \left\{ \eta_r \bar{f}_j \left(\frac{i\lambda_j}{\kappa_j} \nabla + \frac{\tilde{h}_j}{\lambda_j} B_j \right) (\eta_r f_j) \right\},$$

and hence,

$$\int_{B(0,2r)} \left| \left(\frac{i\lambda_j}{\kappa_j} \nabla + \frac{\tilde{h}_j}{\lambda_j} B_j \right) (\eta_r f_j) \right|^2 \geq \int_{B(0,2r)} \left| \left(\frac{i\lambda_j}{\kappa_j} \nabla + \frac{\tilde{h}_j}{\lambda_j} \hat{A} \right) (\eta_r f_j) \right|^2 -$$

$$- 2 \frac{\tilde{h}_j}{\lambda_j} M_j \left[\int_{B(0,2r)} \eta_r^2 \left| f_j \right|^2 \right]^{1/2} \left[\int_{B(0,2r)} \left| \left(\frac{i\lambda_j}{\kappa_j} \nabla + \frac{\tilde{h}_j}{\lambda_j} B_j \right) (\eta_r f_j) \right|^2 \right]^{1/2} -$$

$$- \left(\frac{\tilde{h}_j}{\lambda_j} \right)^2 M_j^2 \int_{B(0,2r)} \eta_r^2 |f_j|^2 \,. \quad \text{(2.33a)}$$

where

$$M_j = \sup_{x \in B(x,s_j)} \left| B_j - \hat{A} \right| \qquad \text{(2.33b)}$$

In [14, 7] it was shown that

$$\int_{\mathbb{R}^2} \left| \left(\frac{i\lambda_j}{\kappa_j} \nabla + \frac{\tilde{h}_j}{\lambda_j} \hat{A} \right) (\eta_r f_j) \right|^2 \geq \frac{\tilde{h}_j}{\kappa_j} \int_{\mathbb{R}^2} \eta_r^2 |f_j|^2 \qquad \text{(2.34)}$$

Combining the above with (2.32) and (2.33a) we obtain

$$\left(\frac{\tilde{h}_j}{\kappa_j} - 1 \right) \int_{B(0,2r)} \eta_r^2 |f_j|^2 \leq \frac{\lambda_j^2}{\kappa_j^2} \int_{B(0,2r)} \left| \nabla \eta_r \right|^2 f_j^2 + M_j \frac{\tilde{h}_j}{\lambda_j} \int_{B(0,2r)} 2\eta_r^2 f_j^2 + \frac{\lambda_j^2}{\kappa_j^2} \left| \nabla \eta_r \right|^2 f_j^2 + \left(\frac{\tilde{h}_j}{\lambda_j} \right)^2 M_j^2 \int_{B(0,2r)} \eta_r^2 | \qquad \text{(2.35)}$$

By (2.24b) we have

$$M_j \leq \frac{\lambda_j^2 C_n}{\kappa_j^2 s_j^n} \qquad \forall n \in \mathbb{N}.$$

By (2.24c) we have, for sufficiently large j,

$$\left| \tilde{h}_j - h_{ex}(\kappa_j) \right| \leq \frac{1}{2} \left| h_{ex}(\kappa_j) - \kappa_j \right| = \frac{1}{2} \frac{\lambda_j^2}{\kappa_j}$$

and hence by (2.35) we have

$$\int_{B(0,r)} |f_j|^2 \leq C \int_{B(0,2r)} \left| \nabla \eta_r \right|^2 f_j^2 \qquad \text{(2.36)}$$

From which we can easily obtain

$$\lim_{j\to\infty} \int_{B(0,r)} |f_j|^2 = 0 \tag{2.37}$$

for all $r > 0$, contradicting (2.30), and therefore proving (2.28).

In order to obtain exponential decay in C^α norm we write first the equation for $\phi(z) = \psi(x_0 + z/\kappa)e^{-iA(x_0)\cdot z}$,.

$$\nabla^2\phi = 2\tilde{A} \cdot \left(i\nabla + \tilde{A}\right)\phi - \phi\left(1 - |\phi|^2 + \left|\tilde{A}_j\right|^2\right) \tag{2.38}$$

where

$$\tilde{A}(z) = A(x_0 + z/\kappa) - A(x_0).$$

It is possible to show, using the identity (2.32) and the boundednes of $\tilde{A}(z)$ in $B(0,2)$, that

$$\int_{B(0,2)} \left|2\tilde{A} \cdot \left(i\nabla + \tilde{A}\right)\phi - \phi\left(1 - |\phi|^2 + \left|\tilde{A}_j\right|^2\right)\right|^2 dz \leq C \int_{B(0,2)} \phi^2 dz$$

Using standard elliptic estimates we then have

$$\|\phi\|_{H^2[B(0,1)]} \leq C\|\phi\|_{L^2[B(0,2)]}.$$

where C is independent ofκ, λ, and x_0. Choosing $\delta = 1/\kappa$ in (2.28) we obtain

$$\|\phi\|_{L^2[B(0,2)]} \leq Ce^{-\beta\lambda d(x_0,\partial\Omega)}$$

Sobolev embedding then implies

$$\|\phi\|_{L^\infty[B(0,1)]} \leq Ce^{-\beta\lambda d(x_0,\partial\Omega)}, \tag{2.39}$$

which proves (2.1a) for $\alpha = 0$.

We now write the equation for $\tilde{A}(z)$,

$$\nabla \times H = \frac{1}{\kappa^2} \Im \left[\bar{\phi} \left(i\nabla + \tilde{A} \right) \phi \right] \tag{2.40}$$

where $H(z) = \nabla \times \tilde{A}(z)$. By (2.2c) we then have

$$\|\nabla H\|_{L^\infty[B(0,1)]} \leq \frac{C}{\kappa^2} e^{-\beta\lambda d(x_0,\partial\Omega)},$$

and hence,

$$\|\nabla h\|_{L^\infty[B(x_0,1/\kappa)]} \leq C e^{-\beta\lambda d(x_0,\partial\Omega)}. \tag{2.41}$$

We now utilize the coordinate system defined in (2.7). Suppose first that $\eta = d(x_0, \partial\Omega) < \delta$. Then, we can integrate (2.41) with respect to η to obtain

$$|h(s,\delta) - h(s,\eta)| \leq \frac{C}{\lambda} e^{-\beta\lambda\eta}. \tag{2.42}$$

Using (2.41) it is also easy to show that

$$\exists \tilde{h}_\lambda : \; d(x_0, \partial\Omega) \geq \delta \Rightarrow |h(x_0) - \tilde{h}_\lambda| \leq C e^{-\beta\lambda\delta}$$

The last two inequalities when combined yield

$$\exists \tilde{h}_\lambda : \; |h(x_0) - \tilde{h}_\lambda| \leq C \max\left(e^{-\beta\lambda\delta}, \frac{1}{\lambda} e^{-\beta\lambda d(x_0,\partial\Omega)} \right). \tag{2.43}$$

Which proves (2.1b).

To prove (2.1c), and (2.1a) for $\alpha \geq 1$ we use (2.38) and (2.40) together with bootstraping and Sobolev embedding. To prove (2.1d) we use (2.42) for $\eta = 0$, which gives

$$|h_{ex} - \tilde{h}_\lambda| \leq \frac{C}{\lambda}.$$

Then we use the same arguments as in lemma 2.6 to show

$$|h_{ex} - \tilde{h}_\lambda| \leq \frac{C\lambda}{\kappa}$$

□

3 Conclusion

In [16] Pan obtains that in the limit $\kappa \to \infty$

$$\int_\Omega \left\{ |\psi|^2 + \left| \frac{1}{\kappa}\nabla\psi - iA\psi \right|^2 \right\} e^{\beta\sqrt{\kappa(h_{ex}-\kappa)}d(x,\partial\Omega)} dx \leq \frac{C}{\sqrt{\kappa(h_{ex}-\kappa)}} \tag{3.1}$$

whenever $h_{ex} - \kappa \gg 1$, for some $\beta > 0$ which is independent of κ. In the present contribution we extend the validity of the above result to external fields satisfying

$$h_{ex} - \kappa \gg \frac{1}{\kappa}$$

We also obtain, in theorem 2.1, convergence in C^α norms in contrast to the above L^2 convergence which is proved in [16]. It should be mentioned, however, that once L^2 convergence is obtained, it is possible to prove (2.39) and (2.41) and then proceed using bootstraping and Sobolev embedding. The main advantage of the results in this work is therefore the greater range of external fields for which exponential rate of decay is guaranteed. This is facilitated by better a-priory estimates of the magnetic field: while in [16] it is first proved that $|h-h_{ex}| \leq C$ in Ω whereas here, (2.24), provides a much better estimate on the magnetic field.

In addition to (3.1) it is demontrated in [16] that for $h_{ex} - \kappa \gg 1$ the energy of the global minimizer is evenly distributed along the bondary (for a more precise definition the reader is refered to [16]). In view of the better estimated of h in the present contribution it appears reasonable to believe that the validity of this result can be extended to external fields satisfying $h_{ex} - \kappa \gg 1/\kappa$. However, since the analysis in [16] is heavily based on the assumption $h_{ex} - \kappa \gg 1$, significant modification is necessary before it can be applied to a greater range of applied magnetic fields.

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