Abrikosov lattices in finite domains

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Abstract

In 1957 Abrikosov published his work on periodic solutions to the linearized Ginzburg-Landau equations. Abrikosov's analysis assumes periodic boundary conditions, which are quite different than the natural boundary conditions the minimizer of the Ginzburg-Landau energy functional should satisfy. In the present work we prove that the global minimizer of the fully non-linear functional can be approximated, in every rectangular subset of the domain, by one of the periodic solution to the linearized Ginzburg-Landau equations in the plane. Furthermore, we prove that the energy of this solution is close to the minimum of the energy over all Abrikosov's solutions in that rectangle.

1 Introduction

Consider a planar superconducting body which is placed at a sufficiently low temperature (below the critical one) under the action of an applied magnetic field . Its energy is given by the Ginzburg-Landau energy functional which can be represented in the following dimensionless form [11]

$$E = \int_\Omega \left(-|\Psi|^2 + \frac{|\Psi|^4}{2} + |h - h_{ex}|^2 + \left| \frac{i}{\kappa}\nabla\Psi + A\Psi \right|^2 \right) dx_1 dx_2 \tag{1.1}$$

in which $\Psi \in H^1(\Omega, \mathbb{C})$ is the superconducting order parameter, such that $|\Psi|$ varies from $|\Psi| = 0$ (when the material is at a normal state) to $|\Psi| = 1$ (for the purely superconducting state). The magnetic vector potential is denoted by $A \in H^1(\Omega, \mathbb{R}^2)$ (the magnetic field is, then, given by $h = \nabla \times A$), h_{ex} is the constant applied magnetic field, and κ is the Ginzburg-Landau parameter which is a material property. The superconductor lies in Ω, which is a smooth connected domain. Its Gibbs free energy is given by E. Note that E is invariant to the gauge transformation

$$\Psi \to e^{i\kappa\zeta}\Psi \qquad ; \qquad A \to A + \nabla\zeta \; . \tag{1.2}$$

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where ζ is any smooth function. Thus, we confine ourselves in the sequel to competitors from the space

$$\mathcal{H} = \left\{ (\psi, A) \in H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^2) \mid (A - \hat{A}) \cdot \hat{n} \quad \hat{A} = h_{ex} x_1 \hat{i}_2 \right\} , \tag{1.3}$$

where $\hat{i}_2$ is a unit vector in the x_2 direction.

For sufficiently large magnetic fields it is well known, both from experimental observations [20] and from theoretical predictions [14], that superconductivity is destroyed and the material must be in the normal state. If the applied magnetic field is then decreased there is a critical field where the material enters the superconducting phase once again. This field is called "the onset field" and is denoted by H_{C_3}. At the bifurcation from the normal state, superconductivity remains concentrated near the boundary, which is why this phenomenon has been termed "surface superconductivity" [23, 8, 9, 18, 12, 16].

In the absence of boundaries the critical field at which superconductivity nucleates is denoted by H_{C_2} and is smaller than H_{C_3} ($H_{C_3} \approx 1.7\kappa$ whereas $H_{C_2} = \kappa$). Furthermore, the bifurcating modes are periodic lattices, named after Abrikosov [2, 10, 4] which have been observed experimentally [13]. It has been conjectured, therefore, by Rubinstein [22] that superconductivity remains concentrated near the boundary for $H_{C_2} < h_{ex} < H_{C_3}$. When $h_{ex} \approx H_{C_2}$ (either for κ large or for large domains) a bifurcation of Abrikosov's lattices far away from the wall was conjectured [22].

Recently, it has been proved both in the large κ limit [21, 7], and in the large domain limit [5] that as long as $H_{C_2} < h_{ex} < H_{C_3}$ superconductivity remains concentrated near the boundaries. However, the second part of the conjecture in [22] is still open. In [24] it is shown for the global minimizer of (1.1) (ψ_κ, A_κ), that ψ_κ diminishes as $h_{ex} \uparrow H_{C_2}$ away from the boundaries. However, the exact structure of ψ_κ is that limit has never been found.

In [6] the bifurcation of periodic solutions from the one-dimensional surface superconductivity solution introduced in [21] was studied. Nevertheless, the analysis in [6] was performed in a half-plane and under the assumption that the solutions are periodic in the direction parallel to the boundary.

In the present contribution we focus on the emergence of Abrikosov's lattices in finite domains in $\mathbb{R}^2$. We prove that after the second bifurcation, when h_{ex} is slightly smaller than κ, the global minimizer of (1.1) in $\mathcal{H}$, can be approximated in an appropriately chosen rectangle in Ω, by one of Abrikosov's solutions. Furthermore, we prove that the energy of this solution is close to the minimum of the energy over all Abrikosov's solutions in that rectangle.

Thus, we prove the following result.

Theorem 1.1 *Let*

$$\kappa^{-1/5} \ll \epsilon = [1 - h_{ex}/\kappa]^{1/2} \ll \frac{1}{\log \kappa}$$

Let further (ψ_κ, A_κ) *denote the global minimizer of* (1.1) *in* $\mathcal{H}$. *Then, denote by R a rectangle in Ω whose side lengths are given by*

$$L_1 = \frac{\omega N}{\sqrt{h_{ex}\kappa}} \quad ; \quad L_2 = \frac{2\pi}{\omega\sqrt{h_{ex}\kappa}}$$

where $N(\kappa,\epsilon)\in\mathbb{N}$ *and* $\omega(\kappa,\epsilon)\in\mathbb{R}$ *are such that*

$$\frac{1}{\kappa\epsilon^5}\ll(L_1L_2)^{1/2}\quad;\quad r_1<\frac{L_2}{L_1}<r_2$$

where r_1 *and* r_2 *are independent of* κ*and* ϵ. *Finally, let*

$$\mathcal{P}_R=\left\{u\in H^1_{loc}(\mathbb{R}^2)\;\middle|\;\begin{array}{l}u(x_1+L_1,x_2)=e^{ih_{ex}\kappa L_1x_2}u(x_1,x_2)\ a.e.\\ u(x_1,x_2+L_2)=u(x_1,x_2)\ a.e.\end{array}\right\},$$

$$\mathcal{U}_R=\left\{u\in\mathcal{P}_R\;\middle|\;\int_R\left(\frac{i}{\kappa}\nabla+\hat{A}\right)u\cdot\left(-\frac{i}{\kappa}\nabla+\hat{A}\right)\bar{\phi}=\frac{h_{ex}}{\kappa}\int_R u\bar{\phi}\quad\forall\phi\in C_c^\infty(R)\right\}$$

where $\hat{A}$ *is given in* (1.3). *Then, there exists* $u_0(\kappa,\epsilon)\in\mathcal{U}_R$ *such that*

$$\int_R|\psi-u_0|^2\leq C\epsilon^2\int_R|\psi|^2 \tag{1.4a}$$

$$\mathcal{J}_R(u_0)\leq\inf_{v\in\mathcal{U}_R}\mathcal{J}_R(v)[1-\delta]<0\quad\forall v\in\mathcal{U}_R \tag{1.4b}$$

where

$$\delta\leq\epsilon^{\alpha/4}\quad\forall\alpha<1 \tag{1.4c}$$

and

$$\mathcal{J}_R(u)=\int_R|u|^4-\epsilon^2|u|^2. \tag{1.4d}$$

Furthermore,

$$\|A_\kappa-\hat{A}\|_{H^1(\Omega)}\leq C\epsilon^3\quad;\quad\|A_\kappa-\hat{A}\|_{H^2(\Omega)}\leq C\epsilon^2 \tag{1.4e}$$

This result proves that we can approximate ψ_κ in every rectangular subset R of Ω, and possibly even in diminishingly small rectangles (as $\kappa\to\infty$), by some function u_0 in $\mathcal{U}_R$, which is the space of Abrikosov's periodic solutions in R. Furthermore, the theorem shows that u_0 can be found by studying the minimization problem of $\mathcal{J}_R$ in $\mathcal{U}_R$, which is a finite dimensional subspace.

The rest of this contribution is arranged as follows: in the next section we review some of the results obtained for the linear periodic problem, analyzed first by Abrikosov [2]. In § 3 we obtain some a-priory estimates that are valid for any critical point of (1.1). In § 4 we obtain upper and lower bounds for (1.1) in R which enable the proof of theorem 1.1. Finally, in § 5 we briefly summarize the main results of this work and emphasize some additional key points.

2 The periodic problem

Consider the problem

$$\left(\frac{i}{\kappa}\nabla + \hat{A}\right)^2 u = \frac{h_{ex}}{\kappa} u \quad \text{in } \mathbb{R}^2 \tag{2.1}$$

where

$$\hat{A} = h_{ex} x_1 \hat{i}_2 \,. \tag{2.2}$$

Let $\omega \in \mathbb{R}$, and let

$$L_1 = \frac{\omega N}{\sqrt{h_{ex}\kappa}} \quad ; \quad L_2 = \frac{2\pi}{\omega\sqrt{h_{ex}\kappa}}$$

where $N \in \mathbb{N}$. The periodic boundary conditions u should satisfy are given by

$$\begin{cases} v(x_1 + L_1, x_2) = e^{i\kappa h_{ex} L_1 x_2} v(x_1, x_2) \\ v(x_1, x_2 + L_2) = v(x_1, x_2) \end{cases} .$$

We can now apply another transformation

$$x \to \sqrt{\kappa h_{ex}}\, x$$

to obtain

$$\left(i\nabla + x_1 \hat{i}_2\right)^2 v = v \tag{2.3a}$$

$$v(x_1 + L_1', x_2) = e^{i\omega N x_2} v(x_1, x_2) \tag{2.3b}$$

$$v(x_1, x_2 + L_2') = v(x_1, x_2) \tag{2.3c}$$

where in the new coordinates

$$L_1' = \omega N \quad ; \quad L_2' = \frac{2\pi}{\omega}.$$

It is easy to check that the phase change around ∂R where $R = [0, L_1'] \times [0, L_2']$ is $2\pi N$. .

The general solution of (2.3a) and (2.3c) is given in the form

$$v = \sum_{n=-\infty}^{\infty} g_n(x_1) e^{i\omega n x_2}$$

where g_n satisfies

$$g_n'' - [(x_1 - n\omega)^2 - 1] g_n = 0 \,,$$

whose general solution is given by

$$g_n = C_n e^{-\frac{1}{2}(x_1 - n\omega)^2} + D_n G(x - n\omega) \,, \tag{2.4}$$

where G can be expressed in terms of Parabolic Cylinder functions [1].

From (2.3b) we conclude that for all $x_1 \in \mathbb{R}$ we must have

$$g_n(x_1 + \omega N) = g_{n-N}(x_1)\,,$$

or, equivalently,

$$C_n = C_{n-N} \quad ; \quad D_n = D_{n-N}\,.$$

By the lemma of Riemann-Lebesgue we must have

$$g_n(x_1) \xrightarrow[|n|\to\infty]{} 0 \quad \forall x_1 \in \mathbb{R}$$

and since $G(x_1)$ is unbounded in $\mathbb{R}$, we must have $D_n = 0$ for all $n \in \mathbb{Z}$.

Consequently, the general solution of (2.3) is given by

$$v = \sum_{n=-\infty}^{\infty} C_n e^{-\frac{1}{2}(x_1 - n\omega)^2} e^{i\omega n x_2}$$

where $C_{n+N} = C_n$ for all n. Thus, we can write that

$$v = \sum_{n=0}^{N-1} C_n f_n \tag{2.5a}$$

where

$$f_n = \sum_{r=-\infty}^{\infty} e^{i(n+rN)\omega x_2} e^{-\frac{1}{2}[x_1 - (n+rN)\omega]^2}\,. \tag{2.5b}$$

Note that

$$f_{n+1}(x_1, x_2) = e^{i\omega x_2} f_n(x_1 - \omega)\,,$$

and hence

$$\|f_n\|_{L^2(R)} = \|f\| \;\forall 0 \le n \le N-1\,. \tag{2.6}$$

We now define the spaces

$$\mathcal{P} = \left\{ u \in H^1_{loc}(\mathbb{R}^2) \;\middle|\; \begin{array}{l} u(x_1 + L_1', x_2) = e^{i\omega N x_2} u(x_1, x_2) \text{ a.e.} \\ u(x_1, x_2 + L_2') = u(x_1, x_2) \text{ a.e.} \end{array} \right\}, \tag{2.7a}$$

$$\mathcal{U} = \left\{ u \in \mathcal{P} \;\middle|\; \int_R \left(i\nabla + x_1 \hat{i}_2\right) u \cdot \left(-i\nabla + x_1 \hat{i}_2\right) \bar{\phi} = \int_R u\bar{\phi} \quad \forall \phi \in C_c^\infty(R) \right\}, \tag{2.7b}$$

and state the following result:

Lemma 2.1 *Let $w \in \mathcal{P}$. Then,*

$$w = w_0 + \tilde{w} \tag{2.8}$$

where $w_0 \in \mathcal{U}_0$ and $\tilde{w} \in \mathcal{U}_0^\perp$ (the orthogonal complement with respect to the $L^2(R)$ inner product). Furthermore,

$$\int_R |\left(-i\nabla + x_1 \hat{i}_2\right) \tilde{w}|^2 \ge 3 \int_R |\tilde{w}|^2. \tag{2.9}$$

Proof: Since the restriction of $\mathcal{P}$ to $H^1(R)$ is a closed subspace of $H^1(R)$, and since $\mathcal{U}_0$ is a closed subspace of $\mathcal{P}$ (2.8) follows immediately.

To prove (2.9) we recall first that $\tilde{w}$ must satisfy (2.3c), and hence it can be represented by the Fourier series

$$\tilde{w} = \sum_{n=-\infty}^{\infty} \tilde{w}_n(x_1)e^{i\omega n x_2}\,, \tag{2.10}$$

where by (2.3b) we have

$$\tilde{w}_n(x_1 + N\omega) = \tilde{w}_{n-N}(x_1)\,. \tag{2.11}$$

Consequently,

$$\int_R |\tilde{w}|^2 = L_2' \sum_{n=-\infty}^{\infty} \int_0^{L_1'} |\tilde{w}_n|^2 dx_1 = L_2' \sum_{n=0}^{N-1} \int_{-\infty}^{\infty} |\tilde{w}_n|^2 dx_1, \tag{2.12}$$

from which we also obtain that $\tilde{w}_n \in L^2(\mathbb{R})$ for all $0 \leq n \leq N-1$.

Since $\tilde{w} \in \mathcal{U}_0^\perp$ we must have

$$\langle \tilde{w}, f_k \rangle \quad \forall 0 \leq k \leq N-1,$$

where f_k is given by (2.5b) . Substituting (2.5b) into the above and making use of (2.10) and (2.11) yields

$$\int_{-\infty}^{\infty} \tilde{w}_n \exp\left\{-\frac{1}{2}(x_1 - n\omega)^2\right\} dx_1 = 0 \tag{2.13}$$

We now make use of (2.10) to obtain

$$\int_R |\left(-i\nabla + x_1\hat{i}_2\right)\tilde{w}|^2 = L_2' \sum_{n=-\infty}^{\infty} \int_0^{L_1'} |\tilde{w}_n' - (x_1 - n\omega)\tilde{w}_n|^2 dx_1 =$$

$$= L_2' \sum_{n=0}^{N-1} \int_{-\infty}^{\infty} |\tilde{w}_n' - (x_1 - n\omega)\tilde{w}_n|^2 dx_1\,, \tag{2.14}$$

from which we obtain that $\tilde{w}_n \in H^1_{mag}(\mathbb{R})$, where

$$H^1_{mag}(\mathbb{R}) = \left\{ w \mid \int_{\mathbb{R}} |w|^2 + |w' - xw|^2 dx \leq \infty \right\}\,.$$

It is well-known [15] that

$$\inf_{\substack{u \in H^1_{mag}(\mathbb{R}) \\ u \perp e^{-x^2/2}}} \frac{\int_{-\infty}^{\infty} |u' - xu|^2 dx}{\int_{-\infty}^{\infty} |u|^2 dx} = 3.$$

Consequently, by (2.14) and (2.12)

$$\int_R |\left(-i\nabla + x_1\hat{i}_2\right)\tilde{w}|^2 \geq 3L_2' \sum_{n=0}^{N-1} \int_{-\infty}^{\infty} |\tilde{w}_n|^2 dx_1 = 3\int_R |\tilde{w}|^2$$

□

3 Apriory estimates

In this section we obtain some apriory estimates which should be satisfied by any solution of the Euler-Lagrange equations and the natural boundary conditions associated with (1.1). Thus, (ψ, A) must satisfy the equations

$$\left(\frac{i}{\kappa}\nabla + A\right)^2 \psi = \psi\left(1 - |\psi|^2\right) \quad , \tag{3.1a}$$

$$-\nabla \times \nabla \times A = \frac{i}{2\kappa}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right) + |\psi|^2 A \quad , \tag{3.1b}$$

together with the boundary conditions

$$\left(\frac{i}{\kappa}\nabla + A\right)\psi \cdot \hat{n} = 0 \quad ; \quad h = h_{ex} \; . \tag{3.1c,d}$$

Since (3.1) are invariant to the gauge transformation (1.2) we fix the Coulomb gauge for A, i.e.,

$$\nabla \cdot A = 0 \text{ in } \Omega \quad ; \quad (A - \hat{A}) \cdot \hat{n} = 0 \text{ on } \partial\Omega \tag{3.2}$$

where $\hat{A}$ is given by (2.2).

The first apriory estimates include the following well-known results:

Lemma 3.1 *Let $h_{ex} \geq \kappa - o(\kappa)$. Then, any solution of* (3.1) *must satisfy*

$$\|\rho\|_{L^\infty(\Omega)} < 1 \tag{3.3a}$$

$$\|h - h_{ex}\|_{C^1(\bar{\Omega})} \leq C \tag{3.3b}$$

$$\left\|\left(\frac{i}{\kappa}\nabla + A\right)\psi\right\|_{L^\infty(\bar{\Omega})} \leq C \, . \tag{3.3c}$$

Here and in the sequel, C is independent of κ and ϵ. *Proof:* See [17, 24].

Let now $\epsilon = (1 - h_{ex}/\kappa)^{1/2}$ be positive and satisfy

$$\kappa^{-1/5} \ll \epsilon \ll \frac{1}{\log \kappa} \, , \tag{3.4}$$

as $\kappa \to \infty$. In this case, according to the following result, every solution must be close to the normal state $\psi \equiv 0$ $h = h_{ex}$:

Lemma 3.2 *Let $|\psi| = \rho$. Any solution of* (3.1) *must satisfy*

$$\int_\Omega \rho^4 \leq C\epsilon^4 \tag{3.5a}$$

$$\int_\Omega |h - h_{ex}|^2 \leq C\epsilon^6 \tag{3.5b}$$

Furthermore, let $\hat{A} : \Omega \to \mathbb{R}^2$ *satisfy* (3.2) *and*

$$\nabla \times \hat{A} = h_{ex}.$$

Then,

$$\|A - \hat{A}\|_{L^p(\Omega)} \leq C(p)\epsilon^3 \tag{3.5c}$$

$$\|A - \hat{A}\|_{H^2(\Omega)} \leq C\epsilon^2 \tag{3.5d}$$

$$\tag{3.5e}$$

for any $p \geq 1$.

Proof: Let

$$f = h - \kappa + \frac{1}{2\kappa}\rho^2 . \tag{3.6}$$

In [5] it was shown that

$$\nabla^2 f - \rho^2 f = \kappa \left|\hat{J}\right|^2 + \left(\kappa - \frac{1}{2\kappa}\right)\rho^4, \tag{3.7}$$

The precise definition of $\hat{J}$ will not concern us. We shall be interested only in its following property

$$\left|\hat{J}\right|^2 \rho^2 = |\nabla f|^2 . \tag{3.8}$$

which is proved in [5].

Integrating (3.7) over Ω yields

$$\int_\Omega \left|\frac{\nabla f}{\rho}\right|^2 + \int_\Omega \rho^4 + \frac{1}{\kappa}\int_\Omega \rho^2(h - h_{ex}) - \epsilon^2 \int_\Omega \rho^2 \leq \frac{1}{\kappa}\int_{\partial\Omega} \frac{\partial f}{\partial n} .$$

Since by (3.3)

$$\frac{1}{\kappa}\left|\int_{\partial\Omega} \frac{\partial f}{\partial n}\right| + \frac{1}{\kappa}\left|\int_\Omega \rho^2(h - h_{ex})\right| \leq \frac{C}{\kappa} ,$$

we obtain

$$\int_\Omega \left|\frac{\nabla f}{\rho}\right|^2 + \int_\Omega \rho^4 \leq \frac{C}{\kappa} + \epsilon^2 \int_\Omega \rho^2 \leq \frac{C}{\kappa} + C\epsilon^2 \left[\int_\Omega \rho^4\right]^{1/2} . \tag{3.9}$$

Using (3.4), (3.5a) easily follows. Furthermore,

$$\int_\Omega |\nabla f| \leq \left[\int_\Omega \rho^2\right]^{1/2} \left[\int_\Omega \left|\frac{\nabla f}{\rho}\right|^2\right]^{1/2} \leq C\epsilon^3 .$$

Hence,

$$\int_\Omega |\nabla f| + \int_{\partial\Omega} |f - (h_{ex} - \kappa)| \leq C\epsilon^3.$$

Poincare inequality [19] then yields,

$$|f-(h_{ex}-\kappa)|_{W^{1,1}(\Omega)} \leq C\epsilon^3.$$

Consequently, by the Gagliardo-Nirenberg inequality we obtain that

$$\int_\Omega |f-(h_{ex}-\kappa)|^2 \leq C\epsilon^6,$$

from which (3.5b) readily follows. To prove (3.5c) we utilize the well-known inequality

$$\|A-\hat{A}\|_{H^1(\Omega)} \leq \int_\Omega |\nabla\times(A-\hat{A})|^2 = \int_\Omega |h-h_{ex}|^2 \tag{3.10}$$

and Sobolev embedding. Finally, in order to prove (3.5d) we observe that by (3.1b)

$$\int_\Omega |\nabla h|^2 \leq \int_\Omega \rho^2 \left|\left(\frac{i}{\kappa}\nabla + A\right)\psi\right|^2.$$

Multiplying (3.1a) by $\rho^2\bar{\psi}$ and integrating by parts we obtain

$$\int_\Omega \rho^2 \left|\left(\frac{i}{\kappa}\nabla + A\right)\psi\right|^2 \leq \int_\Omega \rho^4 .$$

Hence, by (3.5a)

$$\int_\Omega |\nabla h|^2 \leq C\epsilon^4 .$$

Combining the above with (3.10) we obtain

$$\|A-\hat{A}\|_{H^2(\Omega)} \leq C\epsilon^2 .$$

□

We next proceed to obtain some L^∞ estimates.

Lemma 3.3 *Let Ω_δ denote the domain*

$$\Omega_\delta = \{x\in\Omega \mid d(x,\partial\Omega)\geq\delta\} .$$

and let α denote, here and in the sequel, any real number smaller than 1. *Then,*

$$\|\psi\|_{L^\infty(\Omega_{1/\kappa\epsilon})} \leq C_\alpha \epsilon^{\alpha/2} \tag{3.11a}$$

$$\left\|\left(\frac{i}{\kappa} + A\right)\psi\right\|_{L^\infty(\Omega_{1/\kappa\epsilon})} \leq C_\alpha \epsilon^{\alpha/2} \tag{3.11b}$$

$$\|h-h_{ex}\|_{L^\infty(\Omega_{1/\kappa\epsilon})} \leq C_\alpha \epsilon^{\alpha} \tag{3.11c}$$

(3.11d)

Proof: Let χ_r denote a smooth cutoff function satisfying

$$\chi_r(x) = \begin{cases} 0 & |x| > r \\ 1 & |x| < \frac{1}{2} \end{cases} \quad |\nabla \chi_r| \leq \frac{C}{r} \quad |\nabla^2 \chi_r| \leq \frac{C}{r^2} \,. \tag{3.12}$$

Let $x_0 \in \Omega_{2/\kappa\epsilon}$. We multiply (3.7) by $\chi_{1/\kappa\epsilon}(x - x_0)$ and integrate by parts to obtain

$$-\int_{B_{\frac{1}{\epsilon\kappa}}} \nabla\chi \cdot \nabla f - \int_{B_{\frac{1}{\epsilon\kappa}}} \chi\rho^2(h-\kappa) = \kappa \int_{B_{\frac{1}{\epsilon\kappa}}} \chi \left|\frac{\nabla f}{\rho}\right|^2 + \kappa \int_{B_{\frac{1}{\epsilon\kappa}}} \chi\rho^4 \,, \tag{3.13}$$

where $B_r = B(x_0, r)$. For the first integral on the right hand side of (3.13) we have

$$\left|\int_{B_{\frac{1}{\epsilon\kappa}}} \nabla\chi \cdot \nabla f\right| \leq \left[\int_{B_{\frac{1}{\epsilon\kappa}}} |\nabla\chi|^2\rho^2\right]^{1/2} \left[\int_{B_{\frac{1}{\epsilon\kappa}}} \left|\frac{\nabla f}{\rho}\right|^2\right]^{1/2} \leq$$

$$\leq C\epsilon^{1/2}\kappa^{1/2} \left[\int_{B_{\frac{1}{\epsilon\kappa}}} \rho^4\right]^{1/4} \left[\int_{B_{\frac{1}{\epsilon\kappa}}} \left|\frac{\nabla f}{\rho}\right|^2\right]^{1/2} .$$

For the second integral we have, utilizing (3.3b)

$$\left|\int_{B_{\frac{1}{\epsilon\kappa}}} \chi\rho^2(h-\kappa)\right| \leq \int_{B_{\frac{1}{\epsilon\kappa}}} \rho^2|h-h_{ex}| + \int_{B_{\frac{1}{\epsilon\kappa}}} \rho^2|h_{ex}-\kappa| \leq C\epsilon \left[\int_{B_{\frac{1}{\epsilon\kappa}}} \rho^4\right]^{1/2} + C\frac{1}{\epsilon\kappa} \left[\int_{B_{\frac{1}{\epsilon\kappa}}} \rho^4\right]^{1/2}$$

Let

$$X_n = \left[\int_{B_{\frac{2n-1}{\epsilon\kappa}}} \rho^4\right]^{1/2} \quad ; \quad Y_n = \left[\int_{B_{\frac{2n-1}{\epsilon\kappa}}} \left|\frac{\nabla f}{\rho}\right|^2\right]^{1/2} .$$

Then, by (3.13) we have

$$X_0^2 + Y_0^2 \leq C\left[\frac{\epsilon^{1/2}}{\kappa^{1/2}} X_1^{1/2} Y_1 + \frac{\epsilon^2}{\kappa} X_1\right] . \tag{3.14}$$

Since,

$$\left|\frac{\nabla f}{\rho}\right|^2 \leq 2\left|\frac{\nabla h}{\rho}\right|^2 + \frac{2}{\kappa^2}|\nabla\rho|^2 = 2\left|\left(\frac{i}{\kappa}\nabla + A\right)\psi\right|^2 ,$$

we have by (3.3)

$$X_0^2 + Y_0^2 \leq \frac{C}{\epsilon\kappa^2}$$

Since x_0 was arbitrarily chosen, we can cover $B_{1/\epsilon\kappa}$ by a finite number of discs of radius $1/(2\epsilon\kappa)$. Consequently,

$$X_1^2 + Y_1^2 \leq \frac{C}{\epsilon\kappa^2} .$$

Substituting in (3.14) yields

$$X_0^2 + Y_0^2 \leq \frac{C}{\epsilon^{1/4}\kappa^2}\,.$$

The above procedure can recursively be applied to obtain

$$X_0^2 + Y_0^2 \leq \frac{C_n \epsilon^{2\alpha_n}}{\kappa^2}\,,$$

where α_n can be determined by the recurrence relation

$$\alpha_n = \frac{1}{4} + \frac{3}{4}\alpha_{n-1} \quad ; \quad \alpha_0 = -\frac{1}{2}\,.$$

Clearly, $\alpha_n \to 1$, and hence, we can conclude that

$$\int_{B_{\frac{1}{\epsilon\kappa}}} \rho^4 \leq C_\alpha \frac{\epsilon^{2\alpha}}{\kappa^2} \quad \forall \alpha < 1\,. \tag{3.15}$$

We now apply the transformation

$$x \to \kappa(x - x_0)$$

to (3.1a) in $B(x_0, 1/\kappa)$ to obtain

$$\left(\frac{i}{\kappa}\nabla + \tilde{A}\right)^2 \tilde{\psi} = \tilde{\psi}\left(1 - |\tilde{\psi}|^2\right) \text{ in } B(0,1)$$

where $\tilde{\psi}(x) = \psi(\kappa(x - x_0))$, $\tilde{A}(\kappa(x - x_0))$. Standard elliptic estimates [3], together with (1.2), then show that

$$\||\tilde{\psi}|\|_{H^2_{mag}[B(0,1)]} \leq C\|\tilde{\psi}\|_{L^2[B(0,1)]} = C\kappa\|\psi\|_{L^2[B(x_0,1/\kappa)]} \leq C\kappa^{1/2}\|\psi\|_{L^4[B(x_0,1/\kappa)]}.$$

Wherein

$$\|u\|^2_{H^2_{mag}(U)} = \int_U |u|^2 + |(i\nabla + \tilde{A})u|^2 + |(i\nabla + \tilde{A})^2 u|^2$$

Since

$$\|\psi\|^4_{L^4[B(x_0,1/\kappa)]} \leq \int_{B_{\frac{1}{\epsilon\kappa}}} \rho^4 \leq C_\alpha \frac{\epsilon^{2\alpha}}{\kappa^2} \quad \forall \alpha < 1, \tag{3.16}$$

(which is a rather crude estimate, but appears to be difficult to improve) (3.11a) follows immediately from Sobolev embedding. To prove (3.11b) we use bootstrapping and Sobolev embedding. Finally, to prove (3.11c) we notice that by (3.1b)

$$|\nabla h| \leq \rho\left|\left(\frac{i}{\kappa} + A)\right)\psi\right|.$$

Consequently, by (3.11a), (3.11b), (3.3b), and (3.1d) we have

$$|h - h_{ex}| \leq C_\alpha \epsilon^\alpha + \frac{C}{\kappa\epsilon}$$

$$\square$$

We note, once again, that (3.11) is not the optimal estimate. Ideally, one should obtain

$$\|\psi\|_{L^\infty(\Omega_{1/\kappa\epsilon})} + \left\| \left(\frac{i}{\kappa} + A \right) \psi \right\|_{L^\infty(\Omega_{1/\kappa\epsilon})} \leq C\epsilon,$$

however, in view of the crudeness of (3.16), (3.11) is the best we can obtain here.

Let $R(\epsilon, \kappa) \subset \Omega$ denote a rectangle whose sides length are given by

$$L_1 = \frac{\omega N}{\sqrt{h_{ex}\kappa}} \quad ; \quad L_2 = \frac{2\pi}{\omega\sqrt{h_{ex}\kappa}}, \tag{3.17}$$

where $\omega \in \mathbb{R}$, and $N \in \mathbb{N}$ are chosen such that

$$r_1 \leq \frac{L_1}{L_2} \leq r_2 \quad ; \quad \frac{1}{\kappa\epsilon^5} \ll (L_1 L_2)^{1/2} \ll 1,$$

where r_1 and r_2 are constant as $\kappa \to \infty$ and $\epsilon \to 0$ according to (3.4). Denote by x_1 and x_2 the coordinates in the respective directions of L_1 and L_2. Let $\mathcal{R}$ denote the image of R under the transformation

$$x \to \sqrt{h_{ex}\kappa}x. \tag{3.18}$$

Let $\mathcal{P}_\mathcal{R}$ denote the restriction of (2.7a) to $H^1(\mathcal{R})$ and $\mathcal{P}_R$ denote its image under the inverse of (3.18), i.e, the restriction to $H^1(R)$ of

$$\left\{ u \in H^1_{loc}(\mathbb{R}^2) \;\middle|\; \begin{array}{l} u(x_1 + L_1, x_2) = \exp\left\{ i\omega N\sqrt{h_{ex}\kappa}x_2 \right\} u(x_1, x_2) \text{ a.e.} \\ u(x_1, x_2 + L_2) = u(x_1, x_2) \text{ a.e.} \end{array} \right\}.$$

Let η denote a smooth cutoff function satisfying

$$\eta = \begin{cases} 1 & d(x, \partial R) \geq \frac{1}{\kappa\epsilon} \\ 0 & x \in \mathbb{R}^2 \setminus R \end{cases} \quad |\nabla\eta| \leq C\kappa\epsilon \quad |\nabla^2\eta| \leq C\kappa^2\epsilon^2 . \tag{3.19}$$

Let (ψ_κ, A_κ) denote the global minimizer (which depends on ϵ as well) of (1.1) in $\mathcal{H}$. (To keep the notation consistent with the one in the next section, we state the rest of the results in this section for the global minimizer, although they could have been stated for any solution of (3.1).) Clearly, $\eta\psi_\kappa \in \mathcal{P}_R$. Let $\mathcal{U}_\mathcal{R}$ denote the restriction of (2.7b) to $H^1(\mathcal{R})$. Let $\mathcal{U}_R$ denote its image under the inverse of (3.18), i.e,

$$\left\{ u \in \mathcal{P}_R \;\middle|\; \int_R \left(i\nabla + \hat{A} \right) u \cdot \left(-i\nabla + \hat{A} \right) \bar{\phi} = \frac{h_{ex}}{\kappa} \int_R u\bar{\phi} \quad \forall \phi \in C_c^\infty(R) \right\}.$$

From the results of § 2 it follows that $\mathcal{U}_R$ is a finite dimensional subspace of $\mathcal{P}_R$. Furthermore, we can now write

$$\eta\psi_\kappa = u_0 + \tilde{u} \tag{3.20}$$

where $u_0 \in \mathcal{U}_R$ and $\tilde{u} \in \mathcal{U}_R^\perp$. The next lemma estimates the $L^2(R)$ of $\tilde{u}$.

Lemma 3.4 *Let $\tilde{u}$ be defined by* (3.20). *Then,*

$$\int_R |\tilde{u}|^2 \leq C\epsilon^2 \int_R |\psi_\kappa|^2 + C_\alpha (L_1 L_2)^\alpha \epsilon^5 \,. \tag{3.21}$$

Proof: We first multiply (3.1a) by $\eta^2\bar{\psi}_\kappa$ and integrate over R to obtain

$$\int_R \left|\left(\frac{i}{\kappa}\nabla + A_\kappa\right)(\eta\psi_\kappa)\right|^2 = \frac{1}{\kappa^2}\int_R |\nabla\eta|^2|\psi_\kappa|^2 + \int_R \eta^2|\psi_\kappa|^2(1-|\psi_\kappa|^2)\,. \tag{3.22}$$

We now write

$$\int_R \left|\left(\frac{i}{\kappa}\nabla + A_\kappa\right)(\eta\psi_\kappa)\right|^2 = \int_R \left|\left(\frac{i}{\kappa}\nabla + \hat{A}\right)(\eta\psi_\kappa)\right|^2$$
$$+ \int_R \eta^2(A_\kappa - \hat{A})\cdot\left[\frac{i}{\kappa}\left(\bar{\psi}_\kappa\nabla\psi_\kappa - \psi_\kappa\nabla\bar{\psi}_\kappa\right) + |\psi_\kappa|^2 A_\kappa\right] - \int_R \eta^2|\psi_\kappa|^2|A_\kappa - \hat{A}|^2 \tag{3.23}$$

For the second integral on the right-hand-side of (3.23) we have

$$I_2 = \left|\int_R \eta^2(A_\kappa - \hat{A})\cdot\left[\frac{i}{\kappa}\left(\bar{\psi}_\kappa\nabla\psi_\kappa - \psi_\kappa\nabla\bar{\psi}_\kappa\right) + |\psi_\kappa|^2 A_\kappa\right]\right| \leq$$
$$\leq \left[\int_R \eta^2|\psi_\kappa|^2|A_\kappa - \hat{A}|^2\right]^{1/2}\left[\int_R\left|\left(\frac{i}{\kappa}\nabla + A_\kappa\right)(\eta\psi_\kappa)\right|^2\right]^{1/2}.$$

By (3.22) and Hölder inequality we thus have,

$$I_2 \leq C\|A_\kappa - \hat{A}\|_{L^4(R)}\|\psi_\kappa\|_{L^4(R)}\|\psi_\kappa\|_{L^2(R)} \leq C_p(L_1L_2)^{\frac{1}{2}-\frac{1}{4p}}\|A_\kappa - \hat{A}\|_{L^{4p}(R)}\|\psi_\kappa\|^2_{L^4(R)}\ \forall p > 1$$

Since by (3.5c)

$$\|A_\kappa - \hat{A}\|_{L^p(R)} \leq \|A_\kappa - \hat{A}\|_{L^p(\Omega)} \leq C\epsilon^3\,,$$

we obtain

$$I_2 \leq C_\alpha(L_1L_2)^\alpha\epsilon^5 \quad \forall\alpha < 1\,. \tag{3.24}$$

For the last integral on the right-hand-side of (3.23) we have

$$\left|\int_R \eta^2|\psi_\kappa|^2|A_\kappa - \hat{A}|^2\right| \leq \|A_\kappa - \hat{A}\|^2_{L^4(R)}\|\psi_\kappa\|^2_{L^4(R)} \leq C_\alpha(L_1L_2)^\alpha\epsilon^8\,. \tag{3.25}$$

Consequently, by (3.23), (3.24), and (3.25),

$$\int_R \left| \left(\frac{i}{\kappa} \nabla + A_\kappa \right) (\eta \psi_\kappa) \right|^2 = \int_R \left| \left(\frac{i}{\kappa} \nabla + \hat{A} \right) (\eta \psi_\kappa) \right|^2 + O((L_1 L_2)^\alpha \epsilon^5) \,. \tag{3.26}$$

Combining the above with (3.22) and (3.20) we obtain

$$\int_R \left| \left(\frac{i}{\kappa} \nabla + \hat{A} \right) \tilde{u} \right|^2 - \int_R |\tilde{u}|^2 + \int_R \eta^2 |\psi_\kappa|^4 \cong \frac{1}{\kappa^2} \int_R |\nabla \eta|^2 |\psi_\kappa|^2 + \epsilon^2 \int_R |u_0|^2 + O((L_1 L_2)^\alpha \epsilon^5) \,. \tag{3.27}$$

By (2.9), however,

$$\int_R \left| \left(\frac{i}{\kappa} \nabla + \hat{A} \right) \tilde{u} \right|^2 \geq 3 \frac{h_{ex}}{\kappa} \int_R |\tilde{u}|^2 \,.$$

Consequently,

$$\int_R |\tilde{u}|^2 + \int_R \eta^2 |\psi_\kappa|^4 \leq \frac{1}{\kappa^2} \int_R |\nabla \eta|^2 |\psi_\kappa|^2 + \epsilon^2 \int_R |u_0|^2 + C_\alpha (L_1 L_2)^\alpha \epsilon^5 \,. \tag{3.28}$$

In view of (3.19) we obtain

$$\frac{1}{\kappa^2} \int_R |\nabla \eta|^2 |\psi_\kappa|^2 \leq C \epsilon^2 \int_R |\psi_\kappa|^2 \,,$$

and since $\|u_0\|_{L^2(R)} \leq \|\psi_\kappa\|_{L^2(R)}$, we can combine the above with (3.27) to obtain (3.21).

□

We note that the error term in (3.21) is indeed small compared with the first term on the right-hand-side of (3.21), when α is sufficiently close to 1. We shall demonstrate this point later when we obtain a lower bound for $\|\psi_\kappa\|_{L^2(R)}$ at the end of § 4.

Lemma (3.4) basically shows that ψ_κ is indeed close, in $L^2(R)$ sense to u_0. However, in order to obtain a minimization problem for u_0 we need an estimate of (1.1). To this end, we need the following L^∞ estimate

Lemma 3.5 *Let $\tilde{u}$ be defined in* (3.20). *Then,*

$$\|\tilde{u}\|_{L^\infty(R)} \leq C_\alpha \epsilon^\alpha \tag{3.29a}$$

$$\left\| \left(\frac{i}{\kappa} \nabla + \hat{A} \right) \tilde{u} \right\|_{L^\infty(R)} \leq C_\alpha \epsilon^\alpha \tag{3.29b}$$

Proof: Since the proof is rather lengthy we divide it into several steps.

step 1: Let χ_r be given by (3.12), and let $x_0 \in R_{1/\kappa\epsilon}$ (see lemma 3.3 for the definition of R_δ). Clearly, $\chi_{1/2\kappa\epsilon}(x - x_0)\psi_\kappa \in \mathcal{P}_R$. Denote by $u_0^{(0)}$ and $\tilde{u}^{(0)}$ the respective projection of $\chi_{1/2\kappa\epsilon}(x - x_0)\psi_\kappa$ on $\mathcal{U}_R$ and $\mathcal{U}_R^\perp$.

step1.1: Prove that

$$\int_R |\tilde{u}^{(0)}|^2 \leq C_\alpha \frac{\epsilon^{2\alpha}}{\kappa^2}\,. \tag{3.30}$$

To prove (3.30) we repeat the same steps leading to (3.28) to obtain

$$\int_R |\tilde{u}^{(0)}|^2 + \int_R \chi^2|\psi_\kappa|^4 \leq \frac{1}{\kappa^2}\int_R |\nabla\chi|^2|\psi_\kappa|^2 + \epsilon^2 \int_R |u_0^{(0)}|^2 + \frac{C_\alpha}{\kappa^{2\alpha}\epsilon^{2\alpha}}\epsilon^5$$

(where χ stands for $\chi_{1/2\kappa\epsilon}(x-x_0)$). Consequently, in view of (3.12), (3.17), and (3.15) we have

$$\int_R |\tilde{u}^{(0)}|^2 \leq C\epsilon^2 \int_{B\left(x_0,\frac{1}{\kappa\epsilon}\right)} |\psi_\kappa|^2 + \frac{C_\alpha}{\kappa^{2\alpha}\epsilon^{2\alpha}}\epsilon^5 \leq C_\alpha \frac{\epsilon^{2\alpha}}{\kappa^2}\,.$$

step 1.2: Prove that

$$\|\tilde{u}^{(0)}\|_{L^\infty(R)} \leq C_\alpha \epsilon^\alpha \tag{3.31a}$$

$$\left\| \left(\frac{i}{\kappa} + \hat{A}\right)\tilde{u}^{(0)} \right\|_{L^\infty(R)} \leq C_\alpha\epsilon^\alpha\,. \tag{3.31b}$$

We use standard elliptic estimates to prove (3.31). Since $\tilde{u}^{(0)} = \chi\psi_\kappa - u_0^{(0)}$ we have in view of (3.1a)

$$\begin{aligned}\left(\frac{i}{\kappa}\nabla + \hat{A}\right)^2 \tilde{u}^{(0)} - \frac{h_{ex}}{\kappa}\tilde{u}^{(0)} = \epsilon^2\chi\psi_\kappa - \chi|\psi_\kappa|^2\psi_\kappa - \frac{1}{\kappa^2}\psi_\kappa\nabla^2\chi - |A_\kappa - \hat{A}|^2\chi\psi_\kappa + \\ + 2(A_\kappa - \hat{A})\cdot\left(\frac{i}{\kappa}\nabla + A_\kappa\right)(\chi\psi_\kappa) + 2\frac{i}{\kappa}\nabla\chi\cdot\left(\frac{i}{\kappa}\nabla + A_\kappa\right)\psi_\kappa\end{aligned} \tag{3.32}$$

Furthermore since, $\tilde{u^{(0)}} \in \mathcal{P}_R$, we can extend it periodically to $\mathbb{R}^2$, i.e.,

$$\exp\left\{-i\omega\frac{N}{\sqrt{h_{ex}\kappa}}x_2\right\}\tilde{u^{(0)}}(x_1+L_1,x_2) = \tilde{u^{(0)}}(x_1,x_2+L_2) = \tilde{u^{(0)}}(x_1,x_2)\,. \tag{3.33}$$

The periodic extension of $\tilde{u^{(0)}}$ satisfies (3.32) for every $x \in \mathbb{R}^2$ if the right hand side of it is extended in exactly the same manner.

Applying (3.18) to (3.32) we obtain

$$\begin{aligned}\left(i\nabla + x_1\hat{i}_2\right)^2 \tilde{u}^{(0)} - \tilde{u}^{(0)} = \frac{1}{1-\epsilon^2}[\epsilon^2\chi\tilde{\psi}_\kappa - \chi|\tilde{\psi}_\kappa|^2\tilde{\psi}_\kappa] - \tilde{\psi}_\kappa\nabla^2\chi - |(1-\epsilon^2)^{-1/2}\tilde{A}_\kappa - x_1\hat{i}_2|^2\chi\tilde{\psi}_\kappa + \\ + 2((1-\epsilon^2)^{-1/2}\tilde{A}_\kappa - x_1\hat{i}_2)\cdot\left(i\nabla + (1-\epsilon^2)^{-1/2}\tilde{A}_\kappa\right)(\chi\tilde{\psi}_\kappa) + 2i\nabla\chi\cdot\left(i\nabla + (1-\epsilon^2)^{-1/2}\tilde{A}_\kappa\right)\tilde{\psi}_\kappa\end{aligned} \tag{3.34}$$

where $(\tilde{\psi}_\kappa, \tilde{A}_\kappa)$ denote (ψ_κ, A_κ) in the stretched coordinates (3.18). To apply standard elliptic estimates we need an L^2 estimate of the right-hand-side of (3.34) in $B(x,1)$ for every $x \in \mathbb{R}^2$. By (3.11a) and (3.15) we have that

$$\int_{B(x,1)} \chi^2 |\psi_\kappa|^6 \leq C_\alpha \epsilon^{5\alpha} \quad \forall \alpha < 1\,,$$

and by (3.12) and (3.11a)

$$\int_{B(x,1)} |\psi_\kappa \nabla^2 \chi|^2 \leq C\epsilon^4 \int_{B(x,1)} |\psi_\kappa|^2 \leq C_\alpha \epsilon^{5\alpha}\,.$$

In view of (3.5d) and (3.11) we also have

$$\int_{B(x,1)} |(1-\epsilon^2)^{-1/2} A_\kappa - x_1 \hat{i}_2|^2 \cdot \left| \left(i\nabla + (1-\epsilon^2)^{-1/2} A_\kappa \right) \psi_\kappa \right|^2 \leq C\epsilon^{5\alpha}\,.$$

Furthermore,

$$\int_{B(x,1)} |\nabla \chi|^2 \left| \left(i\nabla + (1-\epsilon^2)^{-1/2} A_\kappa \right) \psi_\kappa \right|^2 \leq C\epsilon^{3\alpha}\,.$$

Finally,

$$\int_{B(x,1)} |(1-\epsilon^2)^{-1/2} A_\kappa - x_1 \hat{i}_2|^2 \chi^2 |\psi_\kappa|^2 \leq C_\alpha \epsilon^{5\alpha}\,.$$

Combining the above and (3.30), we may rely on the framework in [3] to obtain

$$\|\tilde{u}^{(0)}\|_{H^2(B(x,1))} \leq C_\alpha \epsilon^\alpha \quad \forall \alpha < 1\,.$$

Sobolev embedding then yields (3.31a). Bootstrapping and Sobolev embedding (3.31b).

Step 2: Prove (3.29).

We first note that (3.29) and (3.31) are different: while $\tilde{u}$ is the projection of $\eta\psi_\kappa$ on $\mathcal{U}_R^\perp$, $\tilde{u}^{(0)}$ is the projection of $\chi\psi_\kappa$ on the same space. To obtain (3.29) we thus need to relate χ and η. Let then $\{x_i\}_{i=1}^M$ denote a set of points in R satisfying

1. $R_{1/\kappa\epsilon} \subseteq \bigcup_{i=1}^M B\left(x_i, \frac{1}{\kappa\epsilon}\right)$
2. $B\left(x_i, \frac{1}{4\kappa\epsilon}\right) \bigcap B\left(x_j, \frac{1}{4\kappa\epsilon}\right) = \emptyset$ if $i \neq j$.

Let $\{\chi_i\}_{i=1}^M$ denote a set of C^∞ functions satisfying

1. $\operatorname{supp} \chi_i \subseteq B\left(x_i, \frac{1}{\kappa\epsilon}\right)$
2. $\sum_{i=1}^M \chi_i = 1$, $\forall x \in R_{1/\kappa\epsilon}$.
3. $|\nabla \chi_i| \leq \frac{C}{\kappa\epsilon}$ $|\nabla^2 \chi_i| \leq \frac{C}{\kappa^2\epsilon^2}$

Let

$$\eta = \sum_{i=1}^{M} \chi_i \, . \tag{3.35}$$

Clearly η satisfies (3.19) and hence, we may use it in (3.20) to define u_0 and $\tilde{u}$. Furthermore, let $\tilde{u}^{(i)}$ denote the projection of $\chi_i\psi_\kappa$ on $\mathcal{U}_R^\perp$. Then,

$$\tilde{u} = \sum_{i=1}^{M} \tilde{u}^{(i)} \, .$$

Furthermore,

$$|\tilde{u}| \leq \sum_{i=1}^{M} |\tilde{u}^{(i)}| \, , \tag{3.36a}$$

$$\left|\left(\frac{i}{\kappa} + \hat{A}\right)\tilde{u}\right| \leq \sum_{i=1}^{M} \left|\left(\frac{i}{\kappa} + \hat{A}\right)\tilde{u}^{(i)}\right| \, . \tag{3.36b}$$

Since M is a large number, we seek an estimate for $\tilde{u}^{(i)}$ when $|x - x_i| > 2/\kappa\epsilon$.

step 2.1: Prove that

$$|\tilde{u}^{(i)}(x)| \leq C_\alpha \epsilon^{\alpha/2} N \exp\left\{-\frac{1}{4}h_{ex}\kappa\left[d_p^2(x,x_i) - \frac{2}{\kappa^2\epsilon^2}\right]\right\} \tag{3.37a}$$

$$\left|\left(\frac{i}{\kappa} + \hat{A}\right)\tilde{u}^{(i)}(x)\right| \leq CN \exp\left\{-\frac{1}{4}\kappa^2\left[d_p^2(x,x_i) - \frac{2}{\kappa^2\epsilon^2}\right]\right\} \tag{3.37b}$$

where

$$d_p(x,x_i) = \min_{j,k=-1,0,1} |x - x_i - (kL_1, jL_2)| \tag{3.37c}$$

Since $u_0^{(i)} = -\tilde{u}^{(i)}$ for every $x \in R \setminus B(x_i, 1/\kappa\epsilon)$ we prove (3.37) for $u_0^{(i)}$. Recall from (2.5) that

$$u_0^{(i)} = \sum_{n=0}^{N-1} C_n f_n(x)$$

where f_n is given by

$$f_n = \sum_{r=-\infty}^{\infty} e^{i(n+rN)\omega\sqrt{h_{ex}\kappa}x_2} e^{-\frac{1}{2}[\sqrt{h_{ex}\kappa}x_1 - (n+rN)\omega]^2} \, . \tag{3.38}$$

Since $u_0^{(i)}$ is the projection of $\chi_i\psi_\kappa$ on $\mathcal{U}_R$ we have

$$C_n = \frac{h_{ex}\kappa\omega}{(2\pi)^{3/2}} \int_R \chi_i \psi \overline{f_n} \, .$$

Let

$$\hat{\psi}_i^r = \frac{\sqrt{h_{ex}\kappa}\omega}{2\pi} \int_0^{L_2} e^{-i\omega n\sqrt{h_{ex}\kappa}x_2} \chi_i \psi dx_2 \,.$$

Let $x_i = (x_{i1}, x_{i2})$. Then, since $\hat{\psi}_i^r(x_1)$ is supported in $(x_{i1} - 1/\kappa\epsilon, x_{i1} + 1/\kappa\epsilon)$ we have

$$C_n = \sqrt{\frac{h_{ex}\kappa}{2\pi}} \sum_{r=-\infty}^{\infty} \int_{x_{i1}-\frac{1}{\kappa\epsilon}}^{x_{i1}+\frac{1}{\kappa\epsilon}} \hat{\psi}_i^r e^{-\frac{1}{2}[\sqrt{h_{ex}\kappa}x_1 - (n+rN)\omega]^2} dx_1$$

By (3.11) and (3.12) we have that

$$|\hat{\psi}_i^r| \leq C_\alpha \epsilon^{\alpha/2} \frac{1}{\kappa\epsilon L_2} \quad \forall \alpha < 1 \,,$$

(recall that $L_2 \gg 1/\kappa\epsilon$). Thus

$$C_n \leq C_\alpha \epsilon^{\alpha/2} \sqrt{h_{ex}\kappa} \sum_{r=-\infty}^{\infty} \int_{-\frac{1}{\kappa\epsilon}}^{\frac{1}{\kappa\epsilon}} e^{-\frac{1}{2}[\sqrt{h_{ex}\kappa}(x_1 - x_{i1}) - (n+rN)\omega]^2} dx_1$$

Let

$$d_{p1}(x_1, y) = \min_{k=-1,0,1} |x_1 - y - kL_1|$$

Then,

$$C_n \leq C_\alpha \epsilon^{\alpha/2} \exp\left\{ -\frac{1}{2} h_{ex}\kappa \left(d_{p1}^2(x_{i1}, n\omega/\sqrt{h_{ex}\kappa}) - \frac{1}{\epsilon^2\kappa^2} \right) \right\} . \tag{3.39}$$

By (3.38) we have the estimate

$$|f_n(x)| \leq C \exp\left\{ -\frac{1}{2} h_{ex}\kappa d_{p1}^2 \left(x_1, n\omega/\sqrt{h_{ex}\kappa} \right) \right\} . \tag{3.40}$$

Consequently,

$$|u_0^{(i)}(x)| \leq C_\alpha \epsilon^{\alpha/2} \sum_{n=0}^{N-1} \left\{ -\frac{1}{2} h_{ex}\kappa \left[d_{p1}^2\big(x_1, n\omega/\sqrt{h_{ex}\kappa}\big) + d_{p1}^2\big(x_{i1}, n\omega/\sqrt{h_{ex}\kappa}\big) \ - \frac{1}{\epsilon^2\kappa^2} \right] \right\} .$$

Since

$$d_{p1}^2(x_1, x_{i1}) \leq 2\Big[d_{p1}^2\big(x_1, n\omega/\sqrt{h_{ex}\kappa}\big) + d_{p1}^2\big(x_{i1}, n\omega/\sqrt{h_{ex}\kappa}\big) \Big] \,,$$

we have that

$$|u_0^{(i)}(x)| \leq C_\alpha \epsilon^{\alpha/2} N \left\{ -\frac{1}{4} h_{ex}\kappa \left[d_{p1}^2\big(x_1, x_{i1}\big) - \frac{2}{\epsilon^2\kappa^2} \right] \right\} . \tag{3.41}$$

To prove that $u_0^{(i)}$ decays in the x_2 direction we note that

$$w_0^{(i)} = e^{-ih_{ex}\kappa x_1 x_2} u_0^{(i)}$$

satisfy the problem

$$\begin{cases} \left(\frac{i}{\kappa}\nabla - h_{ex}x_2\hat{i}_1\right)^2 w_0^{(i)} = \frac{h_{ex}}{\kappa}w_0^{(i)} \\ w_0^{(i)}(x_1+L_1,x_2) = w_0^{(i)}(x_1,x_2) \\ w_0^{(i)}(x_1,x_2+L_2) = w_0^{(i)}(x_1,x_2)e^{i\kappa h_{ex}L_2x_1} \end{cases} .$$

Consequently, $w_0^{(i)}$ must decay in the x_2 direction according to (3.41). We thus obtain

$$|u_0^{(i)}(x)| \leq C_\alpha \epsilon^{\alpha/2} N\left\{-\frac{1}{4}h_{ex}\kappa\left[d_{p1}^2(x_2,x_{i2}) - \frac{2}{\epsilon^2\kappa^2}\right]\right\} . \tag{3.42}$$

Combining (3.42) and (3.41) yields (3.37a), from which one can easily prove (3.37b) using standard elliptic estimates.

Step 2.2: Prove (3.29).

Substituting (3.31) and (3.37) into (3.36) we obtain

$$|\tilde{u}(x)| \leq C_\alpha\left[\epsilon^\alpha + Ne^{-\frac{1}{\epsilon^2}}\right] .$$

Since $L_1L_2 \leq |\Omega|$ we have

$$N \leq C\epsilon^2\kappa^2 .$$

Consequently, in view of (3.4) (3.29) is proved.

□

4 Upper and lower bounds

All the results of the previous section could have been formulated for any solution of the Euler-Lagrange equations. In this section we concentrate, however, on the energy functional

$$E_R(\psi,A) = \int_R\left(-|\psi|^2 + \frac{|\psi|^4}{2} + |h-h_{ex}|^2 + \left|\frac{i}{\kappa}\nabla\psi + A\psi\right|^2\right) . \tag{4.1}$$

We obtain estimates for it in terms of the reduced functional $\mathcal{J}_R$ which appears in theorem 1.1. We start by proving the following upper bound

Lemma 4.1 *Let η be defined by* (3.35), *and let u_0 and $\tilde{u}$ be defined by* (3.20). *Then, for all $v \in \mathcal{U}_R$*

$$E_R(\eta\psi_\kappa, A_\kappa) \leq \mathcal{J}_R(v) + C_\alpha\epsilon^{\alpha-1}\frac{(L_1L_2)^{1/2}}{\kappa} \tag{4.2a}$$

where

$$\mathcal{J}_R(v) = -\epsilon^2 \int_R |v|^2 + \frac{1}{2}\int_R |v|^4 \tag{4.2b}$$

Furthermore,

$$\int_R |\tilde{u}|^4 \leq C_\alpha \epsilon^{2\alpha} \int_R |u_0|^4 + C_\alpha \left[\epsilon^{2\alpha} \frac{(L_1 L_2)^{1/2}}{\kappa} + (L_1 L_2)^\alpha \epsilon^{7\alpha} \right] . \tag{4.3}$$

Proof: Following [24], let $\tilde{\eta}$ denote a smooth cutoff function satisfying

$$\tilde{\eta} = \begin{cases} 1 & x \in R \\ 0 & x \in \mathcal{A}_2 \\ 1 & x \in \Omega \setminus R \ d(x, \partial R) \geq \frac{3}{\kappa\epsilon} \end{cases} \qquad |\nabla \tilde{\eta}| \leq C\kappa\epsilon \quad |\nabla^2 \tilde{\eta}| \leq C\kappa^2\epsilon^2 . \tag{4.4}$$

Where

$$\mathcal{A}_k = \left\{ x \in \Omega \setminus R \,|\, \frac{k-1}{\kappa\epsilon} \leq d(x, \partial R) \leq \frac{k}{\kappa\epsilon} \right\},$$

Let further

$$\tilde{\psi} = \begin{cases} \tilde{\eta} v^p & d(x, \partial R) \leq \frac{3}{2\kappa\epsilon} \text{ or } x \in R \\ \tilde{\eta}\psi_\kappa & d(x, \partial R) \geq \frac{3}{2\kappa\epsilon} \text{ and } x \in \Omega \setminus R \end{cases} \tag{4.5a}$$

where v^p denotes a periodic extension, according to (3.33), of some $v \in \mathcal{U}_R$, and $\tilde{A}$ satisfies

$$\tilde{A} = \begin{cases} \hat{A} & x \in R \cup \mathcal{A}_1 \\ A_\kappa & d(x, \partial R) \geq \frac{2}{\kappa\epsilon} \text{ and } x \in \Omega \setminus R , \end{cases} \tag{4.5b}$$

and,

$$\|\nabla \times \tilde{A} - h_{ex}\|_{L^\infty(\Omega)} \leq C\epsilon^2 . \tag{4.5c}$$

(cf. [24])

In view of the above, since (ψ_κ, A_κ) is the minimizer of E in $\mathcal{H}$ we have

$$E(\tilde{\psi}, \tilde{A}) \geq E(\psi_\kappa, A_\kappa) .$$

Consequently,

$$\begin{aligned} 0 \leq E(\tilde{\psi}, \tilde{A}) - E(\psi_\kappa, A_\kappa) = E_R(v, \hat{A}) - E_R(\psi_\kappa, A_\kappa) + \\ + \int_{\mathcal{A}} \left| \left(\frac{i}{\kappa}\nabla + \tilde{A} \right) \tilde{\psi} \right|^2 - \left| \left(\frac{i}{\kappa}\nabla + A_\kappa \right) \psi_\kappa \right|^2 + \int_{\mathcal{A}} |\nabla \times \tilde{A} - h_{ex}|^2 - |\nabla \times A_\kappa - h_{ex}|^2 \\ + \frac{1}{2}\int_{\mathcal{A}} \left[|\tilde{\psi}|^4 - |\psi_\kappa|^4 \right] - \int_{\mathcal{A}} \left[|\tilde{\psi}|^2 - |\psi_\kappa|^2 \right] \end{aligned} \tag{4.6}$$

where

$$\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3$$

By (4.5) and (3.11) we have

$$\int_{\mathcal{A}} |\nabla \times \tilde{A} - h_{ex}|^2 \leq C_\alpha \epsilon^{4\alpha} \frac{(L_1 L_2)^{1/2}}{\kappa\epsilon} \quad \forall \alpha < 1 \tag{4.7}$$

Moreover, by (4.4) and (3.11) we have

$$\int_{\mathcal{A}_3} \left|\left(\frac{i}{\kappa}\nabla + \tilde{A}\right)\tilde{\psi}\right|^2 - \left|\left(\frac{i}{\kappa}\nabla + A_\kappa\right)\psi_\kappa\right|^2 \leq C_\alpha \epsilon^{\alpha-1} \frac{(L_1 L_2)^{1/2}}{\kappa} . \tag{4.8}$$

To estimate the effect of

$$\int_{\mathcal{A}_1} \left|\left(\frac{i}{\kappa}\nabla + \tilde{A}\right)\tilde{\psi}\right|^2 - \left|\left(\frac{i}{\kappa}\nabla + A_\kappa\right)\psi_\kappa\right|^2 \leq C_\alpha \epsilon^{\alpha-1} \frac{(L_1 L_2)^{1/2}}{\kappa}$$

we choose first $v \in \mathcal{U}_R$ such that

$$\int_R |v|^4 \leq \epsilon^4 L_1 L_2 , \tag{4.9}$$

which yields

$$\int_R |v|^2 \leq \epsilon^2 L_1 L_2$$

In this case

$$\int_{\mathcal{A}_1} \left|\left(\frac{i}{\kappa}\nabla + \hat{A}\right)\tilde{\eta} v^p\right|^2 = \int_{R\cup\mathcal{A}_1} \left|\left(\frac{i}{\kappa}\nabla + \hat{A}\right)\tilde{\eta} v^p\right|^2 \\ - \int_R \left|\left(\frac{i}{\kappa}\nabla + \hat{A}\right) v^p\right|^2 = \int_{\mathcal{A}_1} \tilde{\eta}^2 |v^p|^2 \leq \int_{\mathcal{A}_1} |v^p|^2 .$$

Let

$$v_l^p = v^p(x_1 + l_1, x_2 + l_2) ,$$

where $0 \leq l_1 \leq L_1$ and $0 \leq l_2 \leq L_2$. Clearly,

$$\int_R \left(\int_{\mathcal{A}} |v_l^p|^2\right) dl_1 dl_2 = |\mathcal{A}| \int_R |v|^2 .$$

Consequently,

$$\inf_{(l_1,l_2)\in R} \int_{\mathcal{A}} |v_l^p|^2 \leq \frac{|\mathcal{A}|}{|R|} \int_R |v|^2 \leq C\epsilon \frac{L_1 L_2}{\kappa} .$$

Denote by (l_{1m}, l_{2m}) the values of l_1 and l_2 which minimize the $L^2(\mathcal{A})$ norm of v_l^p and let

$$v_m = v^p(x_1 + l_{1m}, x_2 + l_{2m}) .$$

If we choose $v^p = v_m$ in (4.5a) we obtain

$$\int_{\mathcal{A}_1} \left| \left(\frac{i}{\kappa} \nabla + \hat{A} \right) \tilde{\eta} v_m \right|^2 \leq C\epsilon \frac{L_1 L_2}{\kappa} \,.$$

Combining the above with (4.8) we obtain

$$\int_{\mathcal{A}} \left| \left(\frac{i}{\kappa} \nabla + \tilde{A} \right) \tilde{\psi} \right|^2 \leq C_\alpha \epsilon^{\alpha-1} \frac{(L_1 L_2)^{1/2}}{\kappa} \,. \tag{4.10}$$

Combining (4.6), (4.7), (4.9), and (4.10) we obtain

$$E_R(\psi_\kappa, A_\kappa) \leq E_R(v_m, \hat{A}) + C_\alpha \epsilon^{\alpha-1} \frac{(L_1 L_2)^{1/2}}{\kappa} \quad \forall \alpha < 1 \,.$$

Since

$$E_R(v_m, \hat{A}) = \mathcal{J}_R(v_m) = \mathcal{J}_R(v) \,,$$

(4.2) is proved as long as (4.9) holds. To prove (4.2) for any $v \in \mathcal{U}_R$, we write

$$\mathcal{J}_R(\gamma v) = -\epsilon^2 \gamma^2 \int_R |v|^2 + \frac{1}{2} \gamma^4 \int_R |v|^4 \,.$$

It is easy to show that for a given $v \in \mathcal{U}_R$, $\mathcal{J}_R(\gamma v)$ is minimal for

$$\gamma_0^2 = \epsilon^2 \frac{\int_R |v|^2}{\int_R |v|^4} \,.$$

Let $w = \gamma_0 v$. Then,

$$\int_R |w|^4 = \epsilon^2 \int_R |w|^2 \,, \tag{4.11}$$

and hence

$$\int_R |w|^2 \leq \epsilon^2 L_1 L_2 \,.$$

Consequently,

$$E_R(\psi_\kappa, A_\kappa) \leq \mathcal{J}_R(w) + C_\alpha \epsilon^{\alpha-1} \frac{(L_1 L_2)^{1/2}}{\kappa} \leq \mathcal{J}_R(v) + C_\alpha \epsilon^{\alpha-1} \frac{(L_1 L_2)^{1/2}}{\kappa} \quad \forall \alpha < 1 \,.$$

which proves (4.2).

Let η be given by (3.35). It is easy to show that

$$E_R(\eta \psi_\kappa, A_\kappa) \leq E_R(\psi_\kappa, A_\kappa) + C_\alpha \epsilon^{\alpha-1} \frac{(L_1 L_2)^{1/2}}{\kappa} \,. \tag{4.12}$$

By (3.26), (3.20), and (2.9) we obtain

$$E_R(\eta \psi_\kappa, A_\kappa) \geq -\epsilon^2 \int_R |u_0|^2 + \frac{1}{2} \int_R \eta^4 |\psi_\kappa|^4 + \int_R |\tilde{u}|^2 - C_\alpha (L_1 L_2)^\alpha \epsilon^5 \,. \tag{4.13}$$

Combining the above and (4.2) with $v = u_0$ we obtain that

$$\int_R |\tilde{u}|^2 \leq \frac{1}{2}\int_R |u_0|^4 + C_\alpha \left[\epsilon^{\alpha-1}\frac{(L_1L_2)^{1/2}}{\kappa} + (L_1L_2)^\alpha\epsilon^5\right].$$

By (3.29) we then have

$$\int_R |\tilde{u}|^4 \leq C_\alpha\epsilon^{2\alpha}\int_R |u_0|^4 + C_\alpha \left[\epsilon^{2\alpha}\frac{(L_1L_2)^{1/2}}{\kappa} + (L_1L_2)^\alpha\epsilon^{7\alpha}\right].$$

□

Using (4.3) we can obtain a lower bound for $E_R(\psi_\kappa, A_\kappa)$ in terms of the reduced functional $\mathcal{J}_R$.

Lemma 4.2 *Let η be given by (3.35) and u_0 be given by (3.20). Then,*

$$E_R(\psi_\kappa, A_\kappa) \geq \mathcal{J}_R(u_0) - C_\alpha \left[\epsilon^{\alpha-1}\frac{(L_1L_2)^{1/2}}{\kappa} + (L_1L_2)^\alpha\epsilon^5\right] \tag{4.14}$$

for all $\alpha < 1$.

Proof: By (4.12) and (4.13) we have that

$$E_R(\psi_\kappa, A_\kappa) \geq -\epsilon^2\int_R |u_0|^2 + \frac{1}{2}\int_R \eta^4|\psi_\kappa|^4 - C_\alpha(L_1L_2)^\alpha\epsilon^5 - C_\alpha\epsilon^{\alpha-1}\frac{(L_1L_2)^{1/2}}{\kappa} \tag{4.15}$$

for all $\alpha < 1$.

Since,

$$\eta^4|\psi_\kappa|^4 \geq |u_0|^4 - 4|u_0|^3|\tilde{u}|,$$

we have by (4.3) and Hölder inequality that

$$\int_R \eta^4|\psi_\kappa|^4 \geq \int_R |u_0|^4[1-C_\alpha\epsilon^{\alpha/2}] - C_\alpha \left[(L_1L_2)^\alpha\epsilon^5 + \epsilon^{\alpha-1}\frac{(L_1L_2)^{1/2}}{\kappa}\right]^{1/4}\left[\int_R |u_0|^4\right]^{3/4}. \tag{4.16}$$

By (4.3) we also have that

$$\begin{aligned}\|u_0\|_{L^4(R)} \leq \|\eta\psi_\kappa\|_{L^4(R)} + \|\tilde{u}\|_{L^4(R)} \leq \|\eta\psi_\kappa\|_{L^4(R)} + \\ + C_\alpha \left[\epsilon^{\alpha/2} + (L_1L_2)^\alpha\epsilon^5 + \epsilon^{\alpha-1}\frac{(L_1L_2)^{1/2}}{\kappa}\right]^{1/4}\|u_0\|_{L^4(R)}.\end{aligned} \tag{4.17}$$

To estimate $\|u_0\|_{L^4(R)}$, we thus need an estimate for $\|\eta\psi_\kappa\|_{L^4(R)}$. By (3.28), (3.35), and (2.9) we have that

$$\int_R \eta^4|\psi_\kappa|^4 \leq \epsilon^2\int_R \eta^2|\psi_\kappa|^2 + C_\alpha \left[(L_1L_2)^\alpha\epsilon^5 + \epsilon^{\alpha-1}\frac{(L_1L_2)^{1/2}}{\kappa}\right]$$

(recall that $\|u_0\|_{L^2(R)} \leq \|\eta\psi_\kappa\|_{L^2(R)}$). Consequently,

$$\int_R \eta^4|\psi_\kappa|^4 \leq \epsilon^4 L_1L_2 + C_\alpha\left[(L_1L_2)^\alpha\epsilon^5 + \epsilon^{\alpha-1}\frac{(L_1L_2)^{1/2}}{\kappa}\right]$$

We note that by (3.17) the first term on the right-hand-side of the above inequality is much greater than the second one if α is sufficiently close to 1, such that

$$(L_1L_2)^{1-\alpha} \gg \epsilon\,.$$

Consequently

$$\int_R |\eta\psi_\kappa|^4 \leq 2\epsilon^4 L_1L_2\,,$$

and hence, by (4.17) we obtain

$$\int_R |u_0|^4 \leq 2\epsilon^4 L_1L_2\,.$$

Substituting in (4.16) and then in (4.15) we obtain (4.14).

□

Proof of theorem 1.1: Combining (4.2) and (4.14) we obtain

$$\mathcal{J}_R(u_0) \leq J_R(v) + C_\alpha\left[\epsilon^{\alpha-1}\frac{(L_1L_2)^{1/2}}{\kappa} + (L_1L_2)^\alpha\epsilon^5\right] \tag{4.18}$$

To prove (1.4a) we thus need an estimate for $\inf_{v\in\mathcal{U}_R}\mathcal{J}_R(v)$. Let

$$\beta = \inf_{v\in\mathcal{U}_R}\mathcal{B}(v) = L_1L_2\inf_{v\in\mathcal{U}_R}\frac{\int_R|v|^4}{\left(\int_R|v|^2\right)^2}\,. \tag{4.19}$$

Since $\mathcal{U}_R$ is finite-dimensional, it is easy to show that there exists $w\in\mathcal{U}_R$ satisfying

$$\int_R |w|^4 = \beta L_1L_2\left(\int_R |w|^2\right)^2\,. \tag{4.20}$$

Furthermore, since β is invariant to the transformation $w\to\gamma w$ for every $\gamma\in\mathbb{R}$ we can choose w such that (4.11) is satisfied. Combining (4.11) and (4.20) we obtain

$$\int_R |w|^2 = \epsilon^2\frac{L_1L_2}{\beta}$$

Thus, in view of (4.11)

$$\mathcal{J}_R(w) = -\frac{1}{2}\epsilon^2\int_R |w|^2 = -\epsilon^4\frac{L_1L_2}{2\beta}\,.$$

In [2, 10, 4] $\mathcal{B}(v)$ was calculated in various cases. In particular, it was found that when v is the well-known square lattice, i.e, when

$$v = C \sum_{n=0}^{N-1} f_n \,,$$

where f_n is given by (2.5b), that

$$\mathcal{B}(v) \approx 1.18$$

independently of N and the scale of R. It follows therefore, in view of (4.19) that

$$\inf_{v\in\mathcal{U}_R} \mathcal{J}_R(v) \leq -C\epsilon^4 L_1 L_2 \,, \tag{4.21}$$

where $C > 0$. Substituting (4.21) into (4.18) we obtain (1.4b).

To prove (1.4a) we write

$$\int_R |\psi_\kappa - u_0|^2 \leq 2\left[\int_R |\eta - 1|^2 |\psi_\kappa|^2 + \int_R |\tilde{u}|^2\right] .$$

The right-hand-side of the above inequality can be bounded utilizing (3.21) and (3.19) to obtain

$$\int_R |\psi_\kappa - u_0|^2 \leq C\epsilon^2 \int_R |u_0|^2 + C_\alpha (L_1L_2)^\alpha \epsilon^5 + C_\alpha \epsilon^{\alpha-1} \frac{(L_1L_2)^{1/2}}{\kappa} . \tag{4.22}$$

Since $\|u_0\|_{L^2(R)} \leq \|\psi_\kappa\|_{L^2(R)}$ we need a lower bound for $\|u_0\|_{L^2(R)}$ to complete the proof of (1.4a). To this end we use (1.4b) and (4.21) to obtain

$$\frac{\beta}{2L_1L_2} \|u_0\|^4_{L^2(R)} - \epsilon^2 \|u_0\|^2_{L^2(R)} \leq \mathcal{J}_R(u_0) \leq -C\epsilon^4 L_1 L_2 \,,$$

from which we obtain

$$\|u_0\|^2_{L^2(R)} \geq C\epsilon^2 L_1 L_2 \,.$$

Substituting in (4.22) proves (1.4a).

$\square$

5 Conclusion

Let $R(\kappa, \epsilon) = [0, L_1] \times [0, L_2]$, where L_1 and L_2 are given by (3.17). In the previous sections the following main results were proved:

1. We proved that the $L^2(R)$ distance of ψ_κ from the space of Abrikosov solutions in R, $\mathcal{U}_R$ is much smaller than the $L^2(R)$ norm of ψ_κ.

2. We proved that the energy, which is given by (1.4d), of the projection of $\eta\psi_\kappa$ on $\mathcal{U}_R$, where η is given by (3.35), is approximately the minimum over all $\mathcal{U}_R$ of (1.4d).

We note that the above results do not show that ψ_κ is nearly periodic, inasmuch as every function in $L^2(R)$ can be approximated by a periodic function. Nevertheless, since the energy of the above projection is close to the minimum of $\mathcal{J}_R$ over $\mathcal{U}_R$, we can obtain an approximation of ψ_κ by studying a much simpler minimization problem than the minimization of (1.1) in $\mathcal{H}$. It is widely believed that the minimizer of $\mathcal{J}_R$ in $\mathcal{U}_R$ (which is a finite dimensional space), is the well-known triangular lattice [2, 10, 4], as long as N, in (3.17), is even. If in addition to that, any $u \in \mathcal{U}_R$ whose energy is close to the minimum must be close, in some sense to the triangular lattice, then ψ_κ is indeed nearly periodic.

It is seems worthwhile to note here that the direction of the lattice cannot be determined by the energy considerations applied in the previous sections. Thus if

$$u_0'(\boldsymbol{x}) = u_0(Q\boldsymbol{x}) \quad ; \quad \hat{A}' = h_{ex} Q \begin{bmatrix} 0 \\ x_1 \end{bmatrix},$$

where Q is a 2×2 rotation matrix, then, since the cells affected by the rotation are only those near the boundary, we have that

$$\left| E_R(u_0', \hat{A}') - E_R(u_0, \hat{A}) \right| \leq C \frac{(L_1 L_2)^{1/2}}{\kappa} .$$

Clearly, the above error is indistinguishable by the lower and upper bounds, (4.14) and (4.2), obtained in § 4.

Finally, we note that the limitations (3.4) could have been replaced by the weaker assumptions

$$\frac{1}{\kappa} \ll \epsilon^4 \quad ; \quad (L_1 L_2)^{1/2} \gg \frac{1}{\kappa \epsilon^4} ,$$

if only we could overcome the crudeness of the estimate (3.16). However, to extend the analysis to the case $\epsilon^4 \sim O(1/\kappa)$, a completely different approach is necessary, since in that case the surface energy, which is of $O(1/\kappa)$ is at least equally important to E_R which is of $O(\epsilon^4 L_1 L_2)$.

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