The creeping motion of a small rigid particle near a smooth boundary

Y. ALMOG *

Abstract

The creeping motion of a Newtonian fluid around a particle in a smooth domain is studied. It is proved that the force distribution on the surface of the particle can be approximated, in the limit where the ratio between particle size and the domain's radius of curvature tends to zero, by the force distribution on the same particle near a flat wall. This result is then utilized to show that the velocities of the particle in mobility problems, or the forces acting on it in resistance problems can be approximated by replacing the domain with a flat wall.

1 introduction

We consider the problem of a rigid particle dispersed in a Newtonian fluid, in the absence of inertia forces, in a bounded domain with smooth boundaries. It is widely accepted that

*Faculty of Mathematics, Technion - Israel Institute of Technology, Haifa 32000, Israel

when the both the ratios between either the particle size or its distance from the boundary and the the local radius of curvature on the boundaries tend to zero, the local flow field can be approximated by the creeping motion around the same particle in the presence of a flat wall. Such an assumption was made, for instance, by Brenner & Falade [3], who calculated the force and the torque on a spherical particle, whose velocity and rate of rotation are prescribed, near a smooth boundary. The same assumption was made in many other works as well (cf. [6, 10, 11, 2] to cite a few).

While the underlying assumption in [3] was frequently used in many other formal asymptotic expansions, it was never proved rigorously. In the present contribution, thus, it is proved that the local flow field – represented here by the force distribution on the surface of the particle – can be estimated by the flow field around the same particle near a flat wall. Such a result can be useful in problems involving homogenization techniques, since it is necessary in such problems to bound the error generated by the particles near the wall. The above result can, therefore, provide an upper-bound independent of the boundary's specific geometry.

We prove the above result in cases where the ratio between particle size and its distance from the wall is bounded. Further research is necessary in order to show the validity of our result in cases where the particle's distance from the wall is much smaller than its size (the case $\Delta \to 0$ using the notation in [3]).

Consider a rigid particle suspended in a homogeneous Newtonian fluid of viscosity $\mu = 1$ (if $\mu \neq 1$ we can transform the coordinates into a system were $\mu = 1$). Denote by Ω a convex domain in $\mathbb{R}^3$ with smooth boundaries (in $C^{2,\alpha}$ for some $0 < \alpha < 1$), and by $(\boldsymbol{u}, p)$, respectively, the velocity and pressure field, satisfying in the absence of inertial effects

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0 \quad \text{in } \Omega \setminus B \tag{1.1a}$$

$$\nabla^2 \boldsymbol{u} = \boldsymbol{\nabla} p \quad \text{in } \Omega \setminus B \tag{1.1b}$$

$$\boldsymbol{u} = \boldsymbol{u}^* \quad \text{on } \partial\Omega \tag{1.1c}$$

$$\boldsymbol{u} = \boldsymbol{U} + \boldsymbol{\omega} \times (\boldsymbol{x} - \boldsymbol{x}_0) \quad \text{on } \partial B \tag{1.1d}$$

$$\int_{\partial B} \boldsymbol{f} ds = \boldsymbol{F} \quad \int_{\partial B} (\boldsymbol{x} - \boldsymbol{x}_0) \times \boldsymbol{f} ds = \boldsymbol{L} \tag{(1.1e,f)}$$

wherein $\boldsymbol{f}$ is the surface traction,

$$\boldsymbol{f} = \boldsymbol{\sigma}(\boldsymbol{u}) \cdot \hat{\boldsymbol{n}}, \tag{1.2}$$

where $\boldsymbol{\sigma}(\boldsymbol{u})$ is the stress tensor deriving from $(\boldsymbol{u}, p)$, and $\hat{\boldsymbol{n}}$ is the outward unit normal on $\partial\Omega \bigcup \partial B$. The vectors $\boldsymbol{U}$ and $\boldsymbol{\omega}$ respectively denote the translational and angular velocities of B, $\boldsymbol{F}$ and $\boldsymbol{L}$ respectively denote the force and torque exerted on B by the fluid, and $\boldsymbol{x}_0$ is a fixed locator point in B. Problems where $\boldsymbol{U}$ and $\boldsymbol{\omega}$ are given are known as resistance problems, and those for which $\boldsymbol{F}$ and $\boldsymbol{L}$ are prescribed have been termed mobility problems.

Denote by $(\bar{\boldsymbol{u}}, \bar{p})$ the ambient flow satisfying Stokes problem in Ω in the absence of the particle B, i.e., (1.1a,b) are satisfied in the interior of Ω and (1.1c) is satisfied on $\partial\Omega$. It is known (cf. [9]) that the velocity field possesses the integral representation

$$u_i(\boldsymbol{y}) = \bar{u}_i(\boldsymbol{y}) + \int_{\partial B} T_{ij}(\boldsymbol{x}, \boldsymbol{y}) f_j(\boldsymbol{x}) ds_x, \tag{1.3}$$

and the pressure field is given by

$$p(\boldsymbol{y}) = \bar{p}(\boldsymbol{y}) + \int_{\partial B} P_j(\boldsymbol{x}, \boldsymbol{y}) f_j(\boldsymbol{x}) ds_x. \tag{1.4}$$

In the above, $(\boldsymbol{T}(\boldsymbol{x},\boldsymbol{y}),\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y}))$ is Green's functions of Stokes' problem in Ω, defined by

$$T_{ij} = t_{ij} + \tau_{ij} \quad , \quad P_i = p_i + \pi_i. \tag{1.5}$$

In the latter, $(\boldsymbol{t},\boldsymbol{p})$ denotes the Stokeslet

$$\left.\begin{array}{rcl} t_{ij}(\boldsymbol{x},\boldsymbol{y}) & = & \dfrac{1}{8\pi}\left(\dfrac{\delta_{ij}}{r} + \dfrac{r_i r_j}{r^3}\right) \\ \\ p_i(\boldsymbol{x},\boldsymbol{y}) & = & \dfrac{1}{4\pi}\dfrac{r_i}{r^3} \end{array}\right\} \boldsymbol{r} = \boldsymbol{x} - \boldsymbol{y}\ , \tag{1.6}$$

with $(\boldsymbol{\tau},\boldsymbol{\pi})$ the Stokeslet image, the latter being a regular solution of the Stokes problem satisfying (1.1a,b) in Ω and the boundary condition $\tau_{ij}|_{\partial\Omega_u} = -\, t_{ij}|_{\partial\Omega_u}$, $\sigma_{ij}(\boldsymbol{\tau}_{.k}) \cdot \hat{n}_j|_{\partial\Omega_f} = -\, \sigma_{ij}(\boldsymbol{t}_{.k}) \cdot \hat{n}_j|_{\partial\Omega_f}$, where $\boldsymbol{t}_{.k}$ denotes the vector (t_{1k}, t_{2k}, t_{3k}).

It is easy to show using (1.3) and (1.4) that the stress field may be expressed in the form

$$\sigma_{ij}(\boldsymbol{y}) = \bar{\sigma}_{ij}(\boldsymbol{y}) + \int_{\partial B} (\sigma_y)_{ij}(\boldsymbol{T}_{.k}(\boldsymbol{x},\boldsymbol{y})) f_k(\boldsymbol{x}) ds_x. \tag{1.7}$$

Upon letting $\boldsymbol{y}$ approach the surface of one of the particles, dot-multiplying by the inward normal, and using the 'jump condition' [8], it may be shown [9] that the surface traction $\boldsymbol{f}$ satisfies the boundary integral equation

$$\frac{1}{2} f_i(\boldsymbol{y}) = \bar{f}_i(\boldsymbol{y}) + \int_{\partial B} (\sigma_y)_{ij}(\boldsymbol{T}_{.k}(\boldsymbol{x},\boldsymbol{y})) f_k(\boldsymbol{x}) ds_x \hat{n}_j(\boldsymbol{y}) \tag{1.8}$$

for $\boldsymbol{y} \in \partial B$, wherein $\bar{\boldsymbol{f}}$ is the surface traction due to $(\bar{\boldsymbol{u}}, \bar{p})$, and $\boldsymbol{\sigma}_y(\boldsymbol{T}_{.k})$ is the stress tensor due to $(\boldsymbol{T}_{.k}, \boldsymbol{P}_k)$. Together with (1.1e,f), (1.8) possesses a unique solution [9].

In the next section we prove, for mobility problems that the surface traction on a rigid particle in Ω tends, as the (small) particle approaches the boundary, to the to surface traction on the same particle near a tangent plane to $\partial\Omega$. The ambient surface traction $\bar{\boldsymbol{f}}$ is kept constant in that limit. In § 3 we use the results in § 2 to prove, for mobility problems, that the translational and angular velocities of the rigid particle in Ω respectively tend to its translational and angular velocities in a half-space. For resistance problems we show the same for the force and torque exerted by the fluid on the particle. Finally, in § 4 we briefly summarize the work and address some of the difficulties that arise during the proof of the results in § 2.

2 The surface traction

In this section we prove the following result

Theorem 1 *Let $\Omega \subset \mathbb{R}^3$ be a convex bounded domain with smooth boundaries (in $C^{2,\alpha}$ say). Let $B \subset \Omega$ denote a smooth rigid body. Denote by x_0 the closest point to $\partial\Omega$ on ∂B and by z the closest point to ∂B on$\partial\Omega$, or*

$$d(\partial B, \partial\Omega) = |x - z| \ .$$

Let $z = (0, 0, 0)$ and $x_0 = (0, 0, h)$. The outward normal to $\partial\Omega$ at z is then $-\boldsymbol{e}_z$. Denote by $\Omega^{1/\epsilon}$ the image of Ω under the mapping

$$\boldsymbol{x} \to \frac{\boldsymbol{x}}{\epsilon}$$

If B is independent of ϵ, i.e., the coordinates of any point in B are unaffected by changes in ϵ, then, $\exists \epsilon_0 > 0$ such that for every $0 < \epsilon < \epsilon_0$ and $h > 0$ and for every $\bar{\boldsymbol{f}} \in C(\partial B_\epsilon)$ the *solutions of*

$$\frac{1}{2}f_i^\epsilon(\boldsymbol{y}) = \bar{f}_i(\boldsymbol{y},\epsilon) + \int_{\partial B} (\sigma_y)_{ij}(\boldsymbol{T}^{\Omega^{1/\epsilon}}_{.k}(\boldsymbol{x},\boldsymbol{y}))f_k^\epsilon(\boldsymbol{x})ds_x\hat{n}_j(\boldsymbol{y}) \tag{2.1a}$$

$$\int_{\partial B} \boldsymbol{f}^\epsilon ds = \boldsymbol{F}(\epsilon) \qquad \int_{\partial B} (\boldsymbol{x}-\boldsymbol{x}_0)\times \boldsymbol{f}^\epsilon ds = \boldsymbol{L}(\epsilon) \tag{2.1b,c}$$

and

$$\frac{1}{2}f_i^W(\boldsymbol{y}) = \bar{f}_i(\boldsymbol{y},\epsilon) + \int_{\partial B} (\sigma_y)_{ij}(\boldsymbol{T}^{W}_{.k}(\boldsymbol{x},\boldsymbol{y}))f_k^W(\boldsymbol{x})ds_x\hat{n}_j(\boldsymbol{y}) \tag{2.2a}$$

$$\int_{\partial B} \boldsymbol{f}^W ds = \boldsymbol{F}(\epsilon) \qquad \int_{\partial B} (\boldsymbol{x}-\boldsymbol{x}_0)\times \boldsymbol{f}^W ds = \boldsymbol{L}(\epsilon) \tag{2.1b,c}$$

satisfy

$$\left\|\boldsymbol{f}^\epsilon - \boldsymbol{f}^W\right\|_{L^\infty(\partial B)} \leq C(\alpha,\Omega,h)\epsilon^{1/\alpha}\left\|\boldsymbol{f}^W\right\|_{L^\infty(\partial B)} \quad \forall \alpha > 1\ . \tag{2.3}$$

In the above $\boldsymbol{T}^{\Omega^{1/\epsilon}}$ *denotes the Green's function in* $\Omega^{1/\epsilon}$ *and* $\boldsymbol{T}^W$ *in the half-space* $z > 0$.

To prove the theorem we need first to prove the following auxiliary result which establishes convergence of the Green's function in the limit $\epsilon \to 0$.

Lemma 1 *Let* Ω *and* $\boldsymbol{T}^\Omega$ *be the same as in theorem 1. Denote by* W *the half-space* $z > 0$ *whose boundary is the tangent plane to* $\partial\Omega$ *at* $(-\epsilon t, 0, 0)$. *Let* $\boldsymbol{T}^W$ *be the Green's function in that half-space. Let* $\boldsymbol{y} = (0,0,\epsilon u)$, *where* $u > 0$ *and both* u *and* t *are independent of* ϵ. *Then,* $\exists \epsilon_0 > 0$ *such that*

$$\|\boldsymbol{T}^\Omega - \boldsymbol{T}^W\|_{W^{2,\alpha}(\Omega)}(\boldsymbol{y}) + \|\boldsymbol{P}^\Omega - \boldsymbol{P}^W\|_{W^{2,\alpha}(\Omega)}(\boldsymbol{y}) \leq C(\alpha,\Omega,u,t)\epsilon^{3/\alpha-2}$$

for any $0 < \epsilon < \epsilon_0$ *and* $\alpha > 1$.

proof: Our first step would be to estimate $\boldsymbol{T}^{\Omega}(\boldsymbol{x},\boldsymbol{y})-\boldsymbol{T}^{W}(\boldsymbol{x},\boldsymbol{y})$ for all $\boldsymbol{x} \in \partial\Omega$. As $\boldsymbol{T}^{\Omega}(\boldsymbol{x},\boldsymbol{y}) = 0$ when $\boldsymbol{x}$ is on the boundary and $\boldsymbol{y}$ is in the interior of Ω we need only an estimate of $\boldsymbol{T}^{W}$ on $\partial\Omega$. Denote by $\boldsymbol{x}_p$ the projection of $\boldsymbol{x}$ on the xy plane and by $\boldsymbol{y}_p$ the reflection of $\boldsymbol{y}$ with respect to the xy plane. Let $s = \epsilon^{-1}|\boldsymbol{x}_p|$ and $\delta = \epsilon^{-1}|\boldsymbol{x} - \boldsymbol{x}_p|$. Then,

$$\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{y} = \epsilon s \boldsymbol{e}_r + \epsilon(-u+\delta)\boldsymbol{e}_z \tag{2.4a}$$

$$\boldsymbol{r}_p = \boldsymbol{x} - \boldsymbol{y}_p = \epsilon s \boldsymbol{e}_r + \epsilon(u+\delta)\boldsymbol{e}_z \tag{2.4b}$$

$$\boldsymbol{r}_e = \boldsymbol{x}_p - \boldsymbol{y}_p = \epsilon s \boldsymbol{e}_r + \epsilon u \boldsymbol{e}_z \ . \tag{2.4c}$$

Figure 1 displays the various entities defined in the above. It is well known (cf. [1]) that T^W

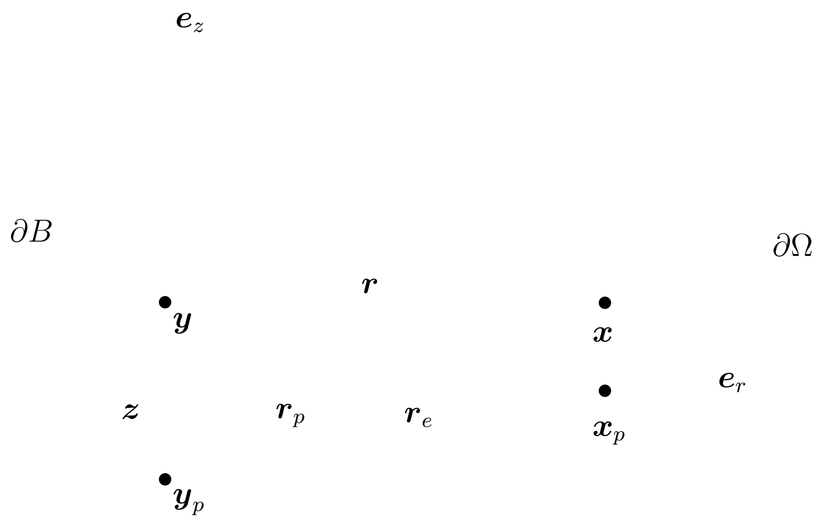

Figure 1: Definition of the various points and displacement vectors which are utilized in the proof of lemma 1 and theorem1

is given by

$$\boldsymbol{T}^W = \left(\frac{1}{r} - \frac{1}{r_p}\right)\boldsymbol{I} + \left(\frac{\boldsymbol{r}\boldsymbol{r}}{r^3} - \frac{\boldsymbol{r}_p\boldsymbol{r}_p}{r_p^3}\right)$$
$$+ 2u\left(\boldsymbol{e}_r\boldsymbol{e}_r - \boldsymbol{e}_z\boldsymbol{e}_z\right)\cdot\boldsymbol{\nabla}\left[\frac{u\boldsymbol{r}_p}{r_p^3} - \frac{1}{r_p}\left(\boldsymbol{I} + \frac{\boldsymbol{r}_p\boldsymbol{r}_p}{r_p^2}\right)\cdot\boldsymbol{e}_z\right] \quad (2.5)$$

Denote by R the minimal radius of curvature of Ω at $(-t,0,0)$. From the definition of Ω it is clear that

$$R \geq C\ .$$

(We shall use the notation C for any arbitrary constant in the sequel.) Hence, the following estimate for δ holds for sufficiently large C and $s < \epsilon^{-\nu}$ for some $0 < \nu < 1$

$$\delta \leq \frac{1}{\epsilon}\left[R - \sqrt{R^2 - \epsilon^2(s+t)^2} \leq C\epsilon\frac{s^2+t^2}{R}\right] \leq C(s^2+t^2)\epsilon\ . \quad (2.6)$$

It is easy to show that for $s < \epsilon^{-\nu}$

$$\left|\frac{1}{r} - \frac{1}{r_p}\right| = \frac{1}{\sqrt{r_e^2+\delta^2}}\frac{\left[1+\frac{2\delta u}{r_e^2+\delta^2}\right]^{\frac{1}{2}} - \left[1-\frac{2\delta u}{r_e^2+\delta^2}\right]^{\frac{1}{2}}}{\left[1-\left(\frac{2\delta u}{r_e^2+\delta^2}\right)^2\right]^{\frac{1}{2}}} \leq \frac{C\delta u}{\epsilon\left[r_e^2+\delta^2\right]^{3/2}} \leq C\frac{(s^2+t^2)u}{\left[s^2+u^2\right]^{3/2}} \leq \frac{C}{\left[s^2+u^2\right]^{1/2}} \quad (2.7a)$$

As $\left|\frac{\partial r_e}{\partial s}\right| \leq C$ we have (the gradient here is tangential to $\partial\Omega$)

$$\left|\boldsymbol{\nabla}\left(\frac{1}{r} - \frac{1}{r_p}\right)\right| \leq C\frac{(s^2+t^2)u}{\epsilon r_e^4} \leq \frac{C}{\epsilon(s^2+u^2)} \quad (2.7b)$$

Similarly,

$$\left|\boldsymbol{\nabla}\boldsymbol{\nabla}\left(\frac{1}{r} - \frac{1}{r_p}\right)\right| \leq C\frac{(s^2+t^2)u}{\epsilon^2 r_e^5} \leq \frac{C}{\epsilon^2\left[s^2+u^2\right]^{3/2}} \quad (2.7c)$$

For $s > \epsilon^{-\nu}$ we have

$$\left|\frac{1}{r} - \frac{1}{r_p}\right| \leq \frac{C}{\epsilon s} \tag{2.8a}$$

$$\left|\boldsymbol{\nabla}\left(\frac{1}{r} - \frac{1}{r_p}\right)\right| \leq \frac{C}{\epsilon^2 s^2} \tag{2.8b}$$

$$\left|\boldsymbol{\nabla\nabla}\left(\frac{1}{r} - \frac{1}{r_p}\right)\right| \leq \frac{C}{\epsilon^3 s^3} \tag{2.8c}$$

By splitting $\partial\Omega$ into the subdomains $|y - y^{'}| \leq \epsilon$ and $|y - y^{'}| > \epsilon$ it is not difficult to show that

$$\int_{\partial\Omega\times\partial\Omega} \frac{|u(y^{'}) - u(y)|^{\alpha}}{|y^{'} - y|^{1+\alpha}} dy^{'} dy \leq C\epsilon \|\boldsymbol{\nabla} u\|^{\alpha}_{L^{\alpha}(\partial\Omega)} + C\epsilon^{1-\alpha}\|u\|^{\alpha}_{L^{\alpha}(\partial\Omega)} \,.$$

Substituting $\boldsymbol{\nabla}\left(\frac{1}{r} - \frac{1}{r_p}\right)$ for u in the above yields, for sufficiently large ν (but still smaller than unity)

$$\left\|\frac{1}{r} - \frac{1}{r_p}\right\|_{W^{2-1/\alpha,\alpha}(\partial\Omega)} \leq C\epsilon^{3/\alpha - 2} \tag{2.9}$$

for any $\alpha \in (1, \infty)$.

To prove that a similar estimate holds for $\boldsymbol{T}^W$ we write it in the form

$$\begin{aligned} \boldsymbol{T}^W = \left(\frac{1}{r} - \frac{1}{r_p}\right)\boldsymbol{I} + \left(\frac{1}{r} - \frac{1}{r_p}\right)\frac{\boldsymbol{rr}}{r^2} + \left(\frac{1}{r} - \frac{1}{r_p}\right)\frac{\boldsymbol{rr}}{rr_p} + \\ + \left(\frac{\boldsymbol{rr} - \boldsymbol{r}_p\boldsymbol{r}_p}{r_p^3}\right) + \\ + 2\epsilon u\left(\boldsymbol{e}_r\boldsymbol{e}_r - \boldsymbol{e}_z\boldsymbol{e}_z\right)\cdot\boldsymbol{\nabla}\left[\frac{\epsilon u \boldsymbol{r}_p}{r_p^3} - \frac{1}{r_p}\left(\boldsymbol{I} + \frac{\boldsymbol{r}_p\boldsymbol{r}_p}{r_p^2}\right)\cdot\boldsymbol{e}_z\right] \end{aligned} \tag{2.10}$$

A straightforward calculation shows that

$$\begin{aligned} \left(\frac{\boldsymbol{rr} - \boldsymbol{r}_p\boldsymbol{r}_p}{r_p^3}\right) + 2\epsilon u\left(\boldsymbol{e}_r\boldsymbol{e}_r - \boldsymbol{e}_z\boldsymbol{e}_z\right)\cdot\boldsymbol{\nabla}\left[\frac{\epsilon u \boldsymbol{r}_p}{r_p^3} - \frac{1}{r_p}\left(\boldsymbol{I} + \frac{\boldsymbol{r}_p\boldsymbol{r}_p}{r_p^2}\right)\cdot\boldsymbol{e}_z\right] = \\ = \frac{2\epsilon^2 u\delta}{r_p^3}\left(\boldsymbol{e}_r\boldsymbol{e}_r - \boldsymbol{e}_z\boldsymbol{e}_z\right)\cdot\left[3\frac{\boldsymbol{r}_p\boldsymbol{r}_p}{r_p^2} - \boldsymbol{I}\right] - 4\frac{\epsilon^2 u\delta}{r_p^3}\boldsymbol{e}_z\boldsymbol{e}_z \end{aligned} \tag{2.11}$$

As $\min(r, r_p) \geq C r_e$ and since

$$\left| \frac{\boldsymbol{r}_p \boldsymbol{r}_p}{r_p^2} \right| \leq C$$
$$\left| \frac{\boldsymbol{r}\boldsymbol{r}}{r^2} \right| \leq C$$
$$\left| \frac{\boldsymbol{r}\boldsymbol{r}}{r r_p} \right| \leq C$$

independently of ϵ, the right-hand-side of (2.10) satisfy the estimates (2.7) and (2.8). Hence,

$$\left\| \boldsymbol{T}^W \right\|_{W^{2-1/\alpha,\alpha}(\partial\Omega)} \leq C(\alpha, \Omega, u, t)\epsilon^{3/\alpha - 2} \tag{2.12}$$

for any $\alpha \in (1, \infty)$.

As $\boldsymbol{T}^W - \boldsymbol{T}^\Omega$ is a regular solution of the Stokes equation for all $\boldsymbol{x} \in \Omega$, standard elliptic estimates for the Stokes equations (cf. lemma IV-6.1 in[4]) then imply that

$$\|\boldsymbol{T}^\Omega - \boldsymbol{T}^W\|_{W^{2,\alpha}(\Omega)} + \|\boldsymbol{P}^\Omega - \boldsymbol{P}^W\|_{W^{2,\alpha}(\Omega)} \leq C(\alpha, \Omega, u, t)\epsilon^{3/\alpha - 2} \tag{2.13a}$$

for all $\alpha \in (1, \infty)$.

$\square$

Proof of theorem 1:

Let $\tilde{\boldsymbol{f}} = \boldsymbol{f}^\epsilon - \boldsymbol{f}^W$. Subtracting (2.2) from (2.1) we obtain

$$\frac{1}{2}\tilde{f}_i(\boldsymbol{y}) = \int_{\partial B} (\sigma_y)_{ij}(\boldsymbol{T}^W_{.k}(\boldsymbol{x}, \boldsymbol{y}))\tilde{f}_k(\boldsymbol{x}) ds_x \hat{n}_j(\boldsymbol{y}) + \int_{\partial B} (\sigma_y)_{ij}(\boldsymbol{T}^{\Omega^{1/\epsilon}}_{.k} - \boldsymbol{T}^W_{.k})(\boldsymbol{x}, \boldsymbol{y}) f^\epsilon_k(\boldsymbol{x}) ds_x \hat{n}_j(\boldsymbol{y}) \tag{2.14a}$$

$$\int_{\partial B} \tilde{\boldsymbol{f}} ds = 0 \quad \int_{\partial B} (\boldsymbol{x} - \boldsymbol{x}_0) \times \tilde{\boldsymbol{f}} ds = 0 \tag{2.14b,c}$$

The second integral on the right-hand side of (2.14a) can be bounded as follows: Denote by B_ϵ the image of B under the mapping

$$\frac{\boldsymbol{x}}{\epsilon} \rightarrow \boldsymbol{x} \ .$$

As

$$\boldsymbol{\sigma}\left(\boldsymbol{T}^{W}_{.k}\right)(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{\epsilon^2}\,\boldsymbol{\sigma}\left(\boldsymbol{T}^{W}_{.k}\right)(\boldsymbol{x}/\boldsymbol{\epsilon},\boldsymbol{y}/\boldsymbol{\epsilon}) \ , \tag{2.15}$$

We have

$$\int_{\partial B}(\sigma_y)_{ij}(\boldsymbol{T}^{\Omega^{1/\epsilon}}_{.k} - \boldsymbol{T}^{W}_{.k})(\boldsymbol{x}/\epsilon,\boldsymbol{y}/\epsilon) f^\epsilon_k(\boldsymbol{x}/\epsilon) ds_x = \int_{\partial B_\epsilon}(\sigma_y)_{ij}(\boldsymbol{T}^{\Omega}_{.k} - \boldsymbol{T}^{W}_{.k})(\boldsymbol{x},\boldsymbol{y}) f^\epsilon_k(\boldsymbol{x}) ds_x$$

By lemma 1,

$$\left|\int_{\partial B_\epsilon}(\sigma_y)_{ij}(\boldsymbol{T}^{\Omega}_{.k} - \boldsymbol{T}^{W}_{.k})(\boldsymbol{x},\boldsymbol{y}) f^\epsilon_k(\boldsymbol{x}) ds_x \hat{n}_j(\boldsymbol{y})\right| \leq$$
$$\leq C\epsilon^{2-1/\alpha}\left\|\boldsymbol{f}^\epsilon\right\|_{L^\infty(\partial B)}\left\{\left\|\boldsymbol{T}^W - \boldsymbol{T}^\Omega\right\|_{W^{2,\alpha}(\Omega)} + \left\|\boldsymbol{P}^W - \boldsymbol{P}^\Omega\right\|_{W^{1,\alpha}(\Omega)}\right\} \leq C\epsilon^{1/\alpha}\left\|\boldsymbol{f}^\epsilon\right\|_{L^\infty(\partial B)} \ . \tag{2.16}$$

Hence,

$$\frac{1}{2}\tilde{f}_i(\boldsymbol{y}) - \int_{\partial B}(\sigma_y)_{ij}(\boldsymbol{T}^{W}_{.k}(\boldsymbol{x},\boldsymbol{y}))\tilde{f}_k(\boldsymbol{x}) ds_x \hat{n}_j(\boldsymbol{y}) = \boldsymbol{g}(\boldsymbol{y}) \tag{2.17a}$$

$$\int_{\partial B}\tilde{\boldsymbol{f}} ds = 0 \qquad \int_{\partial B}(\boldsymbol{x} - \boldsymbol{x}_0)\times\tilde{\boldsymbol{f}} ds = 0 \tag{2.17b}$$

where

$$\|\boldsymbol{g}\|_{L^\infty(\partial B)} \leq C\epsilon^{1/\alpha}\left\|\boldsymbol{f}^\epsilon\right\|_{L^\infty(\partial B)} \tag{2.17c}$$

Since the operator on the left-hand-side of (2.17a) is invertible in the space of $C(\partial B)$ functions satisfying (2.17b) [9, 8], and since the operator is independent of ϵ we have

$$\|\tilde{\boldsymbol{f}}\|_{L^\infty(\partial B)} = \|\boldsymbol{f}^\epsilon - \boldsymbol{f}^W\|_{L^\infty(\partial B)} \leq C\epsilon^{1/\alpha}\left\|\boldsymbol{f}^\epsilon\right\|_{L^\infty(\partial B)} \ . \tag{2.18}$$

It is easy to show, using (2.18) that

$$\|\boldsymbol{f}^{\epsilon}\|_{L^{\infty}(\partial B)} \leq \frac{\left\|\boldsymbol{f}^{W}\right\|_{L^{\infty}(\partial B)}}{1 - C\epsilon^{1/\alpha}} \,. \tag{2.19}$$

Consequently, $\exists \epsilon_0$ such that for all $0 < \epsilon < \epsilon_0$ we have

$$\|\boldsymbol{f}^{\epsilon} - \boldsymbol{f}^{W}\|_{L^{\infty}(\partial B)} \leq C(\alpha, \Omega, h)\epsilon^{1/\alpha} \left\|\boldsymbol{f}^{W}\right\|_{L^{\infty}(\partial B)} \tag{2.20}$$

3 Mobility and resistance problems

In this section we utilize theorem 1 in both mobility and resistance problems. In the first case the force and torque exerted by the fluid on the particle remain fixed, whereas the translational and angular velocities are allowed to vary with ϵ. For resistance problems the velocity and rate of rotation remain fixed, whereas the force and torque vary with ϵ.

For the first case we obtain the following result:

Corollary 1 *Let B and $\Omega^{1/\epsilon}$ be the same as in theorem 1. Denote by W the half-space $z > 0$, and let O be the closet point on $\partial\Omega^{1/\epsilon}$ to B. Let $\boldsymbol{U}_{\epsilon}$ and $\boldsymbol{\omega}_{\epsilon}$ respectively denote the translational and rotational velocities of B for the following mobility problem*

$$\boldsymbol{\nabla} \cdot \boldsymbol{u}^{\epsilon} = 0 \quad \textit{in } \Omega^{1/\epsilon} \setminus B \tag{3.1a}$$

$$\nabla^2 \boldsymbol{u}^{\epsilon} = \boldsymbol{\nabla} p^{\epsilon} \quad \textit{in } \Omega^{1/\epsilon} \setminus B \tag{3.1b}$$

$$\boldsymbol{u}^{\epsilon} = \boldsymbol{u}^{*}\left(\epsilon \boldsymbol{x}\right) \quad \textit{on } \partial\Omega^{1/\epsilon} \tag{3.1c}$$

$$\boldsymbol{u}^{\epsilon} = \boldsymbol{U}_{\epsilon} + \boldsymbol{\omega}_{\epsilon} \times (\boldsymbol{x} - \boldsymbol{x}_0) \quad \textit{on } \partial B \ \boldsymbol{x}_0 \in B \tag{3.1d}$$

$$\int_{\partial B} \boldsymbol{f}^{\epsilon} ds = \boldsymbol{F} \quad \int_{\partial B} (\boldsymbol{x} - \boldsymbol{x}_0) \times \boldsymbol{f}^{\epsilon} ds = \boldsymbol{L} \tag{3.1e,f}$$

Let further $\boldsymbol{U}_W$ *and* $\boldsymbol{\omega}_W$ *denote the velocities of* B *for the mobility problem*

$$\boldsymbol{\nabla}\cdot\boldsymbol{u}^W = 0 \quad in\ W\setminus B \tag{3.2a}$$

$$\nabla^2\boldsymbol{u}^W = \boldsymbol{\nabla}p^W \quad in\ W\setminus B \tag{3.2b}$$

$$\boldsymbol{u}^W = \boldsymbol{u}^*(O) \quad on\ \partial W \tag{3.2c}$$

$$\boldsymbol{u}^W \to \boldsymbol{u}^*(O) \quad as\ |\boldsymbol{x}|\to\infty in\ W \tag{3.2d}$$

$$\boldsymbol{u}^W = \boldsymbol{U}_W + \boldsymbol{\omega}_W\times(\boldsymbol{x}-\boldsymbol{x}_0) \quad on\ \partial B\ \boldsymbol{x}_0\in B \tag{3.2e}$$

$$\int_{\partial B}\boldsymbol{f}^W ds = \boldsymbol{F} \qquad \int_{\partial B}(\boldsymbol{x}-\boldsymbol{x}_0)\times\boldsymbol{f}^W ds = \boldsymbol{L}\ . \tag{3.2e,f}$$

Then, $\exists\epsilon_0>0$ *such that for* $0<\epsilon<\epsilon_0$ *and* $0<\beta<1$ *we have*

$$|\boldsymbol{U}_\epsilon - \boldsymbol{U}_W| \le C(\beta,\Omega,h)\epsilon^\beta \tag{3.3a}$$

$$|\boldsymbol{\omega}_\epsilon - \boldsymbol{\omega}_W| \le C(\beta,\Omega,h)\epsilon^\beta \tag{3.3b}$$

proof: Using the integral presentation (1.3) we can write the velocity field in the form

$$u_i^\epsilon(\boldsymbol{y}) = \bar{u}_i(\epsilon\boldsymbol{y}) + \int_{\partial B} T_{ij}^{\Omega^{1/\epsilon}}(\boldsymbol{x},\boldsymbol{y})f_j^\epsilon(\boldsymbol{x})ds_x =$$
$$= \bar{u}_i(O)+\int_{\partial B}T_{ij}^W f_j^W ds_x+\int_{\partial B}\left(T_{ij}^{\Omega^{1/\epsilon}} - T_{ij}^W\right)f_j^W ds_x+\int_{\partial B}T_{ij}^W(f_j^W - f_j^\epsilon)ds_x+[\bar{u}_i(\epsilon\boldsymbol{y})-\bar{u}_i(O)] \tag{3.4}$$

Denote by $\boldsymbol{f}^{W,\epsilon}$ the solution of

$$\frac{1}{2}f_i^{W,\epsilon}(\boldsymbol{y}) - \int_{\partial B}(\sigma_y)_{ij}(\boldsymbol{T}_{.k}^W(\boldsymbol{x},\boldsymbol{y}))f_k^{W,\epsilon}(\boldsymbol{x})ds_x\hat{n}_j(\boldsymbol{y}) = 2\epsilon D_{ij}(\epsilon\boldsymbol{y})\hat{n}_j(\boldsymbol{y}) \tag{3.5a}$$

$$\int_{\partial B}\boldsymbol{f}^{W,\epsilon}ds = \boldsymbol{F} \qquad \int_{\partial B}(\boldsymbol{x}-\boldsymbol{x}_0)\times\boldsymbol{f}^{W,\epsilon}ds = \boldsymbol{L}\ . \tag{3.5b,c}$$

Where $\boldsymbol{D}$ is the symmetric part of $\boldsymbol{\nabla}\bar{\boldsymbol{u}}$. By theorem 1 $\exists\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$ and $1 < \alpha$

$$\left\|\boldsymbol{f}^{W,\epsilon} - \boldsymbol{f}^{\epsilon}\right\|_{L^\infty(\partial B)} \leq C\epsilon^{1/\alpha}\left\|\boldsymbol{f}^{W,\epsilon}\right\|_{L^\infty(\partial B)} . \tag{3.6}$$

Furthermore, since the linear operator on the left-hand-side (3.5a) is invertible and independent of ϵ, and since $\boldsymbol{f}^W$ satisfies (2.2) we have

$$\left\|\boldsymbol{f}^{W,\epsilon} - \boldsymbol{f}^{W}\right\|_{L^\infty(\partial B)} \leq C\epsilon \quad \forall 0 < \epsilon < \epsilon_0 . \tag{3.7}$$

Combining (3.6) and (3.7) yields, since $\left\|\boldsymbol{f}^W\right\|$ is bounded for $h > 0$,

$$\left\|\boldsymbol{f}^{\epsilon} - \boldsymbol{f}^{W}\right\|_{L^\infty(\partial B)} \leq C\epsilon^{1/\alpha} \quad \forall 0 < \epsilon < \epsilon_0 \; 1 < \alpha . \tag{3.8}$$

Using lemma 1 and Sobolev embedding it is not difficult to show that for $\alpha_1 > 3/2$

$$\begin{aligned}\left|\int_{\partial B}\left(T_{ij}^{\Omega^{1/\epsilon}} - T_{ij}^{W}\right) f_j^W ds_x\right| = \epsilon^{-1}\left|\int_{\partial B_\epsilon}\left(T_{ij}^{\Omega} - T_{ij}^{W}\right) f_j^W ds_x\right| \leq \\ \leq C\epsilon\left\|T_{ij}^{\Omega} - T_{ij}^{W}\right\|_{L^\infty(\Omega)}\left\|\boldsymbol{f}^{\epsilon}\right\|_{L^\infty(\partial B)} \leq C\epsilon^{3/\alpha_1 - 1} ,\end{aligned} \tag{3.9a}$$

and by (3.8) that for any $\alpha_2 > 1$

$$\left|\int_{\partial B} T_{ij}^{W}(f_j^W - f_j^\epsilon) ds_x\right| \leq C\epsilon\left\|\boldsymbol{T}^w\right\|_{L^1(\partial B)} \leq C\epsilon^{1/\alpha_2} , \tag{3.9b}$$

uniformly for all $\boldsymbol{y} \in \partial B$. Furthermore, smoothness of $\bar{\boldsymbol{u}}$ implies that

$$\sup_{\boldsymbol{y}\in\partial B}\left|\bar{\boldsymbol{u}}(\epsilon\boldsymbol{y}) - \bar{\boldsymbol{u}}(O)\right| \leq C\epsilon . \tag{3.9c}$$

Hence,

$$\sup_{\boldsymbol{y}\in\partial B}\left|u_i^\epsilon(\boldsymbol{y}) - \bar{u}_i(O) - \int_{\partial B} T_{ij}^W(\boldsymbol{x},\boldsymbol{y}) f_j^W(\boldsymbol{x}) ds_x\right| \leq C\left(\epsilon^{3/\alpha_1 - 1} + \epsilon^{1/\alpha_2} + \epsilon\right) , \tag{3.10}$$

which is equivalent to the corollary.

□

Note that in the limit $\epsilon \to 0$ the ambient flow tends to be uniform, and a change in the frame of reference would turn the limit problem (3.2) into the problem of finding the velocity of a particle near a flat wall in a quiescent fluid.

It is now possible to consider other limit cases but the one considered in corollary 1, i.e., where the domain size tends to infinity but the particle size remains fixed. In fact, it suffices to require that the ratios between either the particle size or its distance from the boundary and the the local radius of curvature on the boundaries tend to zero, as the following corollary proves

Corollary 2 *Let $\Omega^{\delta/\epsilon}$ and B_δ be the respective images of $\Omega^{1/\epsilon}$ and B under the mapping $\boldsymbol{x} \to \delta\boldsymbol{x}$. Let $\boldsymbol{U}_\epsilon$ and $\boldsymbol{\omega}_\epsilon$ respectively denote the translational and rotational velocities of B_δ for the following mobility problem*

$$\boldsymbol{\nabla} \cdot \boldsymbol{u}^\epsilon = 0 \quad in\ \Omega^{\delta/\epsilon} \setminus B_\delta \tag{3.11a}$$

$$\nabla^2 \boldsymbol{u}^\epsilon = \boldsymbol{\nabla} p^\epsilon \quad in\ \Omega^{\delta/\epsilon} \setminus B_\delta \tag{3.11b}$$

$$\boldsymbol{u}^\epsilon = \boldsymbol{u}^* \quad on\ \partial\Omega^{\delta/\epsilon} \tag{3.11c}$$

$$\boldsymbol{u}^\epsilon = \boldsymbol{U}_\epsilon + \boldsymbol{\omega}_\epsilon \times (\boldsymbol{x} - \boldsymbol{x}_0)\boldsymbol{u}^* \quad on\ \partial B_\delta \tag{3.11d}$$

$$\int_{\partial B_\delta} \boldsymbol{f}^\epsilon ds = \delta \boldsymbol{F} \quad \int_{\partial B_\delta} (\boldsymbol{x} - \boldsymbol{x}_0) \times \boldsymbol{f}^\epsilon ds = \delta^2 \boldsymbol{L} \tag{3.11e,f}$$

Let further $\boldsymbol{U}_W$ *and* $\boldsymbol{\omega}_W$ *denote the velocities of* B *for the mobility problem*

$$\boldsymbol{\nabla} \cdot \boldsymbol{u}^W = 0 \quad \textit{in } W \setminus B_\delta \tag{3.12a}$$

$$\nabla^2 \boldsymbol{u}^W = \boldsymbol{\nabla} p^W \quad \textit{in } W \setminus B_\delta \tag{3.12b}$$

$$\boldsymbol{u}^W = \boldsymbol{u}^*(O) \quad \textit{on } \partial W \tag{3.12c}$$

$$\boldsymbol{u}^W \to \boldsymbol{u}^*(O) + \quad \textit{as } |\boldsymbol{x} \to \infty| \textit{in } W \tag{3.12d}$$

$$\boldsymbol{u}^W = \boldsymbol{U}_W + \boldsymbol{\omega}_W \times (\boldsymbol{x} - \boldsymbol{x}_0) \quad \textit{on } \partial B_\delta \tag{3.12e}$$

$$\int_{\partial B_\delta} \boldsymbol{f}^W ds = \boldsymbol{F} \quad \int_{\partial B_\delta} (\boldsymbol{x} - \boldsymbol{x}_0) \times \boldsymbol{f}^W ds = \boldsymbol{L}\ . \tag{3.12e,f}$$

Then, $\exists \epsilon_0 > 0$ *such that for all* $0 < \epsilon < \epsilon_0$ *and* $0 < \beta < 1$

$$|\boldsymbol{U}_\epsilon - \boldsymbol{U}_W| \leq C(\beta, \Omega, h)\epsilon^\beta \tag{3.13a}$$

$$|\boldsymbol{\omega}_\epsilon - \boldsymbol{\omega}_W| \leq C(\beta, \Omega, h)\epsilon^\beta \tag{3.13b}$$

Proof: Applying the transformation $\boldsymbol{x} \to \boldsymbol{x}/\delta$ to (3.11) and (3.12) we respectively obtain (3.1) and (3.2). Corollary 1 can then be utilized.

$\square$

We note, however, that in contrast to the translational and the angular velocities which are not affected by the change of the limit, the surface traction field considerably varies. For instance, if $\delta = \epsilon$ and either $\boldsymbol{F} \neq 0$ or $\boldsymbol{L} \neq 0$ then $\boldsymbol{f}_\epsilon \sim O(1/\epsilon)$. For neutrally buoyant torque-free particles it is not difficult to show that the surface traction tends to satisfy the integral equation

$$\frac{1}{2}f_i(\boldsymbol{y}) - \int_{\partial B_\epsilon} (\sigma_y)_{ij}(\boldsymbol{T}^W_{.k}(\boldsymbol{x},\boldsymbol{y}))f_k(\boldsymbol{x})ds_x\hat{n}_j(\boldsymbol{y}) = 2D_{ij}(O)\hat{n}_j(\boldsymbol{y}) \quad \boldsymbol{y} \in \partial B_\epsilon \tag{3.14a}$$

$$\int_{\partial B_\epsilon} \boldsymbol{f} ds = 0 \quad \int_{\partial B_\epsilon} (\boldsymbol{x} - \boldsymbol{x}_0) \times \boldsymbol{f} ds = 0\ , \tag{3.14b,c}$$

or, equivalently, to the surface traction on a particle near a wall in a uniform shear flow.

We conclude this section by discussing the case $\delta = 1$ for resistance problems.

Corollary 3 *Let B and $\Omega^{1/\epsilon}$ be the same as in theorem 1. Let $\boldsymbol{F}_\epsilon$ and $\boldsymbol{L}_\epsilon$ respectively denote the force and torque exerted by the fluid on B for the following resistance problem*

$$\boldsymbol{\nabla} \cdot \tilde{\boldsymbol{u}}^\epsilon = 0 \quad \textit{in } \Omega^{1/\epsilon} \setminus B \tag{3.15a}$$

$$\nabla^2 \tilde{\boldsymbol{u}}^\epsilon = \boldsymbol{\nabla} p^\epsilon \quad \textit{in } \Omega^{1/\epsilon} \setminus B \tag{3.15b}$$

$$\tilde{\boldsymbol{u}}^\epsilon = \boldsymbol{u}^*(\epsilon \boldsymbol{x}) \quad \textit{on } \partial\Omega^{1/\epsilon} \tag{3.15c}$$

$$\tilde{\boldsymbol{u}}^\epsilon = \boldsymbol{U} + \boldsymbol{\omega} \times (\boldsymbol{x} - \boldsymbol{x}_0)\tilde{\boldsymbol{u}}^* \quad \textit{on } \partial B \tag{3.15d}$$

$$\int_{\partial B} \boldsymbol{f}^\epsilon ds = \boldsymbol{F}_\epsilon \quad \int_{\partial B} (\boldsymbol{x} - \boldsymbol{x}_0) \times \boldsymbol{f}^\epsilon ds = \boldsymbol{L}_\epsilon \tag{3.1e,f}$$

Let further $\boldsymbol{F}_W$ and $\boldsymbol{L}_W$ denote the force and torque on B for the resistance problem

$$\boldsymbol{\nabla} \cdot \tilde{\boldsymbol{u}}^W = 0 \quad \textit{in } W \setminus B \tag{3.16a}$$

$$\nabla^2 \tilde{\boldsymbol{u}}^W = \boldsymbol{\nabla} p^W \quad \textit{in } W \setminus B \tag{3.16b}$$

$$\tilde{\boldsymbol{u}}^W = \boldsymbol{u}^*(O) \quad \textit{on } \partial W \tag{3.16c}$$

$$\tilde{\boldsymbol{u}}^W \to \boldsymbol{u}^*(O) \quad \textit{as } |\boldsymbol{x} \to \infty| \textit{ in } W \tag{3.16d}$$

$$\tilde{\boldsymbol{u}}^W = \boldsymbol{U} + \boldsymbol{\omega} \times (\boldsymbol{x} - \boldsymbol{x}_0) \quad \textit{on } \partial B \tag{3.16e}$$

$$\int_{\partial B} \boldsymbol{f}^W ds = \boldsymbol{F}_W \quad \int_{\partial B} (\boldsymbol{x} - \boldsymbol{x}_0) \times \boldsymbol{f}^W ds = \boldsymbol{L}_W\ . \tag{3.2e,f}$$

Then, $\exists \epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$ we have

$$|\boldsymbol{F}_\epsilon - \boldsymbol{F}_W| \leq C\epsilon^\beta \tag{3.17a}$$

$$|\boldsymbol{L}_\epsilon - \boldsymbol{L}_W| \leq C\epsilon^\beta \tag{3.17b}$$

Proof: We first reinterpret corollary 1 in terms of mobility matrices. Linearity of Stokes flows implies that

$$\begin{bmatrix} \boldsymbol{U}_\epsilon \\ \\ \boldsymbol{\omega}_\epsilon \end{bmatrix} = \boldsymbol{M}_\epsilon \begin{bmatrix} \boldsymbol{F} \\ \\ \boldsymbol{L} \end{bmatrix} + \mathcal{L}_\epsilon \bar{\boldsymbol{u}} \tag{3.18a}$$

and that

$$\begin{bmatrix} \boldsymbol{U}_W \\ \\ \boldsymbol{\omega}_W \end{bmatrix} = \boldsymbol{M}_W \begin{bmatrix} \boldsymbol{F} \\ \\ \boldsymbol{L} \end{bmatrix} + \mathcal{L}_W \bar{\boldsymbol{u}} \ , \tag{3.18b}$$

wherein $\boldsymbol{M}$ is a 6×6 mobility matrix, which is both symmetric and positive definite [5], and $\mathcal{L}\bar{\boldsymbol{u}}$ is the velocity of the particle in the absence of external forces and torques. The subscripts ϵ and W have the same meaning as in corollary 1. Both $\boldsymbol{M}$ and $\mathcal{L}$ depend on the domain and on particle shape and location. Corollary 1 may now be restated in the form

$$\|\boldsymbol{M}_\epsilon - \boldsymbol{M}_W\| \leq C\epsilon^\beta \quad ; \quad \|\mathcal{L}_W \bar{\boldsymbol{u}} - \mathcal{L}_\epsilon \bar{\boldsymbol{u}}\| \leq C\epsilon^\beta \ . \tag{3.19a,b}$$

Since for $h > 0$, the matrices $\boldsymbol{M}_\epsilon$ for $0 < \epsilon < \epsilon_0$ and $\boldsymbol{M}_W$ are all invertible, we can express $\boldsymbol{M}_\epsilon^{-1} - \boldsymbol{M}_W^{-1}$ in the form [7]

$$\boldsymbol{M}_\epsilon^{-1} - \boldsymbol{M}_W^{-1} = \sum_{k=1}^{\infty} \boldsymbol{M}_W^{-k} \left(\boldsymbol{M}_\epsilon - \boldsymbol{M}_W\right)^k$$

yielding

$$\left\|\boldsymbol{M}_{\epsilon}^{-1}-\boldsymbol{M}_{W}^{-1}\right\| \leq C\epsilon^{\beta} . \tag{3.20}$$

Consider now the resistance problems (3.16) and (3.17). In this case we have, in view of (3.18)

$$\begin{bmatrix} \boldsymbol{F}_{\epsilon} \\ \\ \boldsymbol{L}_{\epsilon} \end{bmatrix} = \boldsymbol{M}_{\epsilon}^{-1} \begin{bmatrix} \boldsymbol{U} \\ \\ \boldsymbol{\omega} \end{bmatrix} - \boldsymbol{M}_{\epsilon}^{-1}\mathcal{L}_{\epsilon}\bar{\boldsymbol{u}} \tag{3.21a}$$

and

$$\begin{bmatrix} \boldsymbol{F}_{W} \\ \\ \boldsymbol{L}_{W} \end{bmatrix} = \boldsymbol{M}_{W}^{-1} \begin{bmatrix} \boldsymbol{F} \\ \\ \boldsymbol{L} \end{bmatrix} - \boldsymbol{M}_{W}^{-1}\mathcal{L}_{W}\bar{\boldsymbol{u}} . \tag{3.21b}$$

The corollary follows immediately from the above relations, (3.10b), and (3.20).

□

4 Conclusion

We have demonstrated in § 2 that the surface traction on a rigid particle near a smooth boundary tends to the surface traction on the same particle near a flat wall. We utilize these results in § 3 to show that the velocity and the rate of rotation of the particle, respectively, tend to the velocity and the rate of rotation of the same particle for mobility problems. The same result is proved for the force and torque in resistance problems.

One major obstacle which needs yet to be overcome is the limit $h \to 0$ (or when the ratio between particle size and its distance from the wall tends to 0). In [3] the formal asymptotic expansion assumes the validity of corollary 3 even when $h \to 0$, despite the fact that the force and torque may tend to ∞ in that limit. Nevertheless, it is (plausibly) assumed that the correction term due to the curvature of the boundary, while not necessarily being negligibly small, is still much smaller than the leading order term.

Using lemma 1, it is not difficult to show that the function g on the right-hand-side of (2.17) satisfies

$$\|\boldsymbol{g}\|_{L^\infty(\partial B)} \leq C h^{3/\alpha-2} \epsilon^{1/\alpha} \left\|\boldsymbol{f}^W\right\|_{L^\infty(\partial B)} . \tag{4.1}$$

However, as the integral operator on the left-hand side of (2.17) depends on h, and since its limit as $h \to 0$ is non-compact, it is not easy to show that the same estimate holds for $\boldsymbol{f}^\epsilon - \boldsymbol{f}^W$. Note that the L_∞ norm of $\boldsymbol{f}^W$ may tend to infinity as $h \to 0$, as is expected from the formal asymptotic expansions in [3].

The convexity requirement can be substantially weakened. We first note that Ω needs to be convex only in some neighborhood of the point z, which is the point closest to the particle B on $\partial\Omega$. If the domain is concave near z, it seems that the same procedure, with a few changes, can be applied in order to obtain the same results.

The results in this work were all stated and proved for a single particle. We can, however, extend them to a multi-particle set, since theorem 1 remains valid. The distance h, from the particle to the wall, will become the distance between this set and $\partial\Omega$.

References

[1] J. R. Blake, *A note on the image systemfor a stokelet in a no-slip boundary*, Proc.

Camb. Phil. Soc., 70 (1971), pp. 303–310.

[2] R. G. Cox, *The motion of suspended particles almost in contact*, Intl. J. Multiphase Flow, 1 (1974), pp. 343–371.

[3] A. Falade and H. Brenner, *First-order wall curvature effects upon the stokes resistance of a spherical particle moving in close proximity to a solid wall*, J. Fluid Mech., 193 (1988), pp. 533–568.

[4] G. P. Galdi, *An Introduction to the Mathematical Theory of the Navier-Stokes Equations*, vol. I, Springer, 1994.

[5] J. Happel and H. Brenner, *Low Reynolds number hydrodynamics*, Martinus Nijhoff, 1983.

[6] G. B. Jeffery, *On a solid rotation of a solid of revolution in a viscous fluid*, Proc. Lond. Math. Soc., 14 (1915), pp. 327–338.

[7] R. Kress, *Linear Integral Equations*, Springer, 2 ed., 1999.

[8] O. A. Ladyzhenskaya, *The Mathematical Theory of Incompressible Viscous Flow*, Gordon & Breach, 1963.

[9] N. Liron and E. Barta, *Motion of a rigid particle in a stokes flow: a new second-kind boundary-integral equation formulation*, J. Fluid Mech., 238 (1992), pp. 579–598.

[10] S. R. Majumdar, *Slow motion of an incompressible viscous liquid generated by the rotation of two spheres in contact*, Mathematika, 14 (1967), pp. 43–46.

[11] M. E. O'Neill, *On asymmetrical slow viscous flows caused by the motion of two equal spheres almost in contact*, Proc. Camb. Phil. Soc, 65 (1969), pp. 543–556.