The onset of superconductivity in long rectangles

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Abstract

The linearized Ginzburg-Landau equations in both semi-infinite strips and rectangles are transformed into equivalent one-dimensional integral equations. Then, the properties of the integral equations are utilized to prove that the onset field for semi-infinite strip is isolated. We solve the integral equations numerically to obtain the onset field for both rectangles and semi-infinite strips. A formal asymptotic expansion of the onset field in the long rectangle limit is obtained as well. Using this formal expansion we show that the onset field converges in this limit faster than any finite exponential rate, and, as a byproduct, that the onset mode in a semi-infinite strip must be asymptotically symmetric.

1 Introduction

Consider a planar superconducting body which is placed in a sufficiently low temperature (below the critical one) under the action of an external magnetic field . It is known both

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from experiments [15] and rigorous analysis [10] that for sufficiently strong magnetic field the normal state would prevail. If the field is then decreased, there is a critical field, depending on the sample's geometry, where the material would enter the superconducting state. For samples with boundaries, this field is known as the onset field. We shall refer to the bifurcating mode of the super-conducting order parameter as the onset mode.

The simplest case in which the bifurcation from the normal state to the superconducting one was calculated is the case of a half-plane [17]. Even in this simple case the onset field is substantially larger than the bifurcation field on the real line [9]. Furthermore, it was found by Saint-James and De-Gennes [17] that superconductivity is concentrated in this case near the boundary. This phenomenon, which appears in the presence of boundaries have been termed, therefore, surface superconductivity. The significance of Saint-James and de Gennes' solution [17] extends far beyond the simple, one-dimensional example of a half-plane. It was proved, first for films [5], then for discs [3], and finally for general two-dimensional domains with smooth boundaries [14, 8], that as the domain's scale tends to infinity the onset field tends to de-Gennes' value. If the boundaries include wedges the onset field will be larger than de-Gennes' value [4, 11, 18, 12].

In the present contribution we focus on semi-infinite strips and long rectangles. In [2] a few relevant results were obtained for these domains. Let $\mathcal{S}$ denote the semi-infinite strip $\{(x,y)| -l \leq x \leq l, 0 \leq y\}$ and let R_L denote the rectangle $\{(x,y)| -l \leq x \leq l, 0 \leq y \leq L\}$. Denote further by $h_{\mathcal{S}}$ the onset field of $\mathcal{S}$, by h_L the onset field of R_L, and by h_{1D} the onset field of the interval $[-l,l]$. The following relevant results were proved in [2]:

1. $h_{\mathcal{S}}$ exists and is greater than h_{1D}.

2. The onset mode of $\mathcal{S}$ decays exponentially fast as $y \to \infty$, or is $O(e^{-\gamma y})$. The value of γ can be easily obtained once $h_{\mathcal{S}}$ is determined.

3. $h_L \geq h_{\mathcal{S}}$. Furthermore, $h_L \xrightarrow[L\to\infty]{} h_{\mathcal{S}}$.

Many of the results in [2] were obtained using Fourier analysis. In the present contribution we employ Fourier analysis once again to obtain integral equations equivalent to the bifurcation problems in $\mathcal{S}$ and R_L. We then utilize the properties of the equation in $\mathcal{S}$ to prove isolation, on the h axis, of $h_{\mathcal{S}}$. We then provide formal arguments, showing that h_L converges faster than any exponentially rate to $h_{\mathcal{S}}$ as $L \to \infty$, and that the onset mode in $\mathcal{S}$ is asymptotically symmetric with respect to x as $y \to \infty$. Finally, we obtain h_L and $h_{\mathcal{S}}$ by numerically solving the formerly derived integral equations. Once $h_{\mathcal{S}}$ is obtained, the exponential rate of decay of the onset mode in $\mathcal{S}$, for large y, is evaluated.

The Ginzburg-Landau energy functional may be represented in the following dimensionless form [7]:

$$E = \int \left(-|\psi|^2 + \frac{|\psi|^4}{2} + |\boldsymbol{H}|^2 + \left| \frac{1}{\kappa} \boldsymbol{\nabla}\psi - i\boldsymbol{A}\psi \right|^2 \right) dxdy \tag{1.1}$$

in which Ψ is the (complex) superconducting order parameter, such that $|\Psi|$ varies from $|\Psi| = 0$ (when the material is at a normal state) to $|\Psi| = 1$ (for the purely superconducting state). The magnetic vector potential is denoted by $\boldsymbol{A}$ (the magnetic field is, then, given by $\boldsymbol{H} = \boldsymbol{\nabla} \times \boldsymbol{A}$), and κ is the Ginzburg-Landau parameter which is a material property. Superconductors for which $\kappa < 1/\sqrt{2}$ are termed type I superconductors, and those for which

$\kappa > 1/\sqrt{2}$ have been termed type II. Note that E is invariant under the gauge transformation

$$\Psi \to e^{i\kappa\theta}\psi; \qquad \boldsymbol{A} \to \boldsymbol{A} + \boldsymbol{\nabla}\theta. \tag{1.2}$$

We look for local minimizers of E in the semi-infinite strip $\mathcal{S}$, or in the rectangle R_L in the the case where the applied magnetic field is constant and perpendicular to the plane. The Euler-Lagrange equations associated with E (the steady state Ginzburg-Landau equations) are given by

$$\left(\frac{i}{\kappa}\boldsymbol{\nabla} + \boldsymbol{A}\right)^2 \Psi = \Psi\left(1 - |\Psi|^2\right), \tag{1.3a}$$

$$-\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) = \frac{i}{2\kappa}\left(\Psi^* \boldsymbol{\nabla}\Psi - \Psi\boldsymbol{\nabla}\Psi^*\right) + |\Psi|^2 \boldsymbol{A}. \tag{1.3b}$$

The natural boundary conditions satisfied on $\partial\mathcal{S}$ for this problem are

$$\left(\frac{i}{\kappa}\boldsymbol{\nabla} + \boldsymbol{A}\right)\psi \cdot \hat{\boldsymbol{n}} = 0, \tag{1.4a}$$

$$\boldsymbol{H} = h\hat{\boldsymbol{z}}. \tag{1.4b}$$

As the Ginzburg-Landau equations are gauge-invariant, we may choose the gauge (following [1, 6]) $\boldsymbol{A} = (0, A(x,y), 0)$. Thus, $\boldsymbol{H} = (0, 0, H(x,y))$ and $H = \partial A/\partial x$. We then linearize (1.3) near the normal state $\Psi \equiv 0$, $A = hx$, to which end we assume the asymptotic expansion

$(\epsilon \ll 1)$

$$\Psi = \epsilon^{1/2}\psi, \tag{1.5a}$$

$$A = hx + \epsilon a, \tag{1.5b}$$

$$h = h^{(0)} + \epsilon h^{(1)} + \ldots, \tag{1.5c}$$

$$a = a^{(0)} + \epsilon a^{(1)} + \ldots, \tag{1.5d}$$

$$\psi = \psi^{(0)} + \epsilon\psi^{(1)} + \ldots. \tag{1.5e}$$

Applying the transformations

$$x^1 = \kappa x, \quad y^1 = \kappa y, \quad h^1 = \frac{h^{(0)}}{\kappa},$$

the linearized form of (1.3a) becomes (we omit the superscripts 1 and (0) in the following)

$$-\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) + 2ihx\frac{\partial\psi}{\partial y} = \psi - h^2x^2\psi. \tag{1.6}$$

The boundary condition (1.4a) for the specific gauge we have chosen becomes after linearization on $\partial\mathcal{S}$

$$\psi_x(\pm l, y) = 0; \qquad i\psi_y(x, 0) + hx\psi(x, 0) = 0. \tag{1.7a,b}$$

On ∂R_L we have to add the boundary condition

$$i\psi_y(x, L) + hx\psi(x, L) = 0. \tag{1.8}$$

The rest of this work is arranged as follows: In the next section we derive integral equations equivalent to (1.6) together with (1.7) in $\mathcal{S}$ or together with (1.7) and (1.8) in R_L. In § 3 we prove isolation of $h_{\mathcal{S}}$, in § 4 we show, using formal arguments, that $h_L \sim h_{\mathcal{S}} + O(e^{-\lambda L})$

for any $\lambda > 0$, and that $\psi(x,y) \sim \bar{\psi}(-x,y)$ as $y \to \infty$. Finally, in § 5, we solve the integral equation from § 2 numerically.

2 Integral equations

Theorem 1 *Let $h = h_8$. Then, any solution of the problem*

$$-\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) + 2ihx\frac{\partial\psi}{\partial y} = \psi - h^2x^2\psi \qquad -l \le x \le l, \quad 0 \le y \tag{2.1a}$$

$$\psi_x(\pm l, y) = 0; \quad i\psi_y(x,0) + hx\psi(x,0) = 0 \tag{2.1b,c}$$

must satisfy

$$\frac{1}{2}\psi(x,0) = \frac{1}{2\pi}PV\int_{-\infty}^{\infty} i\int_{-l}^{l}(hs-\omega)G(x,s,\omega)\psi(s,0)dsd\omega \tag{2.2}$$

where P.V denotes the principal value and

$$G(x,s,\omega) = m(h)\begin{cases} \dfrac{[U(\eta)U'(-\xi(l)) + U(-\eta)U'(\xi(l))]\,[U'(-\xi(-l))U(\xi) + U'(\xi(-l))U(-\xi)]}{U'(\xi(-l))U'(-\xi(l)) - U'(\xi(l))U'(-\xi(-l))}, & x < s, \\ \dfrac{[U(\eta)U'(-\xi(-l)) + U(-\eta)U'(\xi(-l))]\,[U'(-\xi(l))U(\xi) + U'(\xi(l))U(-\xi)]}{U'(\xi(-l))U'(-\xi(l)) - U'(\xi(l))U'(-\xi(-l))}, & x > s, \end{cases} \tag{2.3a}$$

wherein

$$m(h) = \frac{\Gamma(1/2 + a)}{2\sqrt{\pi h}}, \tag{2.3b}$$

$\xi = \xi(x,\omega)$ is given by

$$\xi = \sqrt{\frac{2}{h}}(hx - \omega),$$

and $\eta = \xi(s,\omega)$, $U(x) \equiv U(a,x)$ is a parabolic cylinder function of the first kind, and $a = -1/2h$.

Conversely, suppose for some $h = \tilde{h}_{\mathcal{S}}$, $\exists f \in C^2[-l, l]$ *satisfying* $f'(\pm l) = 0$ *and*

$$\frac{1}{2} f(x) = \frac{1}{2\pi} PV \int_{-\infty}^{\infty} \hat{\phi}(x, \omega) d\omega, \tag{2.4a}$$

where

$$\hat{\phi}(x, \omega) = i \int_{-l}^{l} (hs - \omega) G(x, s, \omega) f(s) ds. \tag{2.4b}$$

Then, $\tilde{h}_{\mathcal{S}} \leq h_{\mathcal{S}}$*. Furthermore, if* $\tilde{h}_{\mathcal{S}} = h_{\mathcal{S}}$*, then the function*

$$\phi(x, y) = \frac{1}{2\pi} PV \int_{-\infty}^{\infty} e^{i\omega y} \hat{\phi}(x, \omega) d\omega \tag{2.5}$$

is a solution of (2.1)

Proof: Suppose first that ψ satisfies (2.1). Upon multiplying (2.1a) by $\exp\{-i\omega y\}$ and integrating by parts we obtain that

$$\hat{\psi}(x, \omega) = \int_{0}^{\infty} e^{-i\omega y} \psi(x, y) dy \tag{2.6}$$

satisfies

$$-\hat{\psi}'' + \left[(hx - \omega)^2 - 1\right] \hat{\psi} = i(hx - \omega)\psi(x, 0) \tag{2.7a}$$

$$\hat{\psi}'(\pm l) = 0, \tag{2.7b}$$

where $\hat{\psi}' = \frac{\partial \hat{\psi}}{\partial x}$. Hence,

$$\hat{\psi}(x, \omega) = i \int_{-l}^{l} (hs - \omega) G(x, s, \omega) \psi(s, 0) ds. \tag{2.8}$$

Note that $G(x,s,\omega)$ is well defined since the denominator in (2.3a) does not vanish for all $\omega \in \mathbb{R}$ when $h > h_{1D}$ (cf. [6, 4, 2]). In view of (2.6) the inverse Fourier transform of $\hat{\psi}$ is

$$\frac{1}{2\pi}\mathrm{PV}\int_{-\infty}^{\infty} e^{i\omega y}\hat{\psi}(x,\omega)d\omega = \begin{cases} \psi(x,y), & y>0, \\ \frac{1}{2}\psi(x,0), & y=0, \\ 0, & y<0. \end{cases} \cdot \tag{2.9}$$

Hence,

$$\frac{1}{2}\psi(x,0) = \frac{1}{2\pi}\mathrm{PV}\int_{-\infty}^{\infty} \hat{\psi}(x,\omega)d\omega. \tag{2.10}$$

Substituting (2.8) together with (2.10) into (2.1b) we obtain (2.2).

Suppose now that for some $h \geq h_S$, $\exists f \in C^2[-l,l]$ satisfying $f'(\pm l) = 0$ and (2.4). In [2] it was demonstrated that

$$\begin{aligned} G(x,s,\omega) =& G(x,s,\omega,h) = \frac{1}{|\omega|}\left[G_1(x,s,\omega,h)e^{-|\omega||x-s|}\right. \\ &\left. + G_2(x,s,\omega,h)e^{-|\omega|(2l-x-s)} + G_3(x,s,\omega,h)e^{-|\omega|(2l+x+s)}\right], \end{aligned} \tag{2.11a}$$

where, as $\omega \to \infty$,

$$G_1 = |\omega|\frac{1}{2}\frac{(\omega-hs)^{a-1/2}}{(\omega-hx)^{a+1/2}}\exp\left\{\frac{1}{2}h(s^2-x^2)\right\}\left[1+O(\frac{1}{\omega^2})\right] \tag{2.11b}$$

$$G_2 = |\omega|\frac{1}{2}\frac{(\omega-hl)^{2a}}{(\omega-hx)^{a+1/2}(\omega-hs)^{a+1/2}}\exp\left\{\frac{1}{2}h(2l^2-s^2-x^2)\right\}\left[1+O(\frac{1}{\omega^2})\right] \tag{2.11c}$$

$$G_3 = |\omega|\frac{1}{2}\frac{(\omega+hl)^{2a}}{(\omega-hx)^{a+1/2}(\omega-hs)^{a+1/2}}\exp\left\{\frac{1}{2}h(2l^2-s^2-x^2)\right\}\left[1+O(\frac{1}{\omega^2})\right], \tag{2.11d}$$

whenever $x < s$. For the cases $x > s$ and $\omega \to -\infty$ we use the identity

$$G(x, s, \omega, h) = G(s, x, \omega, h) = G(-x, -s, -\omega, h). \tag{2.12}$$

Hence, applying Laplace's method it is not difficult to show that for $|\omega| \gg 1$

$$\hat{\phi}(x, \omega) \sim -i\frac{f(x)}{\omega} - ihx\frac{f(x)}{\omega^2} + O(\frac{1}{\omega^3}). \tag{2.13}$$

As $\hat{\phi}$ satisfies

$$-\hat{\phi}'' + \left[(hx - \omega)^2 - 1\right]\hat{\phi} = i(hx - \omega)f(x) \tag{2.14a}$$

$$\hat{\phi}'(\pm l) = 0, \tag{2.14b}$$

and since by (2.13) $f'(\pm l) = 0$, the function

$$\varphi = \hat{\phi} + i\frac{f(x)}{\omega}$$

must satisfy

$$-\varphi'' + \left[(hx - \omega)^2 - 1\right]\varphi = -ihxf + \frac{i}{\omega}\left[(h^2x^2 - 1)f - f''\right] \tag{2.15a}$$

$$\varphi'(\pm l) = 0. \tag{2.15b}$$

Hence,

$$\varphi(x, \omega) = \int_{-l}^{l} G(x, s, \omega)\left\{-ihsf + \frac{i}{\omega}\left[(h^2s^2 - 1)f - f''\right]\right\} ds, \tag{2.16}$$

from which we can deduce the $O(\omega^{-3})$ term in the asymptotic expansion of $\hat{\phi}(x, \omega)$:

$$\hat{\phi}(x, \omega) \sim -i\frac{f(x)}{\omega} - ihx\frac{f(x)}{\omega^2} - i\frac{(h^2x^2 + 1)f + f''}{\omega^3} + O(\frac{1}{\omega^4}). \tag{2.17}$$

Subtituting the above expansion into (2.14a) yields

$$\hat{\phi}''(x,\omega) \sim -i\frac{f''(x)}{\omega} + O(\frac{1}{\omega^2}), \tag{2.18}$$

and hence

$$\hat{\phi}'(x,\omega) \sim -i\frac{f'(x)}{\omega} + O(\frac{1}{\omega^2}). \tag{2.19}$$

Consider then the function $\phi(x,y)$ defined in (2.5). By (2.13) $\phi(x,y)$ is differentiable twice in x and y for any $y \neq 0$. Furthermore,

$$\phi(x,0^+) - \phi(x,0^-) = f(x), \tag{2.20a}$$

$$\frac{\partial\phi}{\partial y}(x,0^+) - \frac{\partial\phi}{\partial y}(x,0^-) = ihxf(x). \tag{2.20b}$$

By (2.16), (2.17), (2.18), and (2.20) the various derivatives of ϕ are given by

$$\phi_x(x,y) = \frac{1}{2\pi}\mathrm{PV}\int_{-\infty}^{\infty} e^{i\omega y}\hat{\phi}'(x,\omega)d\omega, \tag{2.21a}$$

$$\phi_{xx}(x,y) = \frac{1}{2\pi}\mathrm{PV}\int_{-\infty}^{\infty} e^{i\omega y}\hat{\phi}''(x,\omega)d\omega, \tag{2.21b}$$

$$\phi_y(x,y) = \frac{1}{2\pi}\mathrm{PV}\int_{-\infty}^{\infty} e^{i\omega y}\left[i\omega\hat{\phi}(x,\omega) - f(x)\right]d\omega, \tag{2.21c}$$

$$\phi_{yy}(x,y) = \frac{1}{2\pi}\mathrm{PV}\int_{-\infty}^{\infty} e^{i\omega y}\left[-\omega^2\hat{\phi}(x,\omega) - i(hx+\omega)f(x)\right]d\omega. \tag{2.21d}$$

Obviously, by (2.14b) we have

$$\phi_x(\pm l) = 0 \quad \forall y \neq 0. \tag{2.22}$$

Furthermore, by (2.21) we have

$$-\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) + 2ihx\frac{\partial\phi}{\partial y} = \phi - h^2x^2\phi \qquad -l \leq x \leq l, \quad y \neq 0 \tag{2.23}$$

It remains yet to check the conditions satisfied by ϕ for $y = 0^+$. As, by (2.4a),

$$\frac{1}{2}\left[\phi(x,0^+) + \phi(x,0^-)\right] = \frac{1}{2\pi}\mathrm{PV}\int_{-\infty}^{\infty}\hat{\phi}(x,\omega)d\omega = \frac{1}{2}f(x), \tag{2.24}$$

combination with (2.20a) yields

$$\phi(x,0^-) = 0. \tag{2.25}$$

Hence, for $y < 0$, $\phi(x,y)$ is a solution of (2.23) and (2.22) together with the Dirichlet boundary condition $\phi(x,0) = 0$.

Suppose now that the problem in the lower semi-infinite strip $x \in [-l,l]$, $y < 0$ admits only the trivial solution. Then, as $\phi_y(x,0^-) = 0$, we have by (2.20)

$$\phi_y(x,0^+) = ihx\phi(x,0^+) \tag{2.26}$$

and the theorem follows. Hence, it remains necessary to show that no solution other than the trivial one exists to (2.23) and (2.22) for $y < 0$ together with $\phi(x,0) = 0$.

Applying the transformation $y \to -y$ $\phi \to \phi^*$, the problem for $y < 0$ becomes identical with the same problem for $y > 0$. Let then,

$$\lambda(h) = \inf_{\substack{\psi\in H^1(\mathcal{S},\mathbb{C})\\ \|\psi\|_{L^2(\mathcal{S})}=1}} \int_{\mathcal{S}} \left|(\boldsymbol{\nabla} - ihx\hat{\boldsymbol{j}})\psi\right|^2 dxdy, \tag{2.27a}$$

$$\lambda_D(h) = \inf_{\substack{\psi\in H^1(\mathcal{S},\mathbb{C})\\ \|\psi\|_{L^2(\mathcal{S})}=1\ ;\ \psi(x,0)=0}} \int_{\mathcal{S}} \left|(\boldsymbol{\nabla} - ihx\hat{\boldsymbol{j}})\psi\right|^2 .dxdy \tag{2.27b}$$

Suppose first that $h > h_{\mathcal{S}}$. Then, $\lambda_D(h) \geq \lambda(h) > 1$ (cf. [2]), and hence no nontrivial solutions exist to (2.22), (2.23), and $\phi(x,0) = 0$ (cf. [3]).

Hence, the upper critical field of (2.4) $\tilde{h}_{\mathcal{S}}$ must be exactly $h_{\mathcal{S}}$. Furthermore, let $h = h_{\mathcal{S}}$. In this case $\lambda(h_{\mathcal{S}}) = 1$ and hence $\lambda_D(h_{\mathcal{S}}) \geq 1$. If $\lambda_D(h_{\mathcal{S}}) > 1$ then $\phi(x,y) \equiv 0$ for $y < 0$ and hence, (2.5) is indeed a solution of (2.1). Even if $\lambda_D(h_{\mathcal{S}}) = 1$ no nontrivial solution can exist to (2.22), (2.23), and $\phi(x,0) = 0$, otherwise it must be the minimizer of the quadratic forms on the right-hand-side of both (2.27a) and (2.27b), and as no solution but the trivial one can satisfy both (2.25) and (2.26) on $y = 0$ the theorem is proved.

□

In addition to the integral equation it is possible to derive similar integral equations for the rectangle $R_L = \{(x,y)| -l \leq x \leq l\ ;\ 0 \leq y \leq L\}$ as the following theorem states:

Theorem 2 *Let $h = h_L$. Then, any solution of the problem*

$$-\left(\frac{\partial^2\psi^L}{\partial x^2} + \frac{\partial^2\psi^L}{\partial y^2}\right) + 2ihx\frac{\partial\psi^L}{\partial y} = \psi^L - h^2x^2\psi^L \qquad -l \leq x \leq l,\ 0 \leq y, \tag{2.28a}$$

$$\psi^L_x(\pm l, y) = 0 \qquad ; \qquad i\psi^L_y(x,0) + h_Lx\psi^L(x,0) = i\psi^L_y(x,L) + h_Lx\psi^L(x,L) = 0 \tag{2.28b}$$

must satisfy

$$-\frac{1}{2}[\psi^L(x,0) + \psi^L(x,L)] = \frac{1}{L}PV\sum_{n=-\infty}^{\infty}\hat{\psi}_L(x,n\chi), \tag{2.29a}$$

$$-\frac{1}{2}[\psi^L(x,L) - \psi^L(x,0)] = \frac{1}{L}PV\sum_{n=-\infty}^{\infty}\tilde{\psi}_L(x,(n+1/2)\chi), \tag{2.29b}$$

where $\chi = 2\pi/L$ and

$$\hat{\psi}_L(x,\omega) = -i\int_{-l}^{l}(hs-\omega)G(x,s,\omega)[\psi(s,L) - \psi(s,0)]ds, \tag{2.30a}$$

$$\tilde{\psi}_L(x,\omega) = i\int_{-l}^{l}(hs-\omega)G(x,s,\omega)[\psi(s,0) + \psi(s,L)]ds. \tag{2.30b}$$

Conversely, suppose for some h $\exists f, g \in C^2[-l, l]$ *satisfying* $f'(\pm l) = g\prime(\pm l) = 0$ *and*

$$-\frac{1}{2}g = \frac{1}{L}PV \sum_{n=-\infty}^{\infty} \hat{\phi}_L(x, n\chi), \tag{2.31a}$$

$$-\frac{1}{2}f = \frac{1}{L}PV \sum_{n=-\infty}^{\infty} \tilde{\phi}_L(x, (n+1/2)\chi), \tag{2.31b}$$

where

$$\hat{\phi}_L(x, \omega) = -i \int_{-l}^{l} (hs - \omega) G(x, s, \omega) f(s) ds, \tag{2.32a}$$

$$\tilde{\phi}_L(x, \omega) = i \int_{-l}^{l} (hs - \omega) G(x, s, \omega) g(s) ds. \tag{2.32b}$$

Then, $h \leq h_L$*. Furthermore, if* $h = h_L$*, then the function*

$$\phi^L(x, y) = \frac{1}{L}PV \sum_{n=-\infty}^{\infty} \hat{\phi^L}(x, n\chi) e^{in\chi y} \tag{2.33}$$

is a solution of (2.28)

Proof: The proof of (2.30) is very similar to the proof of (2.2) in theorem1. Interested readers can find the details in [2]. The converse statement is proved as follows: Let

$$\varphi^L(x, y) = \frac{1}{L}\text{PV} \sum_{n=-\infty}^{\infty} \tilde{\phi^L}(x, n\chi) e^{in\chi y}, \tag{2.34a}$$

$$\Psi^L = \phi^L + \varphi^L e^{\frac{1}{2}i\chi y}. \tag{2.34b}$$

Following the same steps used in the proof of the converse statement in theorem 1 it is not difficult to show that

$$-\left(\frac{\partial^2 \Psi^L}{\partial x^2} + \frac{\partial^2 \Psi^L}{\partial y^2}\right) + 2ihx\frac{\partial \Psi^L}{\partial y} = \Psi^L - h^2 x^2 \Psi^L \qquad -l \leq x \leq l,\ 0 \leq y \tag{2.35a}$$

$$\psi^L_x(\pm l, y) = 0 \qquad ; \qquad \psi^L(x, 0) = \Psi^L(x, L) = 0\ . \tag{2.35b}$$

One can then define

$$\lambda^L(h) = \inf_{\substack{\psi\in H^1(R_L,\mathbb{C}) \\ \|\psi\|_{L^2(R_L)}=1}} \int_{R_L} \left|(\boldsymbol{\nabla} - ihx\hat{\boldsymbol{j}})\psi\right|^2 dxdy \tag{2.36a}$$

$$\lambda_D^L(h) = \inf_{\substack{\psi\in H^1(R_L,\mathbb{C}) \\ \|\psi\|_{L^2(R_L)}=1\,;\,\psi(x,0)=\psi(x,L)=0}} \int_{R_L} \left|(\boldsymbol{\nabla} - ihx\hat{\boldsymbol{j}})\psi\right|^2 dxdy \tag{2.36b}$$

As R_L is compact, $\lambda^L < \lambda_D^L$, and consequently, non-trivial solutions for (2.35) can exist only if $h < h_L$. Let then $h \geq h_L$ and suppose that the system (2.31) is satisfied by appropriate f and g. Then, by (2.32a) and (2.33) we have

$$\phi^L(x,L) - \phi^L(x,0) = f(x), \tag{2.37a}$$

$$\frac{\partial \phi^L}{\partial y}(x,L) - \frac{\partial \phi}{\partial y}(x,0) = ihxf(x). \tag{2.37b}$$

and hence

$$\phi_y^L(x,y)\Big|_{y=0}^{L} = ihx\,\phi^L(x,y)\Big|_{y=0}^{L}. \tag{2.38}$$

As $\Psi_L \equiv 0$ for $h \geq h_L$ we have

$$\phi_y(x,0) + \phi_y(x,L) = \left[\varphi_y^L(x,y) + \frac{1}{2}i\chi\varphi(x,y)\right]\Bigg|_{y=0}^{L}. \tag{2.39}$$

However, by (2.32b) and (2.34)

$$\varphi^L(x,L) - \varphi^L(x,0) = g(x), \tag{2.40a}$$

$$\frac{\partial \varphi^L}{\partial y}(x,L) - \frac{\partial \varphi}{\partial y}(x,0) = i\left(hx - \frac{1}{2}\chi\right)g(x). \tag{2.40b}$$

Since by (2.31a)

$$\phi(x,0) + \phi(x,L) = g, \tag{2.41}$$

the theorem is proved.

□

3 Isolation of the onset field

In the following we demonstrate an application of Theorem 1: we show that the onset field $h_{\mathcal{S}}$ for the semi-infinite strip $\mathcal{S}$ is isolated. Explicitly we prove the following result

Theorem 3 *There exists $\delta > 0$ such that $\forall h \in (h_{\mathcal{S}} - \delta, h_{\mathcal{S}})$ (2.1) is satisfied by the trivial solution only.*

It should first be noted that in some cases the onset field is not isolated. Consider, for instance, the onset problem for an infinite strip

$$-\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) + 2ihx\frac{\partial\psi}{\partial y} = \psi - h^2x^2\psi, \qquad -l \le x \le l, \quad y \in \mathbb{R}, \tag{3.1a}$$

$$\psi_x(\pm l, y) = 0. \tag{3.1b}$$

Substituting the ansatz

$$\psi(x,y) = F(x)e^{-i\omega y}, \tag{3.2}$$

we obtain

$$F'' - \left[(hx-\omega)^2 - 1\right]F = 0, \quad F'(\pm l) = 0. \tag{3.3a,b}$$

Let $h(\omega)$ denote the largest value of h, for given $\omega \in \mathbb{R}$, for which (3.3) admits non-trivial solutions. It can be shown (cf. [5]) that the onset field in this case,

$$h_{1D} = h(\omega_0) = \sup_{\omega\in\mathbb{R}} h(\omega). \tag{3.4}$$

In addition to (3.4) it was shown (cf. [5]) that $h(\omega)$ is continuous, and that for every $\omega \ne \pm\omega_0$ $h(\omega) \ne h_{1D}$. Hence, $h(\omega)$ is not constant near ω_0 and therefore, h_{1D} is not isolated.

In order to prove theorem 3 we need the following auxiliary result:

Lemma 1 *The integral operator $\mathcal{L} : C[-l,l] \to C[-l,l]$ defined by*

$$\mathcal{L}f = \frac{1}{2\pi} PV \int_{-\infty}^{\infty} d\omega \int_{-l}^{l} (hs - \omega) G(x, s, \omega) f(s) ds \tag{3.5}$$

is compact, and holomorphic in h for $h > h_{1D}$.

Proof: To prove the lemma we show that

$$\mathcal{L}f = \int_{-l}^{l} K(x, s, h) f(s) ds \tag{3.6}$$

where $K(x, s, h)$ is continuous in both x and s and holomorphic in h. To this end we write $\mathcal{L}$ in the form

$$\mathcal{L}f = \frac{1}{2\pi} \int_{0}^{\infty} d\omega \int_{-l}^{l} \left[(hs - \omega) G(x, s, \omega) + (hs + \omega) G(x, s, -\omega) \right] f(s) ds. \tag{3.7}$$

We then apply to (3.7) the asymptotic expansion (2.11) together with the identity (2.12) to obtain, for $x \leq s$ and $\omega \gg 1$,

$$\begin{aligned}(hs - \omega) G(x, s, \omega) + (hs + \omega) G(x, s, -\omega) \sim \\ \sinh\left\{ \frac{h}{2}(x^2 - s^2) \right\} e^{-\omega(s-x)} + \frac{1}{\omega} \left\{ (s - x) F_1(x, s) e^{-\omega(s-x)} + \right. \\ \left. +(2l - s - x) F_2(x, s) e^{-\omega(2l-s-x)} + (2l + s + x) F_3(x, s) e^{-\omega(2l+s+x)} \right\} + O(\omega^{-2}).\end{aligned} \tag{3.8}$$

The exact form of the functions F_1, F_2, and F_3 can be obtained but need not concern us since they are all bounded. Similar expansion can be obtained for $x \geq s$.

By (3.8)

$$\int_{0}^{\infty} \left[(hs - \omega) G(x, s, \omega) + (hs + \omega) G(x, s, -\omega) - \sinh\left\{ \frac{h}{2}(x^2 - s^2) \right\} e^{-\omega(s-x)} \right] d\omega \tag{3.9}$$

converges uniformly. The monotone convergence theorem guarantees, therefore, that

$$\int_0^\infty d\omega \int_{-l}^{x} \left[(hs-\omega)G(x,s,\omega)+(hs+\omega)G(x,s,-\omega)\right] f(s)ds =$$
$$\int_{-l}^{x} ds \int_0^\infty \left[(hs-\omega)G(x,s,\omega)+(hs+\omega)G(x,s,-\omega)\right] f(s)d\omega. \quad (3.10)$$

A similar equality can be derived for $x \geq s$.

Let then,

$$K(x,s,h) = \int_0^\infty \left[(hs-\omega)G(x,s,\omega)+(hs+\omega)G(x,s,-\omega)\right] d\omega. \quad (3.11)$$

It is possible to express $K(x,s,h)$ in the form

$$K(x,s,h) = K_1(x,s,h) + K_2(x,s,h), \quad (3.12a)$$

where

$$K_1(x,s,h) = \begin{cases} \frac{\sinh\left\{\frac{h}{2}(x^2-s^2)\right\}}{s-x} & x \neq s, \\ 0 & x = s, \end{cases} \quad (3.12b)$$

and $K_2(x,s,h)$ is continuous in view of the above-established uniform convergence.

To prove that $\mathcal{L}$ is holomorphic in h we show that $K(x,s,h)$ is holomorphic in some neighbourhood of the real line for $h > h_{1D}$. The size of this neghibourhood should be fixed for all $(x,s) \in [-l,l]^2$. Then, theorem VII-2.6 in [13] would prove that $\mathcal{L}$ is holomorphic as well.

Clearly, $K_1(x,s,h)$ is an entire function in h for all x and s. Inasmuch as $(hs-\omega)G(x,s,\omega)+(hs+\omega)G(x,s,-\omega)$ is holomorphic for $h > h_{1D}$, and in view of the uniform convergence of the integral (3.9), $K_2(x, s, h)$ is holomorphic as well for $h > h_{1D}$.

□

To complete the proof of theorem 3 we consider the equation

$$\mathcal{L}(h)f = \lambda f. \tag{3.13}$$

By theorem 1, non-trivial solutions to (3.13) would exist for $h = h_{\mathcal{S}}$, $\lambda = 1/2$. By lemma 1 this eigenvalue has finite multiplicity, and hence, since $\mathcal{L}$ is holomorphic and by theorem VII-1.9 in [13] we obtain that $\lambda = 1/2$ is satisfied either

1. for all $h > h_{1D}$,

or,

2. for a finite set of values in $[h_{1D} + \delta, h_{\mathcal{S}}]$ for any $\delta > 0$.

However, (2.4) does not admit any non-trivial solutions for $h > h_{\mathcal{S}}$ by theorem 1. Hence, the second option is the viable one, which proves theorem 3.

4 The onset field in the long rectangle limit

It was demonstrated in [2] that the onset field of the rectangle R_L tends to the onset field of the semi-infinite strip $\mathcal{S}$, or that $h_L \to h_{\mathcal{S}}$. In the following we attempt to derive the next order term of the asymptotic behaviour in that limit. To this end, we first derive a new set of integral equations in replace of (2.4) and (2.28), which, in contrast to the latter set of equations, is real and self-adjoint.

Consider first the problem in $\mathcal{S}$. It is not difficult to show, using integration by parts, that

$$\frac{1}{2\pi}\mathrm{PV}\int_{-\infty}^{\infty} e^{i\omega y}\left[i\omega\hat{\psi}(x,\omega)-\psi(x,0)\right]d\omega = \begin{cases} \frac{\partial\psi(x,y)}{\partial y}, & y>0, \\ \frac{1}{2}\frac{\partial\psi(x,y)}{\partial y}(x,0), & y=0, \\ 0, & y<0. \end{cases} \cdot \tag{4.1}$$

Substituting the above identity, (2.8), and (2.9) into (2.1c) we obtain

$$\frac{1}{2\pi}\mathrm{PV}\int_{-\infty}^{\infty}\left[(hx-\omega)\int_{-l}^{l}G(x,s,\omega)(hs-\omega)\psi(s,0)ds-\psi(x,0)\right]d\omega=0. \tag{4.2}$$

In a similar manner we can obtain the equation

$$\frac{1}{L}\mathrm{PV}\sum_{n=-\infty}^{\infty}\left[(hx-n\chi)\int_{-l}^{l}G(x,s,n\chi)(hs-n\chi)\left.\psi(s,y)\right|_0^L ds-\left.\psi(x,y)\right|_0^L\right]=0 \tag{4.3}$$

for the problem in R_L.

Equation (4.2), though complicated, has a major advantage on (2.4): it is a real self-adjoint equation, and is, therefore, the most natural way to present (2.1) in Fourier space. Obviously, (4.2) must have real solutions, for if $f(x)$ is a solution then so is $\bar{f}(x)$. The fact that if $f(x)$ is a solution of (2.4) then $\bar{f}(x)$ is not, may seem puzzling but can be easily resolved: It is not difficult to show, following the same procedure used to prove theorem 1, that if $f(x)$ is a solution of (4.2), then

$$\phi(x,y)=\frac{1}{2\pi}\mathrm{PV}\int_{-\infty}^{\infty}e^{i\omega y}\int_{-l}^{l}i(hs-\omega)G(x,s,\omega)f(s)dsd\omega \tag{4.4}$$

is a solution of (2.1). However, if we substitute $f(x)=\bar{\psi}(x,0)$, where $\psi(x,y)$ is a solution of (2.1), we obtain $\phi(x,y) = -\bar{\psi}(x,-y)$, or equivalently

$$\phi(x,y) = \begin{cases} 0, & y > 0, \\ \tfrac{1}{2}\bar{\psi}(x,0), & y = 0, \\ -\bar{\psi}(x,-y), & y < 0. \end{cases} \tag{4.5}$$

We notice that $\phi(x,0^-) \neq 0$, which explains why $\phi(x,0^-)$ cannot solve (2.4). Consequently, there are solutions of (4.2) which do not satisfy (2.4), that when substituted into (4.5) yield solution $\phi(x,y)$ of (2.1) for which $\phi(x,0^+) \neq f(x)$. Hence, keeping the above in mind, we can treat (4.2) as a real self-adjoint equation which is equivalent to (2.1).

We seek an approximation for a solution $(f_L(x), h_L)$ of (4.3). We expect that the leading order term, as $L \to \infty$, would be a solution of (4.2). Hence, it is necessary to approximate the difference between the left-hand-sides of (4.2) and (4.3) in that limit.

In order to approximate the above difference we first rewrite (4.3) in the form

$$\frac{1}{L}\sum_{n=-\infty}^{\infty}\left[\hat{\Phi}_L(x,n\chi) + \hat{\Phi}_L(x,-n\chi)\right] = 0, \tag{4.6}$$

wherein

$$\hat{\Phi}_L(x,\omega) = (h_L x - \omega)\int_{-l}^{l}(h_L s - \omega)G(x,s,\omega)f_L(s)ds - f_L(x). \tag{4.7}$$

It is not difficult to show, using (2.11), that

$$\Phi_L(x,\omega) + \Phi_L(x,-\omega) \sim -\frac{h^2x^2}{\omega^2}f_L(x) \tag{4.8a}$$

and that

$$\frac{\partial\Phi_L}{\partial\omega}(x,\omega) - \frac{\partial\Phi_L}{\partial\omega}(x,-\omega) \sim \frac{2h^2x^2}{\omega^3}f_L(x) \tag{4.8b}$$

as $\omega \to \infty$. Hence,

$$\int_{-\infty}^{\infty} |\Phi_L(x,\omega) + \Phi_L(x,-\omega)| + \left|\frac{\partial \Phi_L}{\partial \omega}(x,\omega) - \frac{\partial \Phi_L}{\partial \omega}(x,-\omega)\right| d\omega < \infty. \tag{4.9}$$

Consequently, $\Phi_L(x,\omega) + \Phi_L(x,-\omega)$ is both absolutely integrable and of bounded variation in ω, and thus, one can apply the Poisson summation formula to obtain

$$\frac{1}{L}\mathrm{PV} \sum_{n=-\infty}^{\infty} \hat{\Phi}_L(x, n\chi) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \mathrm{PV} \int_{-\infty}^{\infty} e^{-i\omega nL} \hat{\Phi}_L(x,\omega) d\omega. \tag{4.10}$$

The function $\hat{\Phi}_L(x,\omega)$ is holomorphic in ω except for a countable number of points where

$$U'\big(\xi(-l,\omega)\big)U'\big(-\xi(l,\omega)\big) - U'\big(\xi(l,\omega)\big)U'\big(-\xi(-l,\omega)\big) = 0. \tag{4.11}$$

For $h > h_{1D}$ there exist a strip of analyticity of $\hat{\Phi}_L(x,\omega)$, $\Im\omega < \gamma$, where the above equation is not satisfied [2]. Hence,

$$\mathrm{PV} \int_{-\infty}^{\infty} e^{-i\omega nL} \hat{\Phi}_L(x,\omega) d\omega \sim O(e^{-\gamma nL}) \qquad \text{as } L \longrightarrow \infty. \tag{4.12}$$

Consequently, we can obtain the following approximate form of: (4.3)

$$\mathrm{PV} \int_{-\infty}^{\infty} \hat{\Phi}_L(x,\omega) d\omega \cong -\mathrm{PV} \int_{-\infty}^{\infty} e^{-i\omega L} \left[\hat{\Phi}_L(x,\omega) + \hat{\Phi}_L(x,-\omega)\right] d\omega + O(e^{-2\gamma L}). \tag{4.13}$$

As the right-hand-side of (4.13) is of $O(e^{-\gamma L})$ we substitute into both sides the formal asymptotic expansion

$$h_L = h_{\mathbb{S}} + e^{-\gamma L} h^{(1)}(L) + O(e^{-2\gamma L}), \tag{4.14a}$$

$$f_L = f_{\mathbb{S}} + e^{-\gamma L} f^{(1)}(L) + O(e^{-2\gamma L}), \tag{4.14b}$$

where $h^{(1)}(L)$ and $f^{(1)}(L)$ are $O(1)$ as $L \to \infty$. Let

$$\hat{\Phi}_{\mathbb{S}}(x,\omega) = i(h_{\mathbb{S}} x - \omega) \int_{-l}^{l} (h_{\mathbb{S}} s - \omega) G(x,s,\omega) f_{\mathbb{S}}(s) ds - f_{\mathbb{S}}(x). \tag{4.15}$$

Using the same complex plane techniques applied in [2], it is not difficult to show that

$$\int_{-\infty}^{\infty} \hat{\Phi}_{\mathcal{S}}(x,\omega) e^{i\omega L} d\omega \cong 2\pi i e^{-\gamma L}\left[e^{i\alpha L}\mathrm{Res}\left\{\omega_+, \hat{\Phi}_{\mathcal{S}}\right\} + e^{-i\alpha L}\mathrm{Res}\left\{\omega_-, \hat{\Phi}_{\mathcal{S}}\right\}\right] + O(e^{-2\gamma L}), \tag{4.16}$$

where $\omega_+ = \alpha + i\gamma$ and $\omega_- = -\alpha + i\gamma$ are the closest points to the real axis in the upper half of the complex plane where (4.11) is satisfied (in the case $\alpha = 0$ which is expected to take place for sufficiently small l we have $\omega_+ = \omega_-$). The following assumptions were made while deriving (4.16):

1. There are at most two poles of $\hat{\Phi}_{\mathcal{S}}(x,\omega)$ on the line $\Im\omega = \gamma$.

2. Either $\mathrm{Res}\left\{\omega_+, \hat{\Phi}_{\mathcal{S}}\right\} \neq 0$ or $\mathrm{Res}\left\{\omega_-, \hat{\Phi}_{\mathcal{S}}\right\} \neq 0$.

If assumption 1 is not satisfied it would be necessary to introduce additional terms into the right-hand-side of (4.16) corresponding to the additional roots of (4.11). If assumption 2 is violated, higher order terms must be obtained.

In addition to the above assumptions it is convenient to assume that both ω_+ and ω_- are simple poles of $\hat{\Phi}_{\mathcal{S}}$, an assumption which can be proved in the limit $l \to \infty$. In [2] it is demonstrated that

$$\mathrm{Res}\left\{\omega_+, \hat{\psi}\right\} = AC_+ g(x), \tag{4.17a}$$

where

$$g(x) = U(\xi(x,\omega_0^+))U'(\xi(l,-\omega_0^+)) + U(-\xi(x,\omega_0^+))U'(-\xi(l,-\omega_0^+)) \tag{4.17b}$$

$$C_+ = \int_{-l}^{l} g(s)(h_{\mathcal{S}}s - \omega_+)f_{\mathcal{S}}(s)ds \tag{4.17c}$$

and A is a constant which can be determined in principle. The other residue is given by

$$\mathrm{Res}\left\{\omega_-, \hat{\psi}\right\} = \bar{A}\bar{C}_-\bar{g}(-x) \tag{4.17d}$$

where

$$C_- = \int_{-l}^{l} g(s)(h_{\mathrm{S}} s - \omega_+) f_{\mathrm{S}}(-s) ds. \tag{4.17e}$$

Substituting (4.14-4.17) into (4.13), and applying the transformation $\omega \to h\omega$ to the integration dummy variable, we obtain, by equating terms of $O(e^{-\gamma L})$,

$$\begin{aligned}
&\mathrm{PV}\int_{-\infty}^{\infty}\left[(x-\omega)\int_{-l}^{l} {h_{\mathrm{S}}}^3 G(x,s,\omega)(s-\omega) f^{(1)}(s)ds - h_{\mathrm{S}} f^{(1)}(x)\right] d\omega = \\
&\qquad = i\left\{Ae^{i\alpha L}\left[(h_{\mathrm{S}}x-\omega_+)C_+ g(x) + (h_{\mathrm{S}}x+\omega_+)C_- g(-x)\right] - \right.\\
&\qquad\quad \left. -\bar{A}e^{-i\alpha L}\left[(h_{\mathrm{S}}x-\omega_-)\bar{C_-}\bar{g}(-x) + (h_{\mathrm{S}}x+\omega_-)\bar{C_+}\bar{g}(x)\right]\right\} \\
&\quad - h^{(1)}\mathrm{PV}\int_{-\infty}^{\infty}\left\{(x-\omega)\int_{-l}^{l}\frac{\partial}{\partial h}\left[h^3 G(x,s,\omega)\right]_{h=h_{\mathrm{S}}}(s-\omega) f_{\mathrm{S}}(s)ds - f_{\mathrm{S}}(x)\right\} d\omega.
\end{aligned} \tag{4.18}$$

Multiplying by $f_{\mathrm{S}}(-x)$ and integrating over $[-l, l]$ yields, after a few manipulations, the solvability condition

$$h^{(1)}\mathrm{PV}\int_{-\infty}^{\infty}\int_{-l}^{l}\left\{(x-\omega)\int_{-l}^{l}\frac{\partial}{\partial h}\left[h^3 G(x,s,\omega)\right]_{h=h_{\mathrm{S}}}(s-\omega) f_{\mathrm{S}}(s)ds - f_{\mathrm{S}}(x)\right\} f_{\mathrm{S}}(-x)dxd\omega = 0 \tag{4.19}$$

It seems plausible to assume here that $h^{(1)} = 0$ as there is no particular reason to believe that the integral it multiplies vanishes. Furthermore, multiplying (4.8) by $f_{\mathrm{S}}(x)$ and integrating we obtain

$$h^{(1)}\mathrm{PV}\int_{-\infty}^{\infty}\int_{-l}^{l}\left\{(x-\omega)\int_{-l}^{l}\frac{\partial}{\partial h}\left[h^3G(x,s,\omega)\right]_{h=h_{\mathcal{S}}}(s-\omega)f_{\mathcal{S}}(s)ds-f_{\mathcal{S}}(x)\right\}f_{\mathcal{S}}(x)dxd\omega =$$

$$i\left[Ae^{i\alpha L}\left(C_+^2-C_-^2\right)-\bar{A}e^{-i\alpha L}\left(\bar{C}_+^2-\bar{C}_-^2\right)\right], \quad (4.20)$$

and hence, if the left-hand side of (4.20) does not vanish, there exists L such that $h_L < h_{\mathcal{S}}$ which contradicts a result which was rigorously proved in [2]. The above identity, thus, leads us to another important conclusion: $C_+ = C_-$. Recall that the asymptotic behaviour of ψ in $\mathcal{S}$ as $y \to \infty$ is

$$\psi \sim Ce^{-\gamma y}\left[C_+g(x)e^{i\alpha y}+\overline{C_-g(-x)e^{i\alpha y}}\right], \quad (4.21)$$

and hence $\psi(x,y) \sim \bar{\psi}(-x,y)$. In other words we can say that ψ is asymptotically symmetric.

If indeed, as we expect, $h^{(1)} = 0$, then repeating the above procedure for higher order terms yields

$$h_L - h_{\mathcal{S}} \sim o(e^-\lambda L) \quad (4.22)$$

for any $\lambda > 0$. Thus, obtaining the leading asymptotic behaviour is a difficult task which is left to future research.

5 Numerical solution

We conclude this section by presenting a numerical solution of (2.31). To achieve this end we first need the estimates

$$\frac{1}{L}\sum_{n=N+1}^{\infty}\left[\hat{\phi}(x,n\chi)+\hat{\phi}(x,-n\chi)\right] \cong -\frac{ihx}{\pi\chi}S_1(N)f+O\left(\frac{1}{N^3\chi^3}\right) \text{ as } N\to\infty \quad (5.1a)$$

and

$$\frac{1}{L}\sum_{n=N+1}^{\infty}\left[\hat{\phi}(x,(n+1/2)\chi)+\hat{\phi}(x,-(n+1/2)\chi)\right]\cong-\frac{ihx}{\pi\chi}S_2(N)g+O\left(\frac{1}{N^3\chi^3}\right)\ \text{as}\ N\to\infty \tag{5.1b}$$

where

$$S_1(N)=\sum_{n=N+1}\frac{1}{n^2}=\frac{\pi^2}{6}-\sum_{n=1}^{N}\frac{1}{n^2} \tag{5.1c}$$

$$S_2(N)=\sum_{n=N+1}\frac{1}{(n+1/2)^2}=\frac{\pi^2}{2}-4-\sum_{n=1}^{N}\frac{1}{(n+1/2)^2}. \tag{5.1d}$$

For $N\chi$ sufficiently large we may, thus, rewrite (2.31) in the approximate form

$$\frac{1}{2}g=\frac{i}{L}\sum_{n=-N}^{N}\int_{-l}^{l}(hs-n\chi)G(x,s,n\chi)f(s)ds-\frac{ihx}{\pi\chi}S_1(N)f \tag{5.2a}$$

$$\frac{1}{2}f=\frac{i}{L}\sum_{n=-N-1}^{N}\int_{-l}^{l}(hs-(n+1/2)\chi)G(x,s,(n+1/2)\chi)g(s)ds-\frac{ihx}{\pi\chi}S_2(N)g \tag{5.2b}$$

The solution of the above system was obtained by applying a quadrature method. For each n the integrals in (5.2) were evaluated utilizing the trapezium rule except for a close neighbourhood of $x=s$. While it is still reasonable to assume $f(s)\approx f(x)$ for $s\in[x-\Delta,x+\Delta]$ it would be inaccurate to claim that $G(x,s,n\chi)\approx G(x,x,n\chi)$ in the same interval unless $n\chi\ll 1/\Delta$ in view of (2.11). It is still possible, however, to use the approximation

$$\int_{x-\Delta}^{x+\Delta}(hs-n\chi)G(x,s,n\chi)f(s)ds\approx f(x)\int_{x-\Delta}^{x+\Delta}(hs-n\chi)G(x,s,n\chi)ds, \tag{5.3}$$

and to evaluate the integral on the right-hand-side using the trapesian rule with a finer grid (which should depend on n). We note that while the above approximation is $O(\Delta)$ accurate, it is not difficult to obtain $O(\Delta^2)$ security by appropriately modifying it.

Once the matrices representing the integrals in (5.2) are calculated, we can represent the system in the form

$$\sum_{k=-J}^{J}\sum_{j=-J}^{J}\left\{\left[K_{mj}^{(1)}(h)--\frac{ihx}{\pi\chi_m}S_1(N)\delta_{mj}\right]\left[K_{jk}^{(2)}(h)--\frac{ihx}{\pi\chi_j}S_2(N)\delta_{jk}\right]-\delta_{mk}\right\}g_k=0$$

$$\forall -J\le m\le J, \quad (5.4)$$

wherein $g_k = g(-l+k\Delta)$, $\Delta = l/J$ and $K^{(1)}$ and $K^{(2)}$ respectively represent the integrals on (5.2a) and (5.2b).

Non-trivial solutions of (5.4) exist if and only if the determinant of the matrix multiplying the vector $[g_{-J},\ldots,g_J]^T$ vanishes. We, therefore, look for the values of h where this determinant vanishes, to which end we apply the Van Wijngaarden-Dekker-Brent Method [16]. The largest root is the onset field h_L. For sufficiently large L this root is also an extremal point of the determinant and then Brent's method of minimization [16] was applied.

After the onset field has been found, (4.11) can be solved. The root, closest to the real axis, of this equation, provides us with the asymptotic behaviour of ψ in $\mathcal{S}$ as $y\to\infty$ according to (4.21). A globally convergent algorithm, base on the quasi-Newton method of minimization was applied [16].

Figure 1 displays the dependence of the various parameters on l for the semi-infinite strip case. The solid line, the dashed line, and the dotted line respectively denote $h_{\mathcal{S}}(l)$, $\alpha(l)$, and $\gamma(l)$. The solution for this case was obtained by letting $L\to\infty$. Practically, $L=20$ was substituted for $l\le 0.5$, $L=20l$ was substituted for $0.5<l\le 1$, and $L=10l$ for $l>1$. In view of the fast exponential decay of $h_L-h_{\mathcal{S}}$ the error is expected to be diminishingly small as the numerical error is expected to be around $2-3$ percents.

Figure 1 provides us with a few interesting observations:

1. The asymptotic value of $h_{\mathbb{S}}$ in the limit $l \to \infty$, which is expected to be the onset field for the quarter plane $\{(x, y)|\ 0 < x, y\}$, is approximately 1.96, in accordance with the calculations of Schweigert & Peters [18], and in small disagreement with those of Jadallah *et al* [11].

2. There is a critical value of l (denote it by l_c) such that for $l \leq l_c$ α vanishes, but for $l > l_c$ α is positive. Similar behaviour is observed in the infinite-strip (which is essentially one-dimensional) case, where for sufficiently small l asymmetric modes cease to exist [5].

3. The rate of decay of ψ in y, or γ, reaches a maximum exactly at $l = l_c$. Apparently, the roots of (4.11) at ω_+ and ω_-, which are simple for $l > l_c$ combine into a double root at $l = l_c$. Then, one of the roots travels downward along the imaginary axis.

4. As $l \to \infty$ α tends to be a linear function of l. It is not difficult to show that $U'(-\xi(l, \omega))$ becomes exponentially small in this limit. Hence, $-\xi(l, \omega)$ must correspond to the root, which is the closest to the real line in the upper half plane, of U'.

5. As $l \to \infty$ $h_{\mathbb{S}}$ tends to infinity, in accordance with results for thin films.

Figure 2 displays the dependence of h_L on L for two different values of l: the thick corresponds to $l = 0.5$ whereas the thin curve correspons to $l = 1$. Note that as $L \to 0$ h_L tends to infinity, which is again in accordance with results for the thin infinite strip and

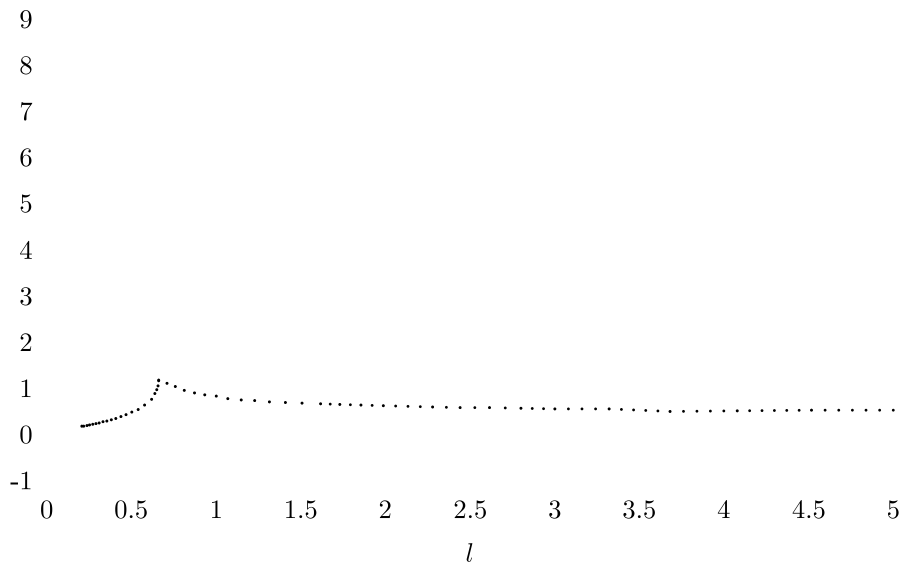

Figure 1: Dependence of h_S (denoted by the solid curve), γ (denoted by the dotted curve), and α (denoted by the dashed curve) on l for the semi-infinite strip case.

semi-infinite strip limits. In addition, one can observe that for $l = 1$ h_L is not monotonic, but attains a local maximum at $L \sim 3.2$ and a local minimum at $L \sim 2.6$.

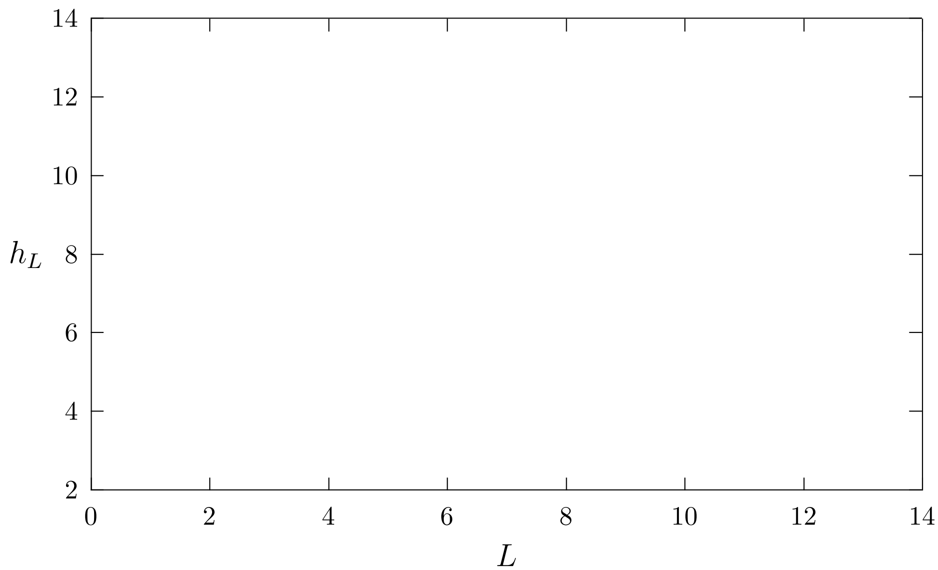

Figure 2: Dependence of h_L on l for the cases $l = 1$, denoted by the thin curve, and the case $l = 0.5$ denoted by the thick curve.

6 Conclusion

We have transformed, using Fourier analysis, in § 2, the linearized Ginzburg-Landau equations in both semi-infinite strips and rectangles into an equivalent set of integral equations: we proved that the upper critical field of the integral equations coincides with the onset field for the relevant domain. These integral equations are by no means the most natural way to present the linearized Ginzburg-Landau equations in Fourier space, as they are non-self-adjoint. Nevertheless, we showed in § 3 that the integral operators are of Fredholm type with kernels holomorphic in h. It was then possible to use these properties to show that the onset field in semi-infinite strips $h_{\mathbb{S}}$ is isolated, i.e., $\exists \delta > 0 : h \in (h_{\mathbb{S}} - \delta, h_{\mathbb{S}}) \Rightarrow$ no non-trivial solution exists.

The integral equations are also suitable for numerical solution. In § 5 the onset field was calculated numerically by solving the integral equations for rectangles. The numerical scheme's main advantage is that it can be applied to arbitrarily long rectangles. When both the width and the length of the rectangle become large the method is still at least copmperable with othe techniques [18, 11].

In addition to the integral equations developed in § 2, another set of integral equations were obtained in § 4 for both rectangles and semi-infinite strips. These equations seem to be inapplicable for numerical purposes, nevertheless they are both real and self-adjoint, and hence, it was natural to apply them in the asymptotic expansion of the onset field in the long rectangle limit (where the length of one side is kept constant) . It was found, by means of this formal expansion, that the onset field converges faster than any exponential rate to the onset field in the corresponding semi-infinite strip. Furthermore, it was demonstrated that the onset mode in a semi-infinite strip is aymptotically symmetric.

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