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Averaging of dilute random media: a rigorous proof of the Clausius-Mossotti formula

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Abstract

We consider a large number of randomly dispersed spherical, identical, inclusions in a bounded domain, with conductivity different than that of the host medium. In the dilute limit, with some mild assumption on the first few marginal probability densities (no periodicity or stationarity are assumed), we prove convergence in H^1 norm of the expectation of the solution of the steady state heat equation, to the solution of an effective medium problem, where the conductivity is given by the Clausius-Mossotti formula. Error estimates are provided as well.

1. Introduction

Consider N spherical inclusions of conductivity σ and radius ϵ immersed in a different medium of conductivity 1. Prescribing the temperature (or the electric potential) on the boundary, the temperature field inside can be described as the unique, weak solution of the problem

$$\begin{cases} \nabla\cdot\big(a(x,\eta_1,\dots,\eta_N)\nabla\phi\big)=0 & \text{in } \Omega\,, \\ \phi=f & \text{on } \partial\Omega\,. \end{cases} \tag{1.1}$$

In the above, $\Omega\subset\mathbb{R}^3$ is bounded and smooth (say $C^{2,\alpha}$ for some positiveα), $\{\eta_i\}_{i=1}^N$ denote the spherical inclusion's centers, and

$$a(x,\eta_1,\dots,\eta_N)=\begin{cases} \sigma & \forall x\in\bigcup_{n=1}^N B(\eta_n,\epsilon)\,, \\ 1 & \forall x\in\Omega\setminus\bigcup_{n=1}^N B(\eta_n,\epsilon)\,, \end{cases} \tag{1.2}$$

and $f\in C^{2,\alpha}(\partial\Omega)$ for some $\alpha>0$.

The particles' centers are assumed to be randomly distributed according to the joint probability density function $f_N(\eta_1,\dots,\eta_N)$, which is assumed to be invariant to permutations of the centers as all particles are identical. Moreover, we assume that the inclusions cannot overlap, i.e.,

$$\exists 1\leq i<j\leq N:\ |\eta_i-\eta_j|<2\epsilon\Rightarrow f_N(\eta_1,\dots,\eta_N)=0\,, \tag{1.3}$$

and that no inclusion can cross the boundary, i.e.,

$$\exists 1 \leq i \leq N : \, d(\eta_i, \partial\Omega) < \epsilon \Rightarrow f_N(\eta_1, \ldots, \eta_N) = 0 \,. \tag{1.4}$$

Let

$$f_k(\eta_1, \ldots, \eta_k) = \int_{\Omega^{N-k}} f_N(\eta_1, \ldots, \eta_k, \eta_{k+1}, \ldots, \eta_N) \, d\eta_{k+1} \cdots d\eta_N \,,$$

denote the $k'th$ order marginal probability density. We assume here boundedness of the first five marginal densities

$$\|f_k\|_{L^\infty(\Omega^k)} \leq C \quad \forall 1 \leq k \leq 5 \,, \tag{1.5}$$

where C is independent of N and ϵ. We denote the expectation of any function $F(x, \cdot) \in L^1(\Omega^N)$, where $x \in \Omega$, by

$$\langle F(x, \cdot) \rangle = \int_{\Omega^N} F(x, \eta_1, \ldots, \eta_N) \, d\eta_1 \cdots d\eta_N \,. \tag{1.6}$$

We focus our attention on the small particle limit in a dilute (or dispersive [7]) medium, i.e.,

$$\bar{\beta} = \frac{4\pi}{3} \frac{N\epsilon^3}{|\Omega|} \ll 1 \,, \text{as } \epsilon \to 0 \,, \tag{1.7}$$

where $\bar{\beta}$ denotes the global volume fraction. Note that N may tend to infinity as $\epsilon \to 0$. As a matter of fact, we further assume that there exists $C > 0$ independent of both ϵ and $\bar{\beta}$ such that

$$\frac{\epsilon}{C} < \bar{\beta} \leq \frac{C}{\ln^4 \epsilon^{-1}} \,. \tag{1.8}$$

The above inequality can alternatively be represented as

$$\frac{C}{\bar{\beta}^2} \leq N \leq \bar{\beta} e^{C\bar{\beta}^{-1/4}} \,,$$

with a different value of C. Note that by (1.8) it follows that $N \geq C\epsilon^{-2}$ and hence $N \to \infty$ as $\epsilon \to 0$.

Define, next, the local volume fraction for all $x \in \Omega$

$$\beta(x) = N \int_{B(x,\epsilon) \cap \Omega_\epsilon} f_1(\eta) \, d\eta \,, \tag{1.9}$$

where

$$\Omega_\epsilon = \{x \in \Omega \,|\, d(x, \partial\Omega) > \epsilon\} \,.$$

Note that $\beta(x)$ is the the probability that $x \in \bigcup_{n=1}^{N} B(\eta_n, \epsilon)$. It follows from (1.5) that

$$\|\beta(\cdot)\|_\infty \leq C\bar{\beta} \,. \tag{1.10}$$

Where $\| \cdot \|_p$ denotes the $L^p(\Omega)$ norm ($p = \infty$ above). When L^p norms are evaluated over domains different than Ω, we shall include them explicitly in the notation.

Under the above assumptions we prove the following theorem

Theorem 1. *Let* $\phi(\cdot, \eta_1, \ldots, \eta_N) \in H^1(\Omega)$ *denote the unique weak solution of* (1.1)*. Suppose that* (1.5) *is satisfied. Let* ϕ_e *denote the solution of the effective medium problem*

$$\begin{cases} \nabla \cdot (a_e \nabla \phi_e) = 0 & \text{in } \Omega \\ \phi_e = f & \text{on } \partial\Omega \, , \end{cases} \tag{1.11}$$

where

$$a_e = 1 + \frac{3(\sigma - 1)}{2 + \sigma} \beta(x) \, . \tag{1.12}$$

Then, in the regime of (1.8)*, we have*

$$\| \langle \phi \rangle - \phi_e \|_{1,2} \leq C(\Omega, \sigma) \bar{\beta}^{3/2} \, , \tag{1.13}$$

where $\| \cdot \|_{1,2}$ *denotes the* $H^1(\Omega)$ *norm.*

Throughout the sequel, we always refer to solutions in a weak sense, including places in the text where we do not state that explicitly. Note that in Remark 2 we extend the result to a regime of $\bar{\beta}$ values which is larger than the one prescribed by (1.8). In return, we need to allow a greater error term on the right-hand-side of (1.13).

The effective medium formula (1.12) was derived by Mossotti (1850) and Clausius (1879) and is therefore known as the Clausius-Mossotti formula. For a formal derivation of this formula the reader is referred to [11], which brings the classical derivation of Maxwell (1873) (cf. [9]). A rigorous proof has been provided in a two-dimensional periodic setting by Rayleigh [14], for a proof in three dimensional periodic medium the reader is referred to [7]. More general periodic settings have also been considered [10, 1].

For random media Kozlov [6] and Papanicolaou & Varadhan [13] proved an almost sure convergence, as $\epsilon \to 0$, of a more general version of (1.1), but for the case when a is stationary (or statistically homogeneous), to an averaged equation with constant coefficients. An extension to the case where a is only locally stationary is presented in [3]. We note that [6, 13, 3] present all an effective qualitative analysis, but do not provide a technique for the quantitative evaluation of the effective medium, in contrast with the present work.

An interesting proof of (1.12) is presented in [7], where an almost-periodic configuration of the particles is assumed. A small random deviation of the inclusions' centers from the lattice points is allowed. The size of each sphere is also allowed to vary randomly on an interval $[0, r]$, where r is much smaller than the period, to secure a dilute medium. The present setting is, of course, more general.

Another approach was adopted by Berlyand and Mityushev [2], where a periodic array of cells is considered. Each cell contains a finite number of randomly dispersed inclusions, whose volume fraction is not necessarily small. They evaluate an effective conductivity, which is averaged both in event space and over the cell. Note that by Theorem 1 the effective medium problem (1.11) approximates the *local* expectation in event space in $H^1(\Omega)$ sense.

The rest of the contribution is dedicated to the proof of Theorem 1. In the next section we derive some preliminary estimates for media with one or two inclusions. Denoting the solution of (1.1) with a single inclusion by ψ_1 and without any inclusions ($a \equiv 1$) by $\bar{\phi}$ we set $\phi_1 = \psi_1 - \bar{\phi}$ to be the single inclusion effect. We estimate $\nabla \phi_1$ far away from the inclusion using classical potential theory. We obtain similar estimates for the two-inclusion problem.

In § 3, we demonstrate that the leading order correction to the average field, is obtained by averaging the sum of the contributions of each single inclusion, neglecting its interaction with the rest of the particles. More precisely, we prove that

$$\| \langle \phi \rangle - \bar{\phi} - N \langle \phi_1 \rangle \|_{1,2} \leq C(\Omega, \sigma) \bar{\beta}^{3/2} \, .$$

In § 4, we show that $\langle\phi_1\rangle$ can be estimated by the average of the solution of a single inclusion problem in $\mathbb{R}^3$ with a constant temperature gradient at infinity. That temperature gradient is given by $\nabla\bar{\phi}$ at the inclusion's center. The average of the temperature field can then be approximated by the solution of (1.11). We note that the classical derivation by Maxwell [9] of the Clausius-Mossotti formula, takes a somewhat similar approach by heuristically evaluating the average of the $\mathbb{R}^3$ solution.

In the last section, we highlight a few points, that were insufficiently emphasized within the analysis. Appendix A provides an estimate satisfied by the Green's function of the Laplace operator with homogeneous Dirichlet boundary conditions in Ω.

2. Preliminaries

In the following we provide some H^1 estimates for the solutions of one-particle and two-particle problems. To achieve this end we first derive, using standard potential theory an integral representation for the solution of (1.1).

2.1. An integral representation

Let ϕ denote a solution of (1.1). As ϕ is harmonic outside the particles we have [5], for all $x\in\Omega\setminus\bigcup_{n=1}^{N}B_n$,

$$\phi(x,\eta_1,\dots,\eta_N)=\int_{\bigcup_{n=1}^{N}\partial B_n^{out}\cup\partial\Omega}\Big[\frac{1}{4\pi|x-\xi|}\frac{\partial\phi}{\partial\nu}(\xi,\eta_1,\dots,\eta_N)-\phi(\xi,\eta_1,\dots,\eta_N)\frac{\partial}{\partial\nu}\Big(\frac{1}{4\pi|x-\xi|}\Big)\Big]\,ds_\xi\,, \tag{2.1}$$

where ds_ξ is an area element around $\xi\in\bigcup_{n=1}^{N}\partial B_n^{out}\cup\partial\Omega$, $\nu(\xi)$ is the outward normal on $\partial\Omega$ and the inner normal on each particle's surface, $B_n=B(\eta_n,\epsilon)$, and $\int_{\partial B_n^{out}}=\lim_{\delta\downarrow\epsilon}\int_{\partial B(\eta_n,\delta)}$. We note that by $\partial/\partial\nu$ we always mean $\nabla_\xi\cdot\nu$. Let $G:\Omega\times\Omega\to\mathbb{R}$ denote the Green's function of the Laplace operator with homogeneous Dirichlet boundary conditions in Ω, i.e.,

$$G(x,\xi)=g(x,\xi)+\frac{1}{4\pi|x-\xi|}\,, \tag{2.2}$$

where $g(\cdot,\xi)$ is harmonic in Ω, for all $\xi\in\Omega$, and satisfies $g=-1/(4\pi|x-\xi|)$ on $\partial\Omega$. By Green's Theorem

$$\int_{\bigcup_{n=1}^{N}\partial B_n^{out}\cup\partial\Omega}\Big[g(x,\xi)\frac{\partial\phi}{\partial\nu}(\xi,\eta_1,\dots,\eta_N)-\phi(\xi,\eta_1,\dots,\eta_N)\frac{\partial g}{\partial\nu}(x,\xi)\Big]\,ds_\xi=0\,.$$

Adding the above to (2.1), while using the fact that $G(x,\cdot)|_{\partial\Omega}=0$, yields

$$\begin{aligned}\phi(x,\eta_1,\dots,\eta_N)=\int_{\bigcup_{n=1}^{N}\partial B_n^{out}}&\Big[G(x,\xi)\frac{\partial\phi}{\partial\nu}(\xi,\eta_1,\dots,\eta_N)-\phi(\xi,\eta_1,\dots,\eta_N)\frac{\partial G}{\partial\nu}(x,\xi)\Big]\,ds_\xi\\ &-\int_{\partial\Omega}\phi(\xi,\eta_1,\dots,\eta_N)\frac{\partial G}{\partial\nu}(x,\xi)\,ds_\xi\,.\end{aligned} \tag{2.3}$$

Let $\bar{\phi}$ denote the unique solution of

$$\begin{cases}\Delta\bar{\phi}=0 & \text{in }\Omega\\ \bar{\phi}=f & \text{on }\partial\Omega\end{cases} \tag{2.4}$$

Note that by the assumptions on $\partial\Omega$ and f we have that $\bar{\phi} \in H^4(\Omega)$ and hence also $\bar{\phi} \in C^2(\Omega)$. Since

$$\bar{\phi} = -\int_{\partial\Omega} f \frac{\partial G}{\partial\nu}\, ds_\xi = -\int_{\partial\Omega} \phi \frac{\partial G}{\partial\nu}\, ds_\xi\,,$$

we obtain by (2.3) that for all $x \in \Omega \setminus \bigcup_{n=1}^{N} B_n$

$$\phi(x,\eta_1,\ldots,\eta_N) = \bar{\phi}(x) + \sum_{n=1}^{N} \int_{\partial B_n^{out}} \Big[G(x,\xi)\frac{\partial\phi}{\partial\nu}(\xi,\eta_1,\ldots,\eta_N) - \phi(\xi,\eta_1,\ldots,\eta_N)\frac{\partial G}{\partial\nu}(x,\xi)\Big]\, ds_\xi\,. \tag{2.5}$$

Applying Green's Theorem once again, using the fact that both $G(x,\cdot)$ and ϕ are harmonic inside B_n for all $1 \le n \le N$, when $x \in \Omega \setminus \bigcup_{n=1}^{N} B_n$, we obtain

$$\int_{\partial B_n^{in}} \Big[G(x,\xi)\frac{\partial\phi}{\partial\nu}(\xi,\eta_1,\ldots,\eta_N) - \phi(\xi,\eta_1,\ldots,\eta_N)\frac{\partial G}{\partial\nu}(x,\xi)\Big]\, ds_\xi = 0 \quad \forall 1 \le n \le N\,,$$

where $\int_{\partial B_n^{in}} = \lim_{\delta\uparrow\epsilon}\int_{\partial B(\eta_n,\delta)}$. Inasmuch as, for all $s \in \partial B(0,1)$, we have

$$\lim_{\delta\downarrow\epsilon}\phi(\eta_n+\delta s) = \lim_{\delta\uparrow\epsilon}\phi(\eta_n+\delta s) \quad ; \lim_{\delta\downarrow\epsilon}\frac{\partial\phi}{\partial\nu}(\eta_n+\delta s) = \sigma\lim_{\delta\uparrow\epsilon}\frac{\partial\phi}{\partial\nu}(\eta_n+\delta s)\,,$$

we obtain that

$$\int_{\partial B_n^{out}} \Big[\frac{1}{\sigma}G(x,\xi)\frac{\partial\phi}{\partial\nu}(\xi,\eta_1,\ldots,\eta_N) - \phi(\xi,\eta_1,\ldots,\eta_N)\frac{\partial G}{\partial\nu}(x,\xi)\Big]\, ds_\xi = 0 \quad \forall 1 \le n \le N\,.$$

Combining the above with (2.5), and then using the fact that ϕ is continuous on ∂B_n for all $1 \le n \le N$, yields

$$\phi(x,\eta_1,\ldots,\eta_N) = \bar{\phi}(x) + (\sigma-1)\sum_{n=1}^{N}\int_{\partial B_n}\phi(\xi,\eta_1,\ldots,\eta_N)\frac{\partial G}{\partial\nu}(x,\xi)\, ds_\xi \quad \forall x \in \Omega\setminus\bigcup_{n=1}^{N} B_n\,. \tag{2.6}$$

2.2. Single inclusion

We next define the one-particle problem. Let

$$\Omega_\epsilon = \{x \in \Omega \,|\, d(x,\partial\Omega) > \epsilon\}\,, \tag{2.7}$$

For every $\eta \in \Omega_\epsilon$, let $\psi_1(\cdot,\eta) : \Omega \to \mathbb{R}$ denote the unique solution of

$$\begin{cases} \nabla\cdot\big(a_1(\cdot,\eta)\nabla\psi_1(\cdot,\eta)\big) = 0 & \text{in } \Omega \\ \psi_1(\cdot,\eta) = f & \text{on } \partial\Omega \end{cases}, \tag{2.8}$$

where

$$a_1(x,\eta) = \begin{cases} \sigma & x \in B(\eta,\epsilon) \\ 1 & x \in \Omega\setminus B(\eta,\epsilon) \end{cases}. \tag{2.9}$$

Set

$$\phi_1(\cdot,\eta) := \psi_1(\cdot,\eta) - \bar{\phi}\,. \tag{2.10}$$

For all $\eta \in \Omega_\epsilon$, define $\phi_0(\cdot,\eta) : \Omega \to \mathbb{R}$ as

$$\phi_0(x,\eta) = (x-\eta)\cdot\nabla\bar{\phi}(\eta) \times \begin{cases} \frac{1-\sigma}{2+\sigma}\frac{\epsilon^3}{|x-\eta|^3} & x \in \Omega\setminus B(\eta,\epsilon) \\ \frac{1-\sigma}{2+\sigma} & x \in B(\eta,\epsilon) \end{cases} . \tag{2.11}$$

Before we attempt to approximate ϕ_1 by ϕ_0 we bring here Theorem 8.16 in [4] in a simplified version.

Theorem 2. *Let u denote a weak solution of*

$$\nabla\cdot(A\nabla u) = \nabla\cdot F\,,$$

in some bounded domain $\Omega \subset \mathbb{R}^3$, where $A \in L^\infty(\Omega)$ is uniformly elliptic and $F \in L^q(\Omega;\mathbb{R}^3)$ for some $q > 3$. Then,

$$\|u\|_\infty \leq C(\|u\|_{L^\infty(\partial\Omega)} + \|F\|_q)\,,$$

where $C = C(|\Omega|, q, A)$.

We can now state the following

Lemma 1. *Let ϕ_1 be given by (2.10). Define for each $\eta \in \Omega_\epsilon$, $u_1(\cdot,\eta) : \Omega \to \mathbb{R}$ by*

$$u_1(x,\eta) = \phi_1(x,\eta) - \phi_0(x,\eta)\,. \tag{2.12}$$

Then,

$$\|u_1(\cdot,\eta)\|_\infty \leq C_q(\Omega,\sigma)\Big(\epsilon^{1+3/q} + \frac{\epsilon^3}{d(\eta,\partial\Omega)^2}\Big)\,, \tag{2.13}$$

for all $q > 3$, and

$$\|u_1(\cdot,\eta)\|_{1,2} \leq C(\Omega,\sigma)\Big(\epsilon^{5/2} + \frac{\epsilon^3}{d(\eta,\partial\Omega)^{3/2}}\Big)\,. \tag{2.14}$$

Proof. By (2.8), (2.10), and (2.12) we have that

$$\nabla\cdot(a_1\nabla u_1) = -\nabla\cdot\big(a_1\nabla(\phi_0 + \bar{\phi})\big)\,. \tag{2.15}$$

Let

$$\psi_0(x,\eta) = (x-\eta)\cdot\nabla\bar{\phi}(\eta) + \phi_0(x,\eta)\,.$$

It is easy to show that $\psi_0(\cdot,\eta)$ is harmonic whenever $|x-\eta| \neq \epsilon$. Furthermore, we have that

$$\frac{\partial\psi_0}{\partial\nu}\Big|_{\partial B(\eta,\epsilon+0)} = \sigma\frac{\partial\psi_0}{\partial\nu}\Big|_{\partial B(\eta,\epsilon-0)}\,,$$

where $F|_{\partial B(\eta,\epsilon\pm0)} = \lim_{\delta\downarrow0} F|_{\partial B(\eta,\epsilon\pm\delta)}$ for any appropriately defined function F, and ν points in the same direction on both surfaces. Consequently,

$$\nabla\cdot\big(a_1[\nabla\phi_0(x,\eta) + \nabla\bar{\phi}(\eta)]\big) = \nabla\cdot\big(a_1\nabla\psi_0\big) = 0\,, \tag{2.16}$$

(cf. also [12] § 3.3). Furthermore, as

$$\nabla\cdot\big(\nabla\bar{\phi}(x) - \nabla\bar{\phi}(\eta)\big) = 0\,,$$

we obtain, using (2.15) and (2.16), that

$$\begin{cases} \nabla\cdot\big(a_1(\cdot,\eta)\nabla u_1(\cdot,\eta)\big) = \nabla\cdot\big([1-a_1(\cdot,\eta)][\nabla\bar{\phi}(\cdot) - \nabla\bar{\phi}(\eta)]\big) & \text{in } \Omega \\ u_1(\cdot,\eta) = -\phi_0(\cdot,\eta) & \text{on } \partial\Omega\,. \end{cases}$$

As (recall that $\bar{\phi} \in C^2(\bar{\Omega})$)

$$\|[1 - a_1(\cdot, \eta)][\nabla\bar{\phi}(\cdot) - \nabla\bar{\phi}(\eta)]\|_q^q = |1 - \sigma|^q \int_{B(\eta,\epsilon)} |\nabla\bar{\phi}(x) - \nabla\bar{\phi}(\eta)|^q \, dx \le C\epsilon^{3+q}$$

and since

$$\|\phi_0(\cdot, \eta)\|_{L^\infty(\partial\Omega)} \le C \frac{\epsilon^3}{d(\eta, \partial\Omega)^2} ,$$

we can use Theorem 2 to complete the proof of (2.13).

To prove (2.14) we set

$$u_1 = u_{1,1} + u_{1,2} ,$$

where

$$\begin{cases} \nabla \cdot (a_1 \nabla u_{1,1}) = \nabla \cdot \big([1 - a_1(x, \eta)][\nabla\bar{\phi}(x) - \nabla\bar{\phi}(\eta)]\big) & \text{in } \Omega \\ u_{1,1} = 0 & \text{on } \partial\Omega . \end{cases}$$

Integration by parts, together with Poincaré's inequality, readily yield that

$$\|u_{1,1}\|_{1,2} \le C\|[1 - a_1(\cdot, \eta)][\nabla\bar{\phi}(\cdot) - \nabla\bar{\phi}(\eta)]\|_2 \le C\epsilon^{5/2} . \tag{2.17}$$

It can be easily verified that $u_{1,2}$ is the minimizer of

$$I(w, \eta) = \int_\Omega a_1(\eta)|\nabla w(\xi, \eta)|^2 \, d\xi ,$$

over the set

$$H^1_{\phi_0}(\Omega) = \{w \in H^1(\Omega) \,|\, w|_{\partial\Omega} = -\phi_0(\cdot, \eta)|_{\partial\Omega}\} .$$

From the uniform ellipticity of a_1 we then get

$$\|\nabla u_{1,2}\|_2 \le C \min_{w \in H^1_{\phi_0}(\Omega)} \|\nabla w\|_2 .$$

Next, we use Poincaré inequality to obtain that some positive C and $\tilde{C}$ exist such that

$$\|u_{1,2}\|_2 \le \|u_{1,2} - w\|_2 + \|w\|_2 \le C\big(\|\nabla u_{1,2}\|_2 + \|w\|_{1,2}\big) \le \tilde{C}\|w\|_{1,2} , \tag{2.18}$$

for all $w \in H^1_{\phi_0}(\Omega)$.

Set then

$$w(t, s) = \zeta(t)\phi_0(s) ,$$

in which $t = d(x, \partial\Omega)$, and s is the projection of x on $\partial\Omega$, which is well-defined for all $t < \delta_0$ (where δ_0 is a property of the smooth boundary). The cutoff function $\zeta \in C^1(\mathbb{R}_+; [0, 1])$ is supported on $[0, \delta]$, for some $\delta < \delta_0$, and satisfies $|\zeta'| \le C/\delta$. We then have

$$\|w\|^2_{1,2} \le C\Big(\delta + \frac{1}{\delta}\Big)\|\phi_0(\cdot, \eta)\|^2_{L^2(\partial\Omega)} + C\delta\|\nabla_s\phi_0(\cdot, \eta)\|^2_{L^2(\partial\Omega)} , \tag{2.19}$$

where ∇_s denotes the tangential derivative on $\partial\Omega$. By (2.11) and the smoothness of $\partial\Omega$ we have

$$\|\phi_0(\cdot, \eta)\|^2_{L^2(\partial\Omega)} \le C\epsilon^6 \int_{\partial\Omega} \frac{ds_\xi}{|\xi - \eta|^4} \le C\epsilon^6 \int_{\mathbb{R}^2} \frac{d\xi}{[|\xi|^2 + d(\eta, \partial\Omega)^2]^2} \le C \frac{\epsilon^6}{d(\eta, \partial\Omega)^2} . \tag{2.20}$$

In a similar manner we obtain that

$$\|\nabla_s\phi_0(\cdot, \eta)\|^2_{L^2(\partial\Omega)} \le C \frac{\epsilon^6}{d(\eta, \partial\Omega)^4} ,$$

which together with (2.20) and (2.19) yields

$$\|w\|_{1,2}^2 \leq C\Big(\delta + \frac{1}{\delta}\Big)\frac{\epsilon^6}{d(\eta,\partial\Omega)^2} + C\delta\frac{\epsilon^6}{d(\eta,\partial\Omega)^4} \,.$$

Upon choosing

$$\delta = \min\big(d(\eta,\partial\Omega),\delta_0\big) \,,$$

we then employ (2.18) to obtain

$$\|u_{1,2}\|_{1,2} \leq C\frac{\epsilon^3}{d(\eta,\partial\Omega)^{3/2}} \,,$$

which together with (2.17) readily yields (2.14).

We next derive an L^2 estimate for $\nabla\phi_1$.

Lemma 2. *Let ϕ_1 be given by* (2.10)*. Then, for all $(\xi,\eta)\in\Omega\times\Omega$,*

$$\|\nabla\phi_1(\cdot,\eta)\|_{L^2\big(B(\xi,\epsilon)\cap\Omega\big)} \leq C(\Omega,\sigma)\frac{\epsilon^{9/2}}{|\xi-\eta|^3} \,. \tag{2.21}$$

Proof. We first use (2.6) with $N=1$ to obtain

$$\begin{aligned} &\frac{1}{\sigma-1}\phi_1(x,\eta) = \int_{\partial B(\eta,\epsilon)}\psi_1(\xi,\eta)\frac{\partial G}{\partial\nu}(x,\xi)\,ds_\xi = \\ &\int_{\partial B(\eta,\epsilon)}\bar\phi(\xi)\frac{\partial G}{\partial\nu}(x,\xi)\,ds_\xi + \int_{\partial B(\eta,\epsilon)}\phi_0(\xi,\eta)\frac{\partial G}{\partial\nu}(x,\xi)\,ds_\xi + \int_{\partial B(\eta,\epsilon)}u_1(\xi,\eta)\frac{\partial G}{\partial\nu}(x,\xi)\,ds_\xi \,, \end{aligned} \tag{2.22}$$

for all $x\in\Omega\setminus B(\eta,\epsilon)$. For the third term we have

$$\Big|\nabla\int_{\partial B(\eta,\epsilon)}u_1(\xi,\eta)\frac{\partial G}{\partial\nu}(x,\xi)\,ds_\xi\Big| \leq \|u_1(\cdot,\eta)\|_\infty\int_{\partial B(\eta,\epsilon)}\|D^2G\|(x,\xi)\,ds_\xi \,,$$

where $\|D^2G\|(x,\xi)$ denotes some appropriate norm of the Hessian matrix $\nabla_x\nabla_\xi G$. In appendix A we show that

$$\|D^2G\|(x,\xi) \leq \frac{C(\Omega)}{|x-\xi|^3} \,. \tag{2.23}$$

Consequently, as $|\partial B(\eta,\epsilon)|\leq C\epsilon^2$, we obtain by (2.13) that

$$\begin{aligned} \Big|\nabla\int_{\partial B(\eta,\epsilon)}u_1(\xi,\eta)\frac{\partial G}{\partial\nu}\,ds_\xi\Big| &\leq C\Big[\epsilon^{1+3/q} + \frac{\epsilon^3}{d(\eta,\partial\Omega)^2}\Big]\frac{\epsilon^2}{d(x,\partial B(\eta,\epsilon))^3} \\ &= C\frac{\epsilon^3}{d(x,\partial B(\eta,\epsilon))^3}\Big[\epsilon^{3/q} + \frac{\epsilon^2}{d(\eta,\partial\Omega)^2}\Big] \,, \end{aligned} \tag{2.24}$$

for all $q>3$.

For the gradient of the second term on the right-hand-side of (2.22) we have

$$\Big|\nabla\int_{\partial B(\eta,\epsilon)}\phi_0(\xi,\eta)\frac{\partial G}{\partial\nu}(x,\xi)\,ds_\xi\Big| \leq \|\phi_0\|_\infty\int_{\partial B(\eta,\epsilon)}\|D^2G\|(x,\xi)\,ds_\xi \leq C\frac{\epsilon^3}{d(x,\partial B(\eta,\epsilon))^3} \,. \tag{2.25}$$

Since $x \in \Omega \setminus B(\eta,\epsilon)$, it follows that $G(x,\cdot)$ is harmonic in $B(\eta,\epsilon)$. Hence, we obtain for the first term on the right-hand-side of (2.22) that

$$\Big|\nabla \int_{\partial B(\eta,\epsilon)} \bar{\phi}(\xi)\frac{\partial G}{\partial \nu}(x,\xi)\, ds_\xi\Big| = \Big|\nabla \int_{\partial B(\eta,\epsilon)} [\bar{\phi}(\xi)-\bar{\phi}(\eta)]\frac{\partial G}{\partial \nu}(x,\xi)\, ds_\xi\Big|$$
$$\leq \epsilon\|\nabla\bar{\phi}\|_\infty \int_{\partial B(\eta,\epsilon)} \|D^2 G\|(x,\xi)\, ds_\xi \leq C\frac{\epsilon^3}{d(x,\partial B(\eta,\epsilon))^3}$$

Combining the above with (2.24) and (2.25) yields

$$|\nabla\phi_1(x,\eta)| \leq C\frac{\epsilon^3}{d(x,\partial B(\eta,\epsilon))^3}\,. \tag{2.26}$$

It follows from (2.26) that (2.21) is valid whenever $|\xi-\eta|>3\epsilon$. An additional bound, uniform in ξ and η, on $\|\nabla\phi_1\|_{L^2(B(\xi,\epsilon))}$ for $|\xi-\eta|\leq 3\epsilon$, would then complete the proof of (2.21). By (2.8) and (2.4) we have that

$$\begin{cases} \nabla\cdot(a_1\nabla\phi_1) = -\nabla\cdot(a_1\nabla\bar{\phi}) = \nabla\cdot([1-a_1]\nabla\bar{\phi}) & \text{in } \Omega \\ \phi_1 = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{2.27}$$

Multiplying (2.27) by ϕ_1 and integrating over Ω we obtain

$$\int_\Omega a_1|\nabla\phi_1|^2\, d\xi = \int_\Omega [1-a_1]\nabla\phi_1\cdot\nabla\bar{\phi}\, d\xi \leq C\|\nabla\phi_1\|_{L^2(B(\eta,\epsilon))}\|\nabla\bar{\phi}\|_{L^2(B(\eta,\epsilon))}\,.$$

Consequently, since $\bar{\phi}\in C^2(\Omega)$,

$$\|\nabla\phi_1\|_2 \leq C\epsilon^{3/2}\,,$$

which readily yields (2.21) for $|\xi-\eta|\leq 3\epsilon$.

Remark 1. One can obtain an effective L^∞ bound on $\nabla\phi_1$ by using the estimates of Li and Vogelius [8]. Consider (2.8) in $D_\epsilon = \Omega\cap B(\eta,4\epsilon)$. Applying the transformation

$$x \to \frac{x-\eta}{\epsilon}\,, \tag{2.28}$$

we obtain (2.8) once again in D_1, which is the image of D_ϵ under the above map. We consider again two different cases: if $D_1 = B(0,4)$ (or equivalently if $\partial D_\epsilon\cap\partial\Omega=\emptyset$) then we can employ Theorem 4.1 in [8] to obtain

$$\|\nabla\tilde{\psi}_1\|_{L^\infty(B(0,3))} \leq C\inf_{z\in\mathbb{R}}\|\tilde{\psi}_1 - z\|_{L^\infty(B(0,4))}\,,$$

where $\tilde{\psi}_1\big((x-\eta)/\epsilon\big) = \psi_1(x)$. By (2.26) and the fact that $\bar{\phi}\in C^2(\Omega)$ we have that

$$\|\nabla\tilde{\psi}_1\|_{L^\infty(\partial B(0,4))} = \epsilon\|\nabla\psi_1\|_{L^\infty(\partial B(\eta,4\epsilon))} \leq C\epsilon\,. \tag{2.29}$$

From the above and the maximum principle we thus get

$$\inf_{z\in\mathbb{R}}\|\tilde{\psi}_1 - z\|_{L^\infty(B(0,4))} \leq \inf_{z\in\mathbb{R}}\|\tilde{\psi}_1 - z\|_{L^\infty(\partial B(0,4))} \leq C\epsilon\,.$$

Consequently, applying the inverse map we obtain that

$$\|\nabla\psi_1\|_{L^\infty(B(\eta,4\epsilon))} \leq C\,.$$

Consider next the case where $B(\eta,4\epsilon)\cap\partial\Omega\neq\emptyset$. Applying again the transformation (2.28) we obtain (2.8) in $D_1\subset B(0,4)$. We can then apply Corollary 1.3 in [8] to obtain

$$\|\nabla\tilde{\psi}_1\|_{L^\infty(D_1)}\leq\inf_{z\in\mathbb{R}}\|\tilde{\psi}_1-z\|_{C^{1,\gamma}(\partial D)}\,. \tag{2.30}$$

Since by (2.29) and the fact that $\|\nabla_s f\|_{L^\infty(\partial\Omega)}\leq C$ (recall that ∇_s denotes the tangential derivative on $\partial\Omega$) we have

$$\|\nabla_s\tilde{\psi}_1\|_{L^\infty(\partial D_1)}\leq C\epsilon$$

we readily obtain that

$$\inf_{z\in\mathbb{R}}\|\tilde{\psi}_1-z\|_{C^{1,\gamma}(\partial D)}\leq C\epsilon\,.$$

Combining the above with (2.30) yields

$$\|\nabla\psi_1\|_{L^\infty(D_\epsilon)}\leq C\,.$$

2.3. Two inclusions

We now proceed to consider a two-particle problem. Let $\psi_2(\cdot,\eta_1,\eta_2):\Omega\to\mathbb{R}$ denote, for every $(\eta_1,\eta_2)\in\Omega_\epsilon\times\Omega_\epsilon$, the unique (weak) solution of

$$\begin{cases}\nabla\cdot(a_2(\cdot,\eta_1,\eta_2)\nabla\psi_2(\cdot,\eta_1,\eta_2))=0 & x\in\Omega\\ \psi_2(\cdot,\eta_1,\eta_2)=f & x\in\partial\Omega\,,\end{cases} \tag{2.31}$$

where

$$a_2(x,\eta_1,\eta_2)=a_1(x,\eta_1)+a_1(x,\eta_2)-1\,. \tag{2.32}$$

We assume in the sequel, without loss of generality, in view of (1.3), that $|\eta_1-\eta_2|\geq 2\epsilon$. For every $(\eta_1,\eta_2)\in\Omega_\epsilon\times\Omega_\epsilon$, we then define $\phi_2(\cdot,\eta_1,\eta_2):\Omega\to\mathbb{R}$ and $v_2(\cdot,\eta_1,\eta_2):\Omega\to\mathbb{R}$ by

$$\psi_2(x,\eta_1,\eta_2)=\phi_2(x,\eta_1,\eta_2)+\bar{\phi}(x)\,,$$

and

$$v_2(\cdot,\eta_1,\eta_2)=\phi_2(x,\eta_1,\eta_2)-\phi_1(x,\eta_1)-\phi_1(x,\eta_2)\,. \tag{2.33}$$

It is easy to show that

$$\begin{cases}\nabla\cdot(a_2\nabla v_2)=-\nabla\cdot\big([a_1(\cdot,\eta_2)-1]\nabla\phi_1(\cdot,\eta_1)+[a_1(\cdot,\eta_1)-1]\nabla\phi_1(\cdot,\eta_2)\big) & \text{in }\Omega\\ v_2(\cdot,\eta_1,\eta_2)=0 & \text{on }\partial\Omega\,.\end{cases} \tag{2.34}$$

Consequently,

$$\|\nabla v_2(\cdot,\eta_1,\eta_2)\|_2\leq C\big\{\|[a_1(\cdot,\eta_2)-1]\nabla\phi_1(\cdot,\eta_1)\|_2+\|[a_1(\cdot,\eta_1)-1]\nabla\phi_1(\cdot,\eta_2)\|_2\big\}\,,$$

and hence by (2.21) we obtain

$$\|\nabla v_2(\cdot,\eta_1,\eta_2)\|_2\leq C\frac{\epsilon^{9/2}}{|\eta_1-\eta_2|^3}\,. \tag{2.35}$$

Next we prove, a two-particle version of (2.21).

Lemma 3. *Let v_2 be given by (2.33). Then, for all $\xi \in \Omega$ and $(\eta_1, \eta_2) \in \Omega_\epsilon \times \Omega_\epsilon$ such that $|\eta_1 - \eta_2| \geq 2\epsilon$ we have*

$$\|\nabla v_2\|_{L^2(B(\xi,\epsilon)\cap\Omega)} \leq C(\Omega,\sigma) \begin{cases} \frac{\epsilon^{15/2}}{|\eta_1-\eta_2|^3}\left[\frac{1}{|\xi-\eta_1|^3} + \frac{1}{|\xi-\eta_2|^3}\right] & \xi \in \Omega \setminus \bigcup_{n=1}^{2} B_n \\ \frac{\epsilon^{9/2}}{|\eta_1-\eta_2|^3} & \xi \in \bigcup_{n=1}^{2} B_n \end{cases} \tag{2.36}$$

Proof. We use (2.6) once again, this time with $N = 2$. By (2.6) with $N = 2$

$$\phi_2(x,\eta_1,\eta_2) = (\sigma - 1)\int_{\partial B_1 \cup \partial B_2} \psi_2(\xi,\eta_1,\eta_2)\frac{\partial G}{\partial \nu}(x,\xi)\, ds \quad \forall x \in \Omega \setminus \bigcup_{n=1}^{2} B_n \tag{2.37}$$

Using the first identity in (2.22) we easily deduce that

$$\begin{aligned} v_2(x,\eta_1,\eta_2) = (\sigma - 1)\Big[\int_{\partial B_1} \phi_1(\xi,\eta_2)\frac{\partial G}{\partial \nu}(x,\xi)\, ds_\xi \\ + \int_{\partial B_2} \phi_1(\xi,\eta_1)\frac{\partial G}{\partial \nu}(x,\xi)\, ds_\xi + \int_{\partial B_1 \cup \partial B_2} v_2(x,\eta_1,\eta_2)\frac{\partial G}{\partial \nu}(x,\xi)\, ds_\xi\Big]. \end{aligned} \tag{2.38}$$

We first attempt to estimate the gradient of the last term on the right-hand-side of (2.38). Let $U_2 = B_1 \cup B_2$ and set

$$\bar{v}_{2,i} = \fint_{B_i} v_2\, d\xi \quad i = 1, 2\,.$$

Since $G(x,\cdot)$ is harmonic inside U_2 for all $x \in \Omega \setminus U_2$ we have that

$$\int_{\partial B_i} \bar{v}_{2,i}\frac{\partial G}{\partial \nu}\, ds = 0 \quad i = 1, 2\,.$$

By Poincaré inequality there exists $C > 0$ such that for any $w \in H^1(B(0,1))$ we have

$$\|w - \bar{w}\|_{L^2(\partial B(0,1))} \leq \|w - \bar{w}\|_{H^{1/2}(\partial B(0,1))} \leq \|w - \bar{w}\|_{H^1(B(0,1))} \leq C\|\nabla w\|_{L^2(B(0,1))}\,,$$

where $\bar{w}$ denotes the average of w over $B(0,1)$. Scaling $B(0,1)$ by a factor of ϵ then yields

$$\|v_2 - \bar{v}_{2,i}\|_{L^2(\partial B_i)} \leq C\epsilon^{1/2}\|\nabla v_2\|_{L^2(B_i)} \quad i = 1, 2\,,$$

where C is independent of ϵ and i. Hence, with the aid of (2.35) we obtain

$$\begin{aligned} \Big|\nabla \int_{\partial U_2} v_2 \frac{\partial G}{\partial \nu}\, ds\Big| \leq C\sum_{i=1}^{2} \|v_2 - \bar{v}_{2,i}\|_{L^2(\partial B_i)}\|D^2 G(x,\cdot)\|_{L^2(\partial B_i)} \leq \\ C\epsilon^{1/2}\|\nabla v_2\|_{L^2(U_2)}\frac{\epsilon}{d(x,U_2)^3} \leq C\frac{\epsilon^6}{|\eta_1 - \eta_2|^3 d(x,U_2)^3}\,. \end{aligned} \tag{2.39}$$

Let

$$\bar{\phi}_1(\eta_2, B_1) = \fint_{B_1} \phi_1(\xi,\eta_2)\, d\xi\,.$$

As above, we have

$$\|\phi_1(\cdot,\eta_2) - \bar{\phi}_1(\eta_2,B_1)\|_{L^2(\partial B_1)} \leq C\epsilon^{1/2}\|\nabla\phi_1(\cdot,\eta_2)\|_{L^2(B_1)}\,.$$

For the first term we thus obtain, with the aid of (2.21), that

$$\Big|\nabla \int_{\partial B_1} \phi_1(\xi,\eta_2)\frac{\partial G}{\partial \nu}\, ds_\xi\Big| \leq C\|\phi_1(\cdot,\eta_2)-\bar{\phi}_1(\eta_2,B_1)\|_{L^2(\partial B_1)}\|D^2G(x,\cdot)\|_{L^2(\partial B_1)}$$
$$\leq C\epsilon^{1/2}\|\nabla\phi_1(\cdot,\eta_2)\|_{L^2(B_1)}\frac{\epsilon}{d(x,U_2)^3} \leq C\frac{\epsilon^6}{|\eta_1-\eta_2|^3 d(x,U_2)^3}\,. \quad (2.40)$$

Let $d(x,U_2)>\epsilon$. We then have

$$\frac{1}{d(x,U_2)^3} \leq C\Big[\frac{1}{|x-\eta_1|^3}+\frac{1}{|x-\eta_2|^3}\Big]\,.$$

Combining the above with (2.38), (2.39), and (2.40) yields for all x such that $d(x,U_2)>\epsilon$

$$|\nabla v_2| \leq C\frac{\epsilon^6}{|\eta_1-\eta_2|^3}\Big[\frac{1}{|x-\eta_1|^3}+\frac{1}{|x-\eta_2|^3}\Big]\,. \quad (2.41)$$

This proves (2.36) for ξ such that $d(\xi,U_2)\geq 2\epsilon$. The case $d(\xi,U_2)<2\epsilon$ readily follows from (2.35).

3. Error estimates

Let ϕ denote a weak solution of (1.1). We first represent ϕ in the following manner

$$\phi(x,\eta_1,\dots,\eta_N) = \bar{\phi}(x) + \sum_{i=1}^N\Big[\phi_1(x,\eta_i)+\frac{1}{2}\sum_{\substack{j=1\\ j\neq i}}^N v_2(x,\eta_i,\eta_j)\Big] + u\,, \quad (3.1)$$

in which ϕ_1 is defined by (2.10) and v_2 by (2.33). We first derive a boundary value problem for u. Clearly,

$$a(x,\eta_1,\dots,\eta_N) = 1+\sum_{n=1}^N[a_1(x,\eta_n)-1]\,.$$

Hence, by (2.4), (2.8), and (2.10)

$$-\nabla\cdot(a\nabla\bar{\phi}) = -\sum_{n=1}^N \nabla\cdot([a_1(x,\eta_n)-1]\nabla\bar{\phi})$$
$$= -\sum_{n=1}^N \nabla\cdot(a_1(x,\eta_n)\nabla\bar{\phi}) = \sum_{n=1}^N \nabla\cdot\big(a_1(x,\eta_n)\nabla\phi_1(x,\eta_n)\big)\,.$$

It follows that

$$-\nabla\cdot\Big(a\Big[\nabla\bar{\phi}+\sum_{i=1}^N\nabla\phi_1(x,\eta_i)\Big]\Big) = \sum_{n=1}^N\nabla\cdot\big(\big[a_1(x,\eta_n)-a\big]\nabla\phi_1(x,\eta_n)\big)\,. \quad (3.2)$$

Using (1.1), (2.4), (3.2), and (3.1) we obtain that

$$-\nabla\cdot(a\nabla u) = \sum_{i=1}^N\nabla\cdot\Big(\big[a-a_1(x,\eta_i)\big]\nabla\phi_1(x,\eta_i)+\frac{1}{2}\sum_{\substack{j=1\\ j\neq i}}^N a\nabla v_2(x,\eta_i,\eta_j)\Big)\,.$$

Using (2.34) we obtain that

$$\nabla\cdot\big(a\nabla v_2(x,\eta_i,\eta_j)\big)=\nabla\cdot\big([a-a_2(x,\eta_i,\eta_j)]\nabla v_2(x,\eta_i,\eta_j)\big)- \\ \nabla\cdot\big([a_1(x,\eta_j)-1]\nabla\phi_1(x,\eta_i)+[a_1(x,\eta_i)-1]\nabla\phi_1(x,\eta_j)\big)\,.$$

Clearly,

$$\frac{1}{2}\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\big([a_1(x,\eta_j)-1]\nabla\phi_1(x,\eta_i)+[a_1(x,\eta_j)-1]\nabla\phi_1(x,\eta_i)\big) \\ =\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}[a_1(x,\eta_j)-1]\nabla\phi_1(x,\eta_i)\,,$$

and hence

$$-\nabla\cdot(a\nabla u)=\sum_{i=1}^{N}\nabla\cdot\Big(\big[a-a_1(x,\eta_i)\big]\nabla\phi_1(x,\eta_i)+ \\ \sum_{\substack{j=1\\ j\neq i}}^{N}[a_1(x,\eta_j)-1]\nabla\phi_1(x,\eta_i)+\frac{1}{2}[a-a_2(x,\eta_i,\eta_j)]\nabla v_2(x,\eta_i,\eta_j)\Big)\,.$$

As

$$a-a_1(x,\eta_i)=\sum_{\substack{j=1\\ j\neq i}}^{N}[a_1(x,\eta_j)-1]$$

we obtain that

$$\begin{cases}-\nabla\cdot(a\nabla u)=\frac{1}{2}\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\nabla\cdot[a-a_2(x,\eta_i,\eta_j)]\nabla v_2(x,\eta_i,\eta_j) & \text{in } \Omega\\ u=0 & \text{on } \partial\Omega\,.\end{cases} \tag{3.3}$$

It immediately follows from (3.3) and Poincaré's inequality that

$$\|u\|_{1,2}\leq C\|\nabla u\|_2\leq C\Bigg\|\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}[a-a_2(\cdot,\eta_i,\eta_j)]\nabla v_2(\cdot,\eta_i,\eta_j)\Bigg\|_2\,. \tag{3.4}$$

We can now begin the estimate of the H^1 norm of the expectation of u. By (3.4) we have

$$\|\langle u\rangle\|_{1,2}\leq\langle\|u\|_{1,2}^2\rangle^{1/2}\leq C\Bigg\langle\Bigg\|\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}[a-a_2(\cdot,\eta_i,\eta_j)]\nabla v_2(\cdot,\eta_i,\eta_j)\Bigg\|_2^2\Bigg\rangle^{1/2}\,.$$

As

$$a-a_2(x,\eta_i,\eta_j)=\sum_{\substack{k=1\\ k\notin\{i,j\}}}^{N}[a_1(x,\eta_k)-1]\,,$$

we obtain that

$$\|\langle u\rangle\|_{1,2} \leq C\Big\langle \Big\| \sum_{i=1}^N \sum_{\substack{j=1\\ j\neq i}}^N \sum_{\substack{k=1\\ k\notin\{i,j\}}}^N [a_1(\cdot,\eta_k)-1]\nabla v_2(\cdot,\eta_i,\eta_j)\Big\|_2^2\Big\rangle^{1/2}$$
$$= C\Big\langle \Big\| \sum_{k=1}^N \sum_{\substack{i=1\\ i\neq k}}^N \sum_{\substack{j=1\\ j\notin\{i,k\}}}^N [a_1(\cdot,\eta_k)-1]\nabla v_2(\cdot,\eta_i,\eta_j)\Big\|_2^2\Big\rangle^{1/2}.$$

Since by (1.3) we have, with probability 1, that

$$[a_1(x,\eta_k)-1][a_1(x,\eta_m)-1] = (\sigma-1)[a_1(x,\eta_k)-1]\delta_{km} \quad \forall x\in\Omega\,,$$

we obtain that

$$\|\langle u\rangle\|_{1,2} \leq C\Big[\sum_{k=1}^N \Big\langle \int_{B(\eta_k,\epsilon)} \Big| \sum_{\substack{i=1\\ i\neq k}}^N \sum_{\substack{j=1\\ j\notin\{i,k\}}}^N \nabla v_2(x,\eta_i,\eta_j)\Big|^2 dx\Big\rangle\Big]^{1/2}.$$

By the multi-particle symmetry of the joint probability distribution we thus have

$$\|\langle u\rangle\|_{1,2} \leq CN^{1/2}\Big[\Big\langle \int_{B(\eta_1,\epsilon)} \Big| \sum_{\substack{i,j=2\\ j\neq i}}^N \nabla v_2(x,\eta_i,\eta_j)\Big|^2 dx\Big\rangle\Big]^{1/2}. \tag{3.5}$$

Next, we evaluate the integrand in (3.5) to obtain, in view of the symmetry of $v_2(x,\eta_1,\eta_2)$ with respect to η_1 and η_2, that

$$\int_{B(\eta_1,\epsilon)} \Big| \sum_{\substack{i,j=2\\ j\neq i}}^N \nabla v_2(x,\eta_i,\eta_j)\Big|^2 dx = \int_{B(\eta_1,\epsilon)} \sum_{\substack{i,j=2\\ j\neq i}}^N |\nabla v_2(x,\eta_i,\eta_j)|^2\, dx +$$
$$4\int_{B(\eta_1,\epsilon)} \sum_{\substack{i,j=2\\ j\neq i}}^N \sum_{\substack{m=2\\ m\notin\{i,j\}}}^N \nabla v_2(x,\eta_i,\eta_j)\cdot\nabla v_2(x,\eta_i,\eta_m)\, dx +$$
$$\int_{B(\eta_1,\epsilon)} \sum_{\substack{i,j=2\\ j\neq i}}^N \sum_{\substack{k,m=2\\ k\notin\{i,j\}\ m\notin\{k,i,j\}}}^N \nabla v_2(x,\eta_i,\eta_j)\cdot\nabla v_2(x,\eta_k,\eta_m)\, dx\,. \tag{3.6}$$

The expectation of the first term on the right-hand-side of the above identity is given, in view of the multi-variable symmetry of f_N, by

$$\Big\langle \int_{B(\eta_1,\epsilon)} \sum_{\substack{i,j=2\\ j\neq i}}^N |\nabla v_2(x,\eta_i,\eta_j)|^2\, dx\Big\rangle = (N-1)(N-2)\Big\langle \int_{B(\eta_1,\epsilon)} |\nabla v_2(x,\eta_2,\eta_3)|^2\, dx\Big\rangle. \tag{3.7}$$

For the second term on the right-hand-side of (3.6) we have

$$\Big\langle \int_{B(\eta_1,\epsilon)} \sum_{\substack{i,j=2\\ j\neq i}}^N \sum_{\substack{m=2\\ m\notin\{i,j\}}}^N \nabla v_2(x,\eta_i,\eta_j)\cdot\nabla v_2(x,\eta_i,\eta_m)\, dx\Big\rangle =$$
$$\frac{(N-1)!}{(N-4)!}\Big\langle \int_{B(\eta_1,\epsilon)} \nabla v_2(x,\eta_2,\eta_3)\cdot\nabla v_2(x,\eta_2,\eta_4)\, dx\Big\rangle. \tag{3.8}$$

Finally, for the third term we get

$$\Big\langle \int_{B(\eta_1,\epsilon)} \sum_{\substack{i,j=2\\ j\neq i}}^{N} \sum_{\substack{k,m=2\\ k\notin\{i,j\}\ m\notin\{k,i,j\}}}^{N} \nabla v_2(x,\eta_i,\eta_j)\cdot\nabla v_2(x,\eta_k,\eta_m)\,dx \Big\rangle =$$
$$\frac{(N-1)!}{(N-5)!}\Big\langle \int_{B(\eta_1,\epsilon)} \nabla v_2(x,\eta_2,\eta_3)\cdot\nabla v_2(x,\eta_4,\eta_5)\,dx \Big\rangle . \quad (3.9)$$

We next estimate each of the above terms separately. For (3.7) use is made of (1.7) next to (2.36) to obtain

$$(N-1)(N-2)\Big\langle \int_{B(\eta_1,\epsilon)} |\nabla v_2(x,\eta_2,\eta_3)|^2\,dx \Big\rangle \le$$
$$CN^2\Big\langle \frac{\epsilon^{15}}{|\eta_2-\eta_3|^6}\Big[\frac{1}{|\eta_2-\eta_1|^3}+\frac{1}{|\eta_3-\eta_1|^3}\Big]^2\Big\rangle$$
$$\le C\epsilon^9\bar\beta^2\int_{\Omega^3}\frac{1}{|\eta_1-\eta_2|^6|\eta_2-\eta_3|^6} f_3(\eta_1,\eta_2,\eta_3)\,d\eta_1\,d\eta_2\,d\eta_3 .$$

Clearly,

$$\int_{\Omega\setminus B(\eta_1,\epsilon)}\int_{\Omega\setminus B(\eta_2,\epsilon)}\frac{1}{|\eta_1-\eta_2|^6|\eta_2-\eta_3|^6}\,d\eta_3 d\eta_2 \le$$
$$\int_{\Omega\setminus B(\eta_1,\epsilon)}\frac{1}{|\eta_1-\eta_2|^6}\,d\eta_2\int_\epsilon^R \frac{1}{r^6}r^2\,dr \le \Big(\int_\epsilon^R\frac{dr}{r^4}\Big)^2 \le \frac{C}{\epsilon^6},$$

where $R=\operatorname{diam}\Omega$. Using the boundedness (1.5) for $k=3$ we then obtain

$$(N-1)(N-2)\Big\langle \int_{B(\eta_1,\epsilon)} |\nabla v_2(x,\eta_2,\eta_3)|^2\,dx \Big\rangle \le C\epsilon^3\bar\beta^2 . \quad (3.10)$$

For the right-hand-side of (3.8) we have

$$\frac{(N-1)!}{(N-4)!}\Big|\Big\langle \int_{B(\eta_1,\epsilon)} \nabla v_2(x,\eta_2,\eta_3)\cdot\nabla v_2(x,\eta_2,\eta_4)\,dx\Big\rangle\Big| \le$$
$$CN^3\Big\langle \|\nabla v_2(x,\eta_2,\eta_3)\|_{L^2(B(\eta_1,\epsilon))}\|\nabla v_2(x,\eta_2,\eta_4)\|_{L^2(B(\eta_1,\epsilon))}\Big\rangle , \quad (3.11)$$

which, with the aid of (2.36) becomes

$$\frac{(N-1)!}{(N-4)!}\Big|\Big\langle \int_{B(\eta_1,\epsilon)} \nabla v_2(x,\eta_2,\eta_3)\cdot\nabla v_2(x,\eta_2,\eta_4)\,dx\Big\rangle\Big| \le$$
$$C\epsilon^6\bar\beta^3\int_{\Omega^4}\Big[\frac{1}{|\eta_2-\eta_1|^3}+\frac{1}{|\eta_3-\eta_1|^3}\Big]\Big[\frac{1}{|\eta_2-\eta_1|^3}+\frac{1}{|\eta_4-\eta_1|^3}\Big]$$
$$\frac{1}{|\eta_2-\eta_4|^3|\eta_2-\eta_3|^3} f_4(\eta_1,\eta_2,\eta_3,\eta_4)\,d\eta_1\,d\eta_2\,d\eta_3\,d\eta_4 .$$

Hence, by the symmetry of the roles of η_3 and η_4,

$$\frac{(N-1)!}{(N-4)!}\Big|\Big\langle \int_{B(\eta_1,\epsilon)} \nabla v_2(x,\eta_2,\eta_3)\cdot\nabla v_2(x,\eta_2,\eta_4)\,dx\Big\rangle\Big| \le$$
$$C\epsilon^6\bar\beta^3\int_{\Omega^4}\Big[\frac{1}{|\eta_2-\eta_1|^6}+\frac{1}{|\eta_3-\eta_1|^6}\Big]\frac{1}{|\eta_2-\eta_4|^3|\eta_2-\eta_3|^3} f_4(\eta_1,\eta_2,\eta_3,\eta_4)\,d\eta_1\,d\eta_2\,d\eta_3\,d\eta_4 \quad (3.12)$$

As before we have that

$$\int_{\Omega\setminus B(\eta_2,,\epsilon)} \frac{1}{|\eta_2-\eta_4|^3}\,d\eta_4 \leq C\int_\epsilon^R \frac{dr}{r} \leq C\ln\epsilon^{-1}\,. \tag{3.13}$$

Furthermore, in a similar manner, one can show that

$$\int_{\Omega\setminus B(\eta_1,\epsilon)}\int_{\Omega\setminus B(\eta_2,\epsilon)} \frac{1}{|\eta_2-\eta_1|^6}\frac{1}{|\eta_2-\eta_3|^3}\,d\eta_3 d\eta_2 \leq \frac{C\ln\epsilon^{-1}}{\epsilon^3}$$

Substituting the above , together with (3.13) into (3.12) then yields

$$\frac{(N-1)!}{(N-4)!}\Big|\Big\langle\int_{B(\eta_1,\epsilon)}\nabla v_2(x,\eta_2,\eta_3)\cdot\nabla v_2(x,\eta_2,\eta_4)\,dx\Big\rangle\Big| \leq C\epsilon^3\bar{\beta}^3\ln^2\epsilon\,. \tag{3.14}$$

Finally, we estimate the right-hand-side of (3.9) in a similar fashion to obtain

$$\frac{(N-1)!}{(N-5)!}\Big\langle\int_{B(\eta_1,\epsilon)}\nabla v_2(x,\eta_2,\eta_3)\cdot\nabla v_2(x,\eta_4,\eta_5)\,dx\Big\rangle \leq C\epsilon^3\bar{\beta}^4\ln^4\epsilon\,. \tag{3.15}$$

We then combine (3.5)-(3.15) to obtain

$$\|\langle u\rangle\|_{1,2} \leq C\bar{\beta}^{1/2}[\bar{\beta}^2+\bar{\beta}^3\ln^2\epsilon+\bar{\beta}^4\ln^4\epsilon]^{1/2}\,. \tag{3.16}$$

By (1.8) we thus obtain

$$\|\langle u\rangle\|_{1,2} \leq C\bar{\beta}^{3/2}\,.$$

We next seek an estimate for the H^1 norm of the second sum on the right-hand-side of (3.1). Let

$$V_2(x,\eta_1,\dots,\eta_N) = \sum_{i=1}^N\sum_{\substack{j=1\\ j\neq i}}^N v_2(x,\eta_i,\eta_j)$$

Clearly,

$$\nabla\Big\langle V_2(x,\eta_1,\dots,\eta_N)\Big\rangle = N(N-1)\Big\langle\nabla v_2(x,\eta_1,\eta_2)\Big\rangle\,.$$

By (2.36) we have for all $\xi\in\Omega$

$$\|\nabla\langle V_2\rangle\|_{L^2(B(\xi,\epsilon)\cap\Omega)} \leq C\epsilon^{3/2}\bar{\beta}^2\Big[\int_{\Omega\setminus B(\xi,\epsilon)}\int_\Omega \frac{1}{|\xi-\eta_1|^3|\eta_1-\eta_2|^3}f_2(\eta_1,\eta_2)\,d\eta_2 d\eta_1 + \frac{1}{\epsilon^3}\int_{B(\xi,\epsilon)\cap\Omega}\int_\Omega\frac{1}{|\eta_1-\eta_2|^3}f_2(\eta_1,\eta_2)\Big]\,d\eta_2 d\eta_1\,. \tag{3.17}$$

Consequently, for all $\xi\in\Omega$

$$\|\nabla\langle V_2\rangle\|_{L^2(B(\xi,\epsilon)\cap\Omega)} \leq C\epsilon^{3/2}\bar{\beta}^2\ln^2\epsilon\,,$$

from which it easily follows, using Poincaré's inequality, that

$$\|\langle V_2\rangle\|_{1,2} \leq C\bar{\beta}^2\ln^2\epsilon\,. \tag{3.18}$$

The foregoing discussion can be summarized as follows

Proposition 1. *Let ϕ denote the (weak) solution of* (1.1). *Then, under the assumptions* (1.5) *and* (1.8) *we have that*

$$\|\langle\phi\rangle-\bar{\phi}-N\langle\phi_1\rangle\|_{1,2} \leq C\bar{\beta}^{3/2}\,. \tag{3.19}$$

Proof. The proof follows immediately from (3.1), (3.5), (3.18), and the fact that

$$\Big\langle\sum_{i=1}^N\phi_1(x,\eta_i)\Big\rangle = N\langle\phi_1\rangle\,.$$

4. Effective medium

To prove Theorem 1 we need to show that the estimate of $\langle\phi\rangle$ provided by (3.19) is a good approximation for the solution of the steady-state heat equation in a continuous medium whose conductivity is a function of both the inclusion's conductivity σ and the volume fraction $\beta(x)$. Consider then the following problem

$$\begin{cases} \nabla \cdot (a_e(x)\nabla\phi_e) = 0 & \text{in } \Omega \\ \phi = f & \text{on } \partial\Omega\,, \end{cases} \tag{4.1a}$$

where

$$a_e(x) = 1 + \gamma(x)\,, \tag{4.1b}$$

in which

$$\|\gamma\|_\infty \leq C\bar{\beta}\,, \tag{4.1c}$$

and

$$a_e > \frac{1}{2}\,, \tag{4.1d}$$

for all x in Ω.

For the solution of (4.1) we can prove the following estimate

Proposition 2. *Let ϕ_e denote the unique solution of* (4.1)*. Then*

$$\|\phi_e - \bar{\phi} + \Delta^{-1}\big(\nabla \cdot (\gamma\nabla\bar{\phi})\big)\|_{1,2} \leq C\bar{\beta}^2\,. \tag{4.2}$$

In the above Δ^{-1} denotes the inverse Laplace operator in $H^1_0(\Omega)$, i.e., for any $F \in H^{-1}(\Omega)$, $w = \Delta^{-1}F$ is the unique (weak) solution of

$$\begin{cases} \Delta w = F & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega \end{cases}.$$

Proof. We bring in brief the proof of this standard result. Set

$$u_e = \phi_e - \bar{\phi} + \Delta^{-1}\big(\nabla \cdot (\gamma\nabla\bar{\phi})\big)\,.$$

Then,

$$\begin{cases} \nabla \cdot (a_e\nabla u_e) = \nabla \cdot \big\{\gamma\nabla\Delta^{-1}\big(\nabla \cdot (\gamma\nabla\bar{\phi})\big)\big\} & \text{in}\Omega \\ u_e = 0 & \text{on } \partial\Omega\,. \end{cases}$$

Consequently, as $a_e > 1/2$ we have that

$$\|\nabla u_e\|_2 \leq C\big\|\gamma\nabla\Delta^{-1}\big(\nabla \cdot (\gamma\nabla\bar{\phi})\big)\big\|_2 \leq C\|\gamma\|_\infty\|\nabla\Delta^{-1}\big(\nabla \cdot (\gamma\nabla\bar{\phi})\big)\|_2 \leq \\ C\|\gamma\|_\infty\|\gamma\nabla\bar{\phi}\|_2 \leq C\|\gamma\|_\infty^2\|\nabla\bar{\phi}\|_2\,.$$

From (4.1c) we then get (4.2).

We next show that $N\langle\phi_1\rangle$ can be expressed as the inverse Laplacian of $\nabla\cdot(\gamma\nabla\bar{\phi})$ for an appropriate choice of γ.

Lemma 4. *Let ϕ_1 be given by* (2.10)*. Then,*

$$\left\|N\langle\phi_1\rangle + \frac{3(\sigma-1)}{2+\sigma}\Delta^{-1}\big(\nabla \cdot (\beta\nabla\bar{\phi})\big)\right\|_{1,2} \leq C\epsilon^{1/2}\bar{\beta} \tag{4.3}$$

Proof. By (2.27) we have

$$-\Delta\phi_1 = \nabla\cdot\big([a_1(x,\eta)-1][\nabla\phi_1(x,\eta)+\nabla\bar{\phi}(x)]\big)\,.$$

Hence, we can employ (2.12) to obtain

$$-\Delta\phi_1 = \nabla\cdot\big([a_1(x,\eta)-1][\nabla\phi_0(x,\eta)+\nabla\bar{\phi}(x)]\big)+\nabla\cdot\big([a_1(x,\eta)-1]\nabla u_1(x,\eta)\big)\,, \tag{4.4}$$

where ϕ_0 is given by (2.11).

Denote by $\phi_{1,1}$ the weak solution of

$$\begin{cases} -\Delta\phi_{1,1} = \frac{3}{2+\sigma}\nabla\cdot\big([a_1(x,\eta)-1]\nabla\bar{\phi}(x)\big) & \text{in } \Omega\,,\\ \phi_{1,1}=0 & \text{on } \partial\Omega\,. \end{cases} \tag{4.5}$$

To facilitate averaging, we represent the above problem in its weak form. Let then $\chi\in C_0^\infty(\Omega)$. The weak formulation of the last equation, with the aid of (2.11), reads

$$\int_\Omega \nabla\chi\cdot\nabla\phi_{1,1}\,dx = -\frac{3}{2+\sigma}\int_\Omega \nabla\chi\cdot[a_1(x,\eta)-1]\nabla\bar{\phi}(x)\,dx\,.$$

Taking the average of the above equation yields

$$\int_\Omega \nabla\chi\cdot\nabla\langle\phi_{1,1}\rangle\,dx = -\frac{3}{2+\sigma}\int_\Omega \nabla\chi\cdot\langle[a_1(x,\eta)-1]\rangle\nabla\bar{\phi}(x)\,dx\,. \tag{4.6}$$

(Recall that as $\phi_{1,1}\in H_0^1(\Omega)$ we have $\langle\nabla\phi_{1,1}\rangle=\nabla\langle\phi_{1,1}\rangle$.) Clearly,

$$\langle[a_1(x,\eta)-1]\rangle = (\sigma-1)\int_{B(x,\epsilon)\cap\Omega_\epsilon} f_1(\eta)\,d\eta = (\sigma-1)\frac{\beta(x)}{N}\,, \tag{4.7}$$

where Ω_ϵ is defined by (2.7) and $\beta\in L^2(\Omega)$ is given by (1.9). By (4.6) and (4.7) it follows that $\langle\phi_{1,1}\rangle$ is a weak solution of

$$\Delta\langle\phi_{1,1}\rangle = -\frac{3(\sigma-1)}{2+\sigma}\nabla\cdot\Big(\frac{\beta(x)}{N}\nabla\bar{\phi}\Big)\,.$$

Hence,

$$N\langle\phi_{1,1}\rangle = -\frac{3(\sigma-1)}{2+\sigma}\Delta^{-1}\big(\nabla\cdot(\beta\nabla\bar{\phi})\big)\,. \tag{4.8}$$

Set

$$\phi_1=\phi_{1,1}+\phi_{1,2}\,. \tag{4.9}$$

By (4.4) and (4.5) we have that

$$\begin{cases} -\Delta\phi_{1,2} = \nabla\cdot\Big([a_1(x,\eta)-1]\Big[\frac{\sigma-1}{2+\sigma}\nabla\bar{\phi}(x)+\nabla\phi_0(x-\eta)+\nabla u_1(x,\eta)\Big]\Big) & \text{in } \Omega\,,\\ \phi_{1,2}=0 & \text{on } \partial\Omega\,. \end{cases} \tag{4.10}$$

By (2.11), since $a_1(x,\eta)=1$ whenever $x\notin B(\eta,\epsilon)$, we have that

$$[a_1(x,\eta)-1]\nabla\phi_0(x,\eta) = -[a_1(x,\eta)-1]\frac{\sigma-1}{2+\sigma}\nabla\bar{\phi}(\eta)\,.$$

Substituting into (4.10) then yields

$$\begin{cases} -\Delta\phi_{1,2} = \frac{\sigma-1}{2+\sigma}\nabla\cdot\big([a_1(x,\eta)-1][\nabla\bar{\phi}(x)-\nabla\bar{\phi}(\eta)]\big)+\nabla\cdot\big([a_1(x,\eta)-1]\nabla u_1(x,\eta)\big) & \text{in } \Omega\,,\\ \phi_{1,2}=0 & \text{on } \partial\Omega\,. \end{cases}$$

As in (4.6) we readily show that

$$\int_\Omega \nabla\chi\cdot\nabla\langle\phi_{1,2}\rangle\,dx = -\int_\Omega \nabla\chi\cdot\langle[a_1(x,\eta)-1]\Big[\frac{\sigma-1}{2+\sigma}[\nabla\bar\phi(x)-\nabla\bar\phi(\eta)]+\nabla u_1(x,\eta)\Big]\rangle\,dx\,.$$

Hence,

$$\|\nabla\langle\phi_{1,2}\rangle\|_2 \le \frac{\sigma-1}{2+\sigma}\|\langle[a_1(\cdot,\eta)-1][\nabla\bar\phi(\cdot)-\nabla\bar\phi(\eta)]\rangle\|_2 + \|\langle[a_1(\cdot,\eta)-1]\nabla u_1(\cdot,\eta)\rangle\|_2\,. \tag{4.11}$$

Since by (1.5) we have that

$$|\langle[a_1(x,\eta)-1][\nabla\bar\phi(x)-\nabla\bar\phi(\eta)]\rangle| \le C\int_{B(x,\epsilon)\cap\Omega_\epsilon}|\nabla\bar\phi(x)-\nabla\bar\phi(\eta)|\,d\eta \le C\epsilon^4$$

uniformly in x, it readily follows that the first term on the right-hand-side of (4.11) obeys a similar estimate

$$\|\langle[a_1(\cdot,\eta)-1][\nabla\bar\phi(\cdot)-\nabla\bar\phi(\eta)]\rangle\|_2 \le C\epsilon^4\,.$$

For the second term we have by(1.5) that

$$\|\langle[a_1(\cdot,\eta)-1]\nabla u_1(\cdot,\eta)\rangle\|_2^2 \le C\int_\Omega\Big[\int_{B(x,\epsilon)\cap\Omega_\epsilon}|\nabla u_1(x,\eta)|\,d\eta\Big]^2dx \le$$
$$C\epsilon^3\int_\Omega\int_{B(x,\epsilon)\cap\Omega_\epsilon}|\nabla u_1(x,\eta)|^2\,d\eta\,dx \le C\epsilon^3\int_\Omega\int_{\Omega_\epsilon}|\nabla u_1(x,\eta)|^2\,d\eta\,dx\,.$$

With the aid of (2.14) we then obtain that

$$\|\langle[a_1(\cdot,\eta)-1]\nabla u_1(\cdot,\eta)\rangle\|_2^2 \le C\epsilon^3\int_{\Omega_\epsilon}\|\nabla u_1(\cdot,\eta)\|_2^2\,d\eta \le$$
$$C\epsilon^3\int_{\Omega_\epsilon}\Big[\epsilon^{5/2}+\frac{\epsilon^3}{d(\eta,\partial\Omega)^{3/2}}\Big]^2d\eta \le C\epsilon^7\,.$$

Hence, from (4.11) we get

$$\|N\nabla\langle\phi_{1,2}\rangle\|_2 \le C\bar\beta\epsilon^{1/2}\,.$$

Combining the above together with (4.8), (4.9), and Poincaré's inequality yields (4.3).

Proof (Proof of Theorem 1). Combining (4.3), (3.19), and (1.8) yields

$$\Big\|\langle\phi\rangle-\bar\phi+\frac{3(\sigma-1)}{2+\sigma}\Delta^{-1}\big(\nabla\cdot(\beta\nabla\bar\phi)\big)\Big\|_{1,2} \le C\bar\beta^{3/2}\,.$$

To complete the proof we need only show that if we choose

$$\gamma = \frac{3(\sigma-1)}{2+\sigma}\beta\,,$$

then both (4.1c) and (4.1d) hold, but this follows immediately from (1.10) and (1.7). The Theorem then follows from (4.2).

Remark 2. One can relax the limitation imposed on $\bar\beta$ by the upper bound of (1.8), by allowing a larger error in (1.13). Thus, if we require that $\bar\beta \le \ln^{-2(2+\delta)}(\epsilon^{-1})$ for some $0<\delta<2$ we can replace the right-hand-side of (1.13) by $C\bar\beta^{1+\delta/(2+\delta)}$. This can be readily verified using (3.16) and (3.18).

5. Concluding remarks

In the following we illuminate a few key points that were only briefly considered, if at all, throughout the previous sections.

Higher order terms: In principle one can continue the expansion (3.1) to include $\mathcal{O}(\bar{\beta}^k)$ terms. Let u be defined by (3.1). The next order term can be obtained by setting $N=3$ in (3.1). We then obtain

$$u = \sum_{n_1,n_2,n_3} v_3(x,\eta_{n_1},\eta_{n_2},\eta_{n_3}) + \dots$$

where the summation is over all the C_N^3 3-combinations of $\{1,\dots,N\}$,

$$v_3(x,\eta_1,\eta_2,\eta_3) = \phi_3(x,\eta_1,\eta_2,\eta_3) - [v_2(x,\eta_1,\eta_2)+ \\ v_2(x,\eta_1,\eta_3) + v_2(x,\eta_2,\eta_3) + \phi_1(x,\eta_1) + \phi_1(x,\eta_2) + \phi_1(x,\eta_3)]\,,$$

and $\phi_3 = \psi_3 - \bar{\phi}$ where

$$\begin{cases} \nabla\cdot(a_3\nabla\psi_3) = 0 & \text{in } \Omega \\ \psi_3 = f & \text{on } \partial\Omega\,. \end{cases}$$

In the above

$$a_3(x,\eta_1,\eta_2,\eta_3) = a_1(x,\eta_1) + a_1(x,\eta_2) + a_1(x,\eta_3) - 2\,.$$

One may recursively construct the series up to an $\mathcal{O}(\bar{\beta}^k)$ error for any $k\in\mathbb{N}$. These, of course, would be asymptotic series in the limit $\bar{\beta}\to 0$: it is very difficult to determine whether they also converge as $k\to\infty$ using the techniques employed in this work.

C^α convergence: Once higher order terms are obtained (at least up to $k=4$) it seems possible to obtain, with the aid of Li & Vogelius' estimates [8] a C^α convergence, for some positive α, of $\langle\phi\rangle$ to ϕ_e, using estimates that are similar to (2.13). Note that to obtain (3.19), it was necessary to introduce into (3.1) not only the first order correction $\sum_{n=1}^N \phi_1(x,\eta_n)$ but also the $\mathcal{O}(\beta^2)$ term $\sum_{i,j} v_2(x,\eta_i,\eta_j)$. The reason for that becomes clear once we notice that in (3.5) we need to raise to the power 1/2 integrals that are supported on a small set. Consequently, instead of getting an $\mathcal{O}(\bar{\beta}^3)$ error we obtain an $\mathcal{O}(\bar{\beta}^{3/2})$ one. If we seek C^α convergence, then, the DeGiorgi-Nash-Moser theory [4] would require us to raise the integrals in (3.5) to the power $1/q$, for $q>3$. Hence, to obtain an error which is much smaller than $\mathcal{O}(\bar{\beta})$ we need to include at least three terms in the expansion (3.1) (and then the error would be at least $\mathcal{O}(\bar{\beta}^{4/q})$ for any $q>3$)

Logarithmic errors: The lower bound in (1.8) appears to be in line with the expectation that N is large, though the analysis can perhaps be performed under a weaker constraint. In contrast, the upper bound appears to result from the crudeness of our estimates. There are two different steps that produce error terms of $\mathcal{O}(\bar{\beta}^2\ln^2\epsilon)$: the first is the derivation of (3.12) and (3.15), and the second is the estimate (3.17). In both cases one can hope that better results are obtained through the derivation of a leading order approximation of ϕ_2 similar to the one obtained for ϕ_1 in (2.14). The approximation can, perhaps, be obtained from the solution of (2.31) in $\mathbb{R}^3$ assuming linear asymptotic behaviour of ψ_2 as $|x|\to\infty$. It seems worthwhile to note here that eliminating the upper bound in (1.8) may facilitate a comparison between the results of stochastic homogenization [6,13] with those obtained in the present work.

Stationarity: An interesting point to note is that the gradients of the various marginal probability densities, or $\|\nabla f_i\|_\infty$ for any $1\le i$, need not be bounded as $\epsilon\to 0$ in order to obtain (1.13). However, if we attempt to replace the average in event space in (1.13) by a local spatial average, i.e., by

$$⨍_{B(x,r)} \phi\, dx\,,$$

for some $\bar{\beta}^{-1/3} \ll r \ll 1$, boundedness of $\|\nabla f_i\|_\infty$ should most probably be assumed for the first few probability densities. Such an assumption can be interpreted as "quasi-stationarity". Note that in [6] stationarity of the probability law had to be assumed in order to apply Birkhoff's Theorem so that equivalence of the different averages is established.

Generalizations: The present technique can certainly be generalized in several different directions: the inclusions' shape may vary, and can be assumed to have a random character. One would still need to solve one-particle and two-particle problems - just with a different shape. One can also consider $a(x)$ which is non-constant both inside and outside the inclusions. The estimates obtained using potential theory would have to be generalized in this case. Problems of linear elasticity seem tractable as well.

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Appendix A. Proof of (2.23)

Let $\Omega \subset \mathbb{R}^3$ be bounded with smooth boundary $\partial\Omega \in C^{2,\alpha}$ for some $\alpha > 0$. For some $\epsilon > 0$, we define Ω_ϵ as

$$\Omega_\epsilon = \{x \in \Omega \,|\, d(x, \partial\Omega) < \epsilon\} .$$

Let $\delta_0 > 0$ be sufficiently small so that for every $y \in \Omega_{\delta_0}$ there exists a unique point $\bar{y} \in \partial\Omega$ so that $d(y, \partial\Omega) = |y - \bar{y}|$. Denote the reflection of y with respect to $\bar{y}$ by y^*. We begin by the following elementary lemma

Lemma 5. *There exists $C > 0$ depending only on Ω such that*

$$\sup_{(x,y)\in\partial\Omega\times\Omega_{\delta_0/2}} \left|\frac{1}{|x-y|} - \frac{1}{|x-y^*|}\right| \le C . \tag{A.1}$$

Proof. Let $y \in \Omega_{\delta_0/2}$. We set the origin of a coordinate system at its projection on $\partial\Omega$, i.e., $\bar{y} = (0, 0, 0)$. We further set $y = (0, 0, d)$ and hence for the reflection point we have $y^* = (0, 0, -d)$. Let $x = (x_1, x_2, x_3) \in \partial\Omega$, and let $r = |x - y|$ and $r^* = |x - y^*|$. By the smoothness of $\partial\Omega$ there exists $C_0(\Omega)$ and $\epsilon_1(\Omega)$ such that

$$|x_3| \le C_0(x_1^2 + x_2^2) \quad x \in B(0, \epsilon_1) .$$

Without loss of generality we can assume $d < \epsilon_1/2$, otherwise $\min(r, r^*) > \epsilon_1/2$ and hence

$$\left|\frac{1}{r} - \frac{1}{r^*}\right| \le \frac{2}{\epsilon_1} .$$

When $(x_1^2+x_2^2+x_3^2)^{1/2} > \epsilon_1$ the above inequality remains valid since it easily follows that $\min(r, r^*) > \epsilon_1/2$.

Let $x \in B(0, \epsilon_1)$. Clearly,

$$\frac{1}{r} - \frac{1}{r^*} = \frac{4dx_3}{r^* r(r + r^*)} .$$

Consider first the case $d^2 \le C_0\delta_0(x_1^2 + x_2^2)$. Here we have fact that

$$\left|\frac{1}{r} - \frac{1}{r^*}\right| \le 4C_0^{3/2}\delta_0^{1/2}\frac{(x_1^2 + x_2^2)^{3/2}}{r^* r(r + r^*)} \le 2C_0^{3/2}\delta_0^{1/2} .$$

For the case $d^2 > C_0\delta_0(x_1^2 + x_2^2)$ we have

$$|x_3| \le \frac{d^2}{\delta_0}\,,$$

which would mean in particular that $|x_3| \le d/2$ and hence that $\min(r, r^*) \ge d/2$. Consequently,

$$\Big|\frac{1}{r} - \frac{1}{r^*}\Big| \le \frac{4}{\delta_0}\frac{d^3}{r^* r(r + r^*)} \le \frac{1}{\delta_0}\,.$$

Proof (Proof of (2.23)**).** *step 1:* We first prove that

$$|G(x,\xi)| \le \frac{C(\Omega)}{|x-\xi|}\,. \tag{A.2}$$

Let g be given by (2.2). Since $g(\cdot,\xi)$ is harmonic for all $\xi \in \Omega$ we have by the maximum principle

$$\|g(\cdot,\xi)\|_\infty \le \|g(\cdot,\xi)\|_{L^\infty(\partial\Omega)} \le \frac{1}{4\pi d(\xi,\partial\Omega)}\,.$$

Consequently,

$$\sup_{(x,\xi)\in\Omega\times\Omega\setminus\Omega_{\delta_0/2}} |x-\xi| G(x,\xi) \le \frac{1}{4\pi} + \frac{\operatorname{diam}\Omega}{2\pi\delta_0} \le C\,. \tag{A.3}$$

Suppose that $\xi \in \Omega_{\delta_0/2}$. Set then

$$g(x,\xi) = h(x,\xi) - \frac{1}{4\pi|x-\xi^*|}\,.$$

Since h is harmonic in Ω we have by the previous lemma that

$$\sup_{\xi\in\Omega\setminus\Omega_{\delta_0/2}} \|h(\cdot,\xi)\|_\infty \le C\,. \tag{A.4}$$

Furthermore, since

$$u = \frac{1}{|x-\xi^*|} - \frac{1}{|x-\xi|}$$

is harmonic in $\Omega\setminus\{\xi\}$ and since $u_{x\to\xi} = -\infty$, we obtain that

$$\frac{1}{|x-\xi^*|} \le \frac{1}{|x-\xi|} + \|u(\cdot,\xi)\|_{L^\infty(\partial\Omega)} \le \frac{1}{|x-\xi|} + C\,. \tag{A.5}$$

Consequently,

$$\sup_{(x,\xi)\in\Omega\times\Omega\setminus\Omega_{\delta_0/2}} |x-\xi| G(x,\xi) \le \frac{1}{4\pi} + \sup_{(x,\xi)\in\Omega\times\Omega\setminus\Omega_{\delta_0/2}} \Big|\frac{|x-\xi|}{4\pi|x-\xi^*|}\Big| + \sup_{(x,\xi)\in\Omega\times\Omega\setminus\Omega_{\delta_0/2}} |x-\xi|\,|h(x,\xi)|\,,$$

which by (A.4) and (A.5) yields

$$\sup_{(x,\xi)\in\Omega\times\Omega\setminus\Omega_{\delta_0/2}} |x-\xi| G(x,\xi) \le C\,.$$

One can now readily verify (A.2) from the above and (A.3).

Step 2: We next show that

$$|\nabla_\xi G(x,\xi)| \le \frac{C}{|x-\xi|^2}\,. \tag{A.6}$$

Let $\delta(\Omega) > 0$ be the same as in Lemma 6.5 in [4]. We set $\delta_2 = \min(\delta, \delta_0)$. Suppose first that $x \in \Omega \setminus \Omega_{\delta_2/2}$. Let $\delta_1 = \min(\delta_2/2, |x-\xi|/2)$. Since G is harmonic in $B(x, \delta_1)$ we can apply to it Theorem 4.8 in [4] to obtain, with the aid of (A.2), that there exists $C > 0$, which is independent of x and ξ such that

$$|\nabla_x G(x,\xi)| \leq \frac{C}{\delta_1} \|G(\cdot,\xi)\|_{L^\infty(B(x,\delta_1)} \leq \frac{C}{|x-\xi|\delta_1}\,.$$

Hence,

$$\sup_{(x,\xi)\in\Omega\setminus\Omega_{\delta_2/2}\times\Omega} \big|\, |x-\xi|^2 \nabla_x G(x,\xi)\, \big| \leq C\,. \tag{A.7}$$

Next, let $x \in \Omega_{\delta_2/2}$. In this case we distinguish again between two different cases

1. $|x-\xi| < 3d(x,\partial\Omega)\,,$
2. $|x-\xi| \geq 3d(x,\partial\Omega)\,.$

In case 1 we us the fact that G is harmonic in $B(x, |x-\xi|/4)$ to obtain, as before, that

$$|\nabla_x G(x,\xi)| \leq \frac{C}{|x-\xi|} \|G(\cdot,\xi)\|_{L^\infty(B(x,|x-\xi|/4)} \leq \frac{C}{|x-\xi|^2}\,. \tag{A.8}$$

In case 2 we denote by $\bar{x}$ the projection of x on $\partial\Omega$. Set further $\delta_3 = 3d(x,\partial\Omega)/2 < \delta$. Let $D = B(\bar{x},\delta_3)\cap\Omega$. Then, since G is harmonic in D we obtain by Lemma 6.5 in [4] that there exists $C > 0$, independent of ξ, such that

$$|\nabla_x G(x,\xi)| \leq \frac{C}{d(x,\partial D\setminus\partial\Omega)} \|G(\cdot,\xi)\|_{L^\infty(D)} \leq \frac{C}{|x-\xi|^2}\,. \tag{A.9}$$

Combining the above, (A.7), and (A.8) yields

$$\sup_{(x,\xi)\in\Omega\times\Omega} \big|\, |x-\xi|^2 \nabla_x G(x,\xi)\, \big| \leq C\,.$$

The symmetry of G then yields (A.6).

Step 3: We now complete the proof of (2.23). To this end we apply the procedure of step 2 to $\partial G(x,\xi)/\partial\xi_i$ for each $i \in \{1,2,3\}$ to obtain that

$$\sup_{(x,\xi)\in\Omega\times\Omega} \big|\, |x-\xi|^3 \nabla_x \partial G(x,\xi)/\partial\xi_i\, \big| \quad \forall i \in \{1,2,3\}\,,$$

(note that $\partial G(x,\xi)/\partial\xi_i|_{x\in\partial\Omega} = 0$). This completes the proof of (2.23).

Remark 3. While Lemma 6.5 in [4] is stated for $B(\bar{x},\rho)\cap\Omega$ for $\rho = \delta$ it is also valid for any $\rho < \delta$ as is evident from the proof. Furthermore, there is no need that G would be harmonic in Ω: it is enough that it is harmonic in D. Finally, while the statement of Lemma 6.5 would require a bound on $\|G(\cdot,\xi)\|_{L^\infty(\Omega)}$, it is again evident from the proof that (A.9) is correct also with $\|G(\cdot,\xi)\|_{L^\infty(D)}$.

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