The Clausius-Mossotti formula in a dilute random medium with fixed volume fraction.

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Abstract

We consider a medium composed of randomly dispersed spherical, identical, inclusions in a bounded domain, with conductivity different than that of the host medium. We study the limit where the number of the inclusions tends to infinity but their volume fraction remains fixed. For small volume fractions, we prove convergence, in $W^{1,p}$ norm ($1 < p < 2$), of the expectation of the solution of the steady state heat equation, to the solution of an effective medium problem, where the conductivity is given by the Clausius-Mossotti formula. This improves a previous result which required that the volume fraction tends to zero as the inclusions' number goes to infinity.

1 Introduction

Consider N spherical inclusions of conductivity σ and radius ϵ immersed in a different medium of conductivity 1. Prescribing the temperature (or the electric potential) on the boundary, the temperature field inside can be described as the unique, weak solution of the problem

$$\begin{cases} \nabla \cdot \big(a(x,\eta_1,\dots,\eta_N)\nabla\phi\big) = 0 & \text{in } \Omega\,, \\ \phi = f & \text{on } \partial\Omega\,. \end{cases} \tag{1.1}$$

In the above, $\Omega \subset \mathbb{R}^3$ is bounded and smooth (say $C^{2,\gamma}$ for some positive γ), $\{\eta_i\}_{i=1}^N$ denote the spherical inclusion's centers, and

$$a(x,\eta_1,\dots,\eta_N) = \begin{cases} \sigma & \forall x \in \bigcup\limits_{n=1}^{N} B(\eta_n,\epsilon)\,, \\ 1 & \forall x \in \Omega \setminus \bigcup\limits_{n=1}^{N} B(\eta_n,\epsilon)\,, \end{cases} \tag{1.2}$$

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and $f \in C^{2,\gamma}(\partial\Omega)$ for some $\gamma > 0$.

The particles' centers are assumed to be randomly distributed according to the joint probability density function $f_N(\eta_1, \dots, \eta_N)$, which is assumed to be invariant to permutations of the centers as all particles are identical. Moreover, we assume that the inclusions cannot overlap, i.e.,

$$\exists 1 \le i < j \le N : \ |\eta_i - \eta_j| < 2\epsilon \Rightarrow f_N(\eta_1, \dots, \eta_N) = 0\,, \tag{1.3}$$

and that no inclusion can cross the boundary, i.e.,

$$\exists 1 \le i \le N : \ d(\eta_i, \partial\Omega) < \epsilon \Rightarrow f_N(\eta_1, \dots, \eta_N) = 0\,. \tag{1.4}$$

Let

$$f_k(\eta_1, \dots, \eta_k) = \int_{\Omega^{N-k}} f_N(\eta_1, \dots, \eta_k, \eta_{k+1}, \dots, \eta_N)\, d\eta_{k+1} \cdots d\eta_N\,,$$

denote the $k'th$ order marginal probability density. We assume here boundedness of the first three marginal densities

$$\|f_k\|_{L^\infty(\Omega^k)} \le C \quad \forall 1 \le k \le 3\,, \tag{1.5}$$

where C is independent of N and ϵ. Note that in [1] the same assumption was made for $1 \le k \le 5$. We denote the expectation of any function $F(x, \cdot) \in L^1(\Omega^N)$, where $x \in \Omega$, by

$$\mathbb{E}_f\big(F(x, \cdot)\big) = \int_{\Omega^N} F(x, \eta_1, \dots, \eta_N) f_N(\eta_1, \dots, \eta_N)\, d\eta_1 \cdots d\eta_N\,. \tag{1.6}$$

To the above assumptions on f_N we add the following set of assumptions

$$f_3(\eta_2, \eta_3/\eta_1) = \tilde{f}_2(\eta_2/\eta_1)\tilde{f}_2(\eta_3/\eta_1) + g(\eta_1, \eta_2, \eta_3)\,, \tag{1.7a}$$

in which g satisfies

$$g(\eta_1, \eta_2, \eta_3) = \begin{cases} -\tilde{f}_2(\eta_2/\eta_1)\tilde{f}_2(\eta_3/\eta_1) & |\eta_2 - \eta_3| \le 2\epsilon \\ 0 & |\eta_2 - \eta_1| \le 2\epsilon \\ 0 & |\eta_3 - \eta_1| \le 2\epsilon\,. \end{cases} \tag{1.7b}$$

Furthermore

$$|g| \le C_\alpha \frac{\epsilon^\alpha}{|\eta_2 - \eta_3|^\alpha}\,, \tag{1.7c}$$

for some $\alpha > 0$. In addition, we have that

$$\tilde{f}_2(\eta_2/\eta_1) = \tilde{C} f_2(\eta_2/\eta_1)\,, \tag{1.7d}$$

We further assume that

$$\tilde{f}_2(\eta_2/\eta_1) = h_1(|\eta_2 - \eta_1|)h_2(\eta_2)\,. \tag{1.7e}$$

where $h_1 : \mathbb{R}_+ \to \mathbb{R}_+$ is bounded, measurable and satisfies $h_1(x) = 0$ for all $x \le 2\epsilon$, and $h_2 \in Lip(\Omega, \mathbb{R}_+)$ has a global Lipschitz norm L which is independent of N and ϵ, and satisfies $h_2(x) = 0$ whenever $d(x, \partial\Omega) \le \epsilon$.

The above assumptions, except for (1.7e), represent a requirement for some mild asymptotic independence of the inclusions' centers η_2 and η_3, when the location of η_1 is given. Note that (1.3) does not allow for independence, and hence, it is necessary to assume (1.7b). Property (1.7c) assumes decay of the interaction between η_2 and η_3 as the distance between them grows. The condition (1.7d) reconciles the asymptotic decay of the interaction (1.7c) with the expectation that

$$\int_\Omega f_3(\eta_2, \eta_3/\eta_1)\, d\eta_3 = f_2(\eta_2/\eta_1)\,.$$

(Note that $\tilde{C}$ must be close to 1 in view of (1.7c).) Finally, (1.7e) assumes short range radially symmetric behaviour of the conditional density $f_2(\eta_2/\eta_1)$. One case in which all the above assumptions are satisfied is the uniform distribution case, i.e,

$$f_N(\eta_1, \dots, \eta_N) = \begin{cases} 0 & \exists 1 \le i < j \le N:\ |\eta_i - \eta_j| < 2\epsilon \\ 0 & \exists 1 \le i \le N:\ d(\eta_i, \partial\Omega) < \epsilon \\ C_N & \text{otherwise}\,, \end{cases}$$

where the value of C_N is obtained through the requirement that $\Pr(\Omega^N) = 1$. Naturally, (1.7) represents a much wider class of probability measures.

We focus our attention on the small particle limit in a dilute (or dispersive [8]) medium, i.e., we first let $\epsilon \to 0$ but keep the volume fraction $\bar{\beta}$ fixed, where

$$\bar{\beta} = \frac{4\pi}{3}\frac{N\epsilon^3}{|\Omega|}\,, \tag{1.8}$$

and then let $\bar{\beta} \to 0$. Note that N must tend to infinity as $\epsilon \to 0$ when $\bar{\beta}$ is fixed. We may allow $\bar{\beta}$ and ϵ to simultaneous tend to 0 as long as for some $1 < p < 2$, there exists $C > 0$, independent of both ϵ and $\bar{\beta}$, such that

$$\frac{\epsilon^{\frac{2p-1}{p}}}{C} < \bar{\beta}\,. \tag{1.9}$$

The above inequality can alternatively be represented as

$$\frac{C}{\bar{\beta}^{\frac{p+1}{2p-1}}} \le N\,,$$

with a different value of C. Note that by (1.9) it follows that $N \geq C\epsilon^{-(p+1)/p}$ and hence $N \to \infty$ as $\epsilon \to 0$. In [1] we have assumed (1.9) with a greater lower bound, and together with an additional upper bound, i.e.,

$$\frac{\epsilon}{C} < \bar{\beta} \leq \frac{C}{\ln^4 \epsilon^{-1}} ,$$

(but (1.7) is not needed there).

Define, next, the local volume fraction for all $x \in \Omega$

$$\beta(x) = N \int_{B(x,\epsilon)\cap\Omega_\epsilon} f_1(\eta)\, d\eta\, , \tag{1.10}$$

where

$$\Omega_\epsilon = \{x \in \Omega \,|\, d(x, \partial\Omega) > \epsilon\}\, . \tag{1.11}$$

Note that $\beta(x)$ is the the probability that $x \in \bigcup_{n=1}^{N} B(\eta_n, \epsilon)$. It follows from (2.8) that

$$\|\beta(\cdot)\|_\infty \leq C\bar{\beta}\, . \tag{1.12}$$

Where $\|\cdot\|_p$ denotes the $L^p(\Omega)$ norm ($p = \infty$ above). When L^p norms are evaluated over domains different than Ω, we shall include them explicitly in the notation.

Under the above assumptions we prove the following theorem

Theorem 1. *Let $\phi(\cdot, \eta_1, \ldots, \eta_N) \in H^1(\Omega)$ denote the unique weak solution of* (1.1). *Suppose that* (1.5) *and* (1.7) *are satisfied. Let ϕ_e denote the solution of the effective medium problem*

$$\begin{cases} \nabla \cdot (a_e \nabla \phi_e) = 0 & \text{in } \Omega \\ \phi_e = f & \text{on } \partial\Omega\, , \end{cases} \tag{1.13}$$

where

$$a_e = 1 + \frac{3(\sigma - 1)}{2 + \sigma}\beta(x)\, . \tag{1.14}$$

Then, in the regime of (1.9), *for the same $1 < p < 2$, we have*

$$\|\mathbb{E}_f(\phi) - \phi_e\|_{1,p} \leq C(\Omega, \sigma)\bar{\beta}^{\frac{p+1}{p} - \frac{1}{2}}\, , \tag{1.15}$$

where $\|\cdot\|_{1,p}$ denotes the $W^{1,p}(\Omega)$ norm.

Throughout the sequel, we always refer to solutions in a weak sense, including places in the text where we do not state that explicitly.

The effective medium formula (1.14) was derived by Mossotti (1850) and Clausius (1879) and is therefore known as the Clausius-Mossotti formula. For a formal derivation of this formula the reader is referred to [12], which brings the classical derivation of Maxwell (1873) (cf. [9]). A rigorous proof has been provided in a two-dimensional periodic setting by Rayleigh [14], for a proof in three dimensional periodic medium the reader is referred to [8]. More general periodic settings have also been considered [10, 2].

An interesting proof of (1.14) is presented in [8], where an almost-periodic configuration of the particles is assumed. A small random deviation of the inclusions' centers from the lattice points is allowed. Berlyand and Mityushev [3], consider a periodic array of cells containing a finite number of randomly dispersed inclusions, whose volume fraction is not necessarily small. They evaluate an effective conductivity, which is averaged both in event space and over the cell.

In [1] a similar result is established with the following differences:

1. The bound is on the H^1 norm, with an error of $\bar{\beta}^{3/2}$, i.e., it is established that

$$\|\mathbb{E}_f(\phi) - \phi_e\|_{1,2} \le C(\Omega, \sigma)\bar{\beta}^{3/2} .$$

Compare the above with (1.15) where the estimates are on weaker norms with larger errors.

2. Boundedness of five marginal densities is assumed, whereas here we need only three.

3. (1.7) is not assumed. Instead we assume that

$$\bar{\beta} \le \frac{C}{\ln^4 \epsilon^{-1}} . \tag{1.16}$$

Had we assumed here here boundedness of five marginal densities, then, it would have been possible (most probably), using the techniques employed in this work, to obtain the following bound

$$\|\mathbb{E}_f(\phi) - \phi_e\|_{1,p} \le C\bar{\beta}^{\frac{p+1}{p}} .$$

Thus, compromising on weaker norms should allow for a smaller error (and so would a higher number of bounded marginal densities). We note that the trade between (1.16) and (1.7) allows for an easier comparison with results from random homogenization theory [7, 13]. Some significant progress in this direction has recently been reported in [11], though for a very different setting.

The rest of the contribution is dedicated to the proof of Theorem 1. In the next section we cite and obtain some regularity results, and derive some preliminary estimates for media with one inclusion. In § 3, we use (1.7) to obtain a bound on the average of the single inclusion solution obtained in § 2. Finally, in § 4, we complete the proof of Theorem 1.

2 Preliminaries

2.1 Some regularity results

The following Proposition is a straightforward result of Section 4.3 in [5].

Proposition 2.1. *Let $1 < p < 2$. There exists a weak solution, $u \in W^{1,p}(\Omega)$, of*

$$\begin{cases} \Delta u = \operatorname{div} f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{2.1}$$

where $f \in L^p(\Omega, \mathbb{R}^3)$. Furthermore,

$$\|\nabla u\|_p \leq C\|f\|_p\,. \tag{2.2}$$

Proof. Clearly,

$$\|\nabla u\|_p = \sup_{\|F\|_q=1} \int_\Omega \nabla u \cdot F\,dx\,,$$

where $q = p/(p-1)$ denotes the Hölder conjugate of p. Let then $\|F\|_q = 1$. Since $q > 2$, we have by § 4.3 in [5] (cf. also Proposition II.6.2 in [4]) that a weak solution $v \in W^{1,q}(\Omega)$ exists for

$$\begin{cases} -\Delta v = \nabla \cdot F & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{2.3}$$

By Theorem 4.6 in [5] (cf. also Corollary II.6.1 in [4]) we have that there exists $C(\Omega, p) > 0$ such that

$$\|\nabla v\|_q \leq C\|F\|_q \leq C\,. \tag{2.4}$$

Suppose first $f \in C^\infty(\Omega, \mathbb{R}^3)$. Then, there exists a unique solution $u \in W^{1,p}(\Omega)$ for (2.1). Multiplying (2.3) by u yields, after integration by parts, with the aid of

$$\Big|\int_\Omega \nabla u \cdot F\,dx\Big| = \Big|\int_\Omega f \cdot \nabla v\,dx\Big| \leq \|f\|_p\|\nabla v\|_q \leq C\|f\|_p$$

Taking the infimum with respect to all $F \in L^q(\Omega, \mathbb{R}^3)$, with unity norm, yields (2.2) for all $f \in C^\infty(\Omega, \mathbb{R}^3)$. The proof of the Proposition for all $f \in L^p(\Omega, \mathbb{R}^3)$ follows by density. □

Consider next the problem,

$$\begin{cases} \operatorname{div} A\nabla u = \operatorname{div} f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{2.5}$$

where $A \in L^\infty(\Omega, M_{3\times3})$ is uniformly elliptic, and $f \in L^p(\Omega, \mathbb{R}^3)$. Here we prove

Lemma 2.1. *Let $u \in H^1(\Omega)$ denote the weak solution of (2.5), with $f \in L^2(\Omega,\mathbb{R}^3)$ and $A \in L^\infty(\Omega, M_{3\times 3})$ which is uniformly elliptic, i.e, there exists $\alpha > 0$ such that*

$$\xi \cdot A(x)\xi \geq \alpha|\xi|^2 \quad \forall \xi \in \mathbb{R}^3 .$$

Let $1 < p < 2$. There exists $C(\alpha, p, \Omega)$ such that

$$\|\nabla u\|_p \leq C\big(\|A - I\|_{\frac{2}{2-p}} \|f\|_2 + \|f\|_p\big) . \tag{2.6}$$

Proof. We write

$$\begin{cases} -\Delta u = \operatorname{div}(A - I)\nabla u + \operatorname{div} f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega . \end{cases}$$

By (2.2) we have

$$\|\nabla u\|_p \leq C(\|(A - I)\nabla u\|_p + \|f\|_p) \leq C(\|(A - I)\|_{2/(2-p)} \|\nabla u\|_2 + \|f\|_p) . \tag{2.7}$$

Multiplying (2.5) by u and integrating by parts leads, however, to

$$\alpha\|\nabla u\|_2 \leq \|f\|_2 ,$$

which, when substituted into (2.7) immediately yields (2.6). $\square$

2.2 Single inclusion

As in [1] we define now the one-particle problem. Let Ω_ϵ be defined by (1.11). For every $\eta \in \Omega_\epsilon$, let $\psi_1(\cdot, \eta) \in H^1(\Omega)$ denote the unique solution of

$$\begin{cases} \nabla \cdot \big(a_1(\cdot, \eta)\nabla\psi_1(\cdot, \eta)\big) = 0 & \text{in } \Omega \\ \psi_1(\cdot, \eta) = f & \text{on } \partial\Omega \end{cases} , \tag{2.8}$$

where

$$a_1(x, \eta) = \begin{cases} \sigma & x \in B(\eta, \epsilon) \\ 1 & x \in \Omega \setminus B(\eta, \epsilon) \end{cases} .$$

Let $\bar{\phi}$ denote the unique solution of

$$\begin{cases} \Delta\bar{\phi} = 0 & \text{in } \Omega \\ \bar{\phi} = f & \text{on } \partial\Omega \end{cases} \tag{2.9}$$

Note that by the assumptions on $\partial\Omega$ and f we have that $\bar{\phi} \in C^2(\bar{\Omega})$. Set then

$$\phi_1(\cdot, \eta) := \psi_1(\cdot, \eta) - \bar{\phi} . \tag{2.10}$$

For all $\eta \in \Omega_\epsilon$, define $\phi_0(\cdot,\eta) : \Omega \to \mathbb{R}$ as

$$\phi_0(x,\eta) = (x-\eta)\cdot\nabla\bar{\phi}(\eta) \times \begin{cases} \frac{1-\sigma}{2+\sigma}\frac{\epsilon^3}{|x-\eta|^3} & x \in \Omega \setminus B(\eta,\epsilon) \\ \frac{1-\sigma}{2+\sigma} & x \in B(\eta,\epsilon) \end{cases}, \tag{2.11}$$

and then set

$$u_1(x,\eta) = \phi_1(x,\eta) - \phi_0(x,\eta)\,. \tag{2.12}$$

We now cite a few results from [1].

Lemma 2.2. *Let ϕ_1 be given by (2.10), and $u_1(\cdot,\eta) : \Omega \to \mathbb{R}$ by (2.12). Then, for all $(\xi,\eta) \in \Omega \times \Omega$ we have*

$$\|\nabla\phi_1(\cdot,\eta)\|_{L^2\left(B(\xi,\epsilon)\cap\Omega\right)} \leq C(\Omega,\sigma)\frac{\epsilon^{9/2}}{|\xi-\eta|^3}\,, \tag{2.13}$$

and

$$\|u_1(\cdot,\eta)\|_{1,2} \leq C(\Omega,\sigma)\Big(\epsilon^{5/2} + \frac{\epsilon^3}{d(\eta,\partial\Omega)^{3/2}}\Big)\,. \tag{2.14}$$

In the sequel we need estimates of u_1 that are more refined then (2.14). To this end we need first the following auxiliary result

Lemma 2.3. *Let $(\xi,\eta) \in \Omega_\epsilon \times \Omega_\epsilon$ and suppose that $|\xi-\eta| \geq 3\epsilon$. Then, there exists $C > 0$, independent of ϵ, η, and ξ, such that*

$$[|\xi-\eta|^2 + 4d(\xi,\partial\Omega)d(\eta,\partial\Omega)] \leq C \inf_{(x,\zeta)\in B(\xi,\epsilon)\times\partial B(\eta,\epsilon)} [|x-\zeta|^2 + 4d(x,\partial\Omega)d(\zeta,\partial\Omega)]\,. \tag{2.15}$$

Proof. Step 1: Prove that there exists $C > 0$ such that for all $x \in B(\xi,\epsilon)$ we have

$$[|x-\eta|^2 + 4d(x,\partial\Omega)d(\eta,\partial\Omega)] \leq C \inf_{\zeta\in\partial B(\eta,\epsilon)} [|x-\zeta|^2 + 4d(x,\partial\Omega)d(\zeta,\partial\Omega)]\,. \tag{2.16}$$

Let $x \in B(\xi,\epsilon)$. Clearly, for every $\zeta \in \partial B(\eta,\epsilon)$ we have

$$d(\eta,\partial\Omega) \leq d(\zeta,\partial\Omega) + \epsilon \quad ; \quad |x-\eta| \leq |x-\zeta| + \epsilon \leq 2|x-\zeta|\,. \tag{2.17a,b}$$

Consider first the case where $\eta \in \Omega_{2\epsilon}$. Here we have

$$d(\eta,\partial\Omega) \leq 2d(\zeta,\partial\Omega)\,,$$

which with the aid of (2.17b) readily yields (2.16).

Suppose next that $d(\eta,\partial\Omega) \leq 2\epsilon$. Here we have for all $\zeta \in \partial B(\eta,\epsilon)$

$$d(\eta,\partial\Omega) \leq 2\epsilon \leq 2|x-\zeta|\,.$$

Furthermore,

$$d(x,\partial\Omega) \leq d(\zeta,\partial\Omega) + |x-\zeta| \leq 3\epsilon + |x-\zeta| \leq 4|x-\zeta|\,.$$

Hence,

$$d(x,\partial\Omega)d(\eta,\partial\Omega) \leq 8|x-\zeta|^2\,,$$

which together with (2.17b) completes the proof of (2.16).

Step 2: Prove that there exists $C>0$ such that

$$[|\xi-\eta|^2 + 4d(\xi,\partial\Omega)d(\eta,\partial\Omega)] \leq C \inf_{x\in B(\xi,\epsilon)} [|x-\eta|^2 + 4d(x,\partial\Omega)d(\eta,\partial\Omega)]\,. \tag{2.18}$$

The proof can be established in a similar manner to the proof of (2.16).

Combining (2.16) and (2.18) yields (2.15). □

We can now prove

Lemma 2.4. *Let ϕ_1 be given by* (2.10)*. Then, for all $(\xi,\eta)\in\Omega_\epsilon\times\Omega_\epsilon$*

$$\|\nabla u_1(\cdot,\eta)\|_{L^2\left(B(\xi,\epsilon)\right)} \leq C\epsilon^{9/2}\Big\{\frac{1}{|\xi-\eta|^3}\Big[\epsilon + \frac{\epsilon^{3/2}}{d(\eta,\partial\Omega)^{3/2}}\Big] + \frac{1}{[|\xi-\eta|^2 + 4d(\xi,\partial\Omega)d(\eta,\partial\Omega)]^{3/2}}\Big\}\,. \tag{2.19}$$

Proof. Let $G:\Omega\times\Omega\to\mathbb{R}$ denote the Green's function of the Laplace operator with homogeneous Dirichlet boundary conditions in Ω, i.e.,

$$G(x,\xi) = \tau(x,\xi) + \frac{1}{4\pi|x-\xi|}\,, \tag{2.20}$$

where $\tau(\cdot,\xi)$ is harmonic in Ω, for all $\xi\in\Omega$, and satisfies $\tau = -1/(4\pi|x-\xi|)$ on $\partial\Omega$. Since (cf. [1])

$$\frac{1}{\sigma-1}\phi_1(x,\eta) = \int_{\partial B(\eta,\epsilon)} \bar\phi(\zeta)\frac{\partial G}{\partial\nu}(x,\zeta)\,ds_\zeta + \int_{\partial B(\eta,\epsilon)} \phi_0(\zeta,\eta)\frac{\partial G}{\partial\nu}(x,\zeta)\,ds_\zeta + \int_{\partial B(\eta,\epsilon)} u_1(\zeta,\eta)\frac{\partial G}{\partial\nu}(x,\zeta)\,ds_\zeta\,,$$

we may use the fact that

$$\frac{1}{\sigma-1}\phi_0(x,\eta) = \int_{\partial B(\eta,\epsilon)} [\bar\phi(\eta) + \zeta\cdot\nabla\bar\phi(\eta) + \phi_0(\zeta,\eta)]\frac{\partial}{\partial\nu}\frac{1}{4\pi|x-\zeta|}\,ds_\zeta$$

for all $x \in \Omega \setminus B(\eta, \epsilon)$, to obtain that

$$\frac{1}{\sigma - 1} u_1(x, \eta) = \int_{\partial B(\eta,\epsilon)} [\bar{\phi}(\zeta) + \phi_0(\zeta, \eta)] \frac{\partial \tau}{\partial \nu}(x, \zeta)\, ds_\zeta + \\ \int_{\partial B(\eta,\epsilon)} [\bar{\phi}(\zeta) - \bar{\phi}(\eta) - \zeta \cdot \nabla \bar{\phi}(\eta)] \frac{\partial}{\partial \nu} \frac{1}{4\pi |x - \zeta|}\, ds_\zeta + \int_{\partial B(\eta,\epsilon)} u_1(\zeta, \eta) \frac{\partial G}{\partial \nu}(x, \zeta)\, ds_\zeta \,. \tag{2.21}$$

We now estimate the various terms on the right-hand-side of (2.21). For the first term we have

$$\int_{\partial B(\eta,\epsilon)} [\bar{\phi}(\zeta) + \phi_0(\zeta, \eta)] \frac{\partial \tau}{\partial \nu}(x, \zeta)\, ds_\zeta = \int_{\partial B(\eta,\epsilon)} [\bar{\phi}(\zeta) - \bar{\phi}(\eta) + \phi_0(\zeta, \eta)] \frac{\partial \tau}{\partial \nu}(x, \zeta)\, ds_\zeta \,.$$

Since

$$\|\bar{\phi}(\zeta) - \bar{\phi}(\eta) + \phi_0(\zeta, \eta)\|_{L^\infty(\partial B(\eta,\epsilon))} \leq C\epsilon \,,$$

we obtain that

$$\left| \nabla \int_{\partial B(\eta,\epsilon)} [\bar{\phi}(\zeta) + \phi_0(\zeta, \eta)] \frac{\partial \tau}{\partial \nu}(x, \zeta)\, ds_\zeta \right| \leq C\epsilon^3 \|D^2 \tau(x, \cdot)\|_{L^\infty(\partial B(\eta,\epsilon))} \,, \tag{2.22}$$

where $\|D^2\tau\|(x,\xi)$ denotes some appropriate norm of the Hessian matrix $\nabla_x \nabla_\xi \tau$. In appendix A, we show that there exists $C(\Omega)$ such that

$$\|D^2 \tau(x, \zeta)\| \leq \frac{C}{[|x - \zeta|^2 + 4d(x, \partial\Omega) d(\zeta, \partial\Omega)]^{3/2}} \,, \tag{2.23}$$

and hence, by (2.22),

$$\left| \nabla \int_{\partial B(\eta,\epsilon)} [\bar{\phi}(\zeta) + \phi_0(\zeta, \eta)] \frac{\partial \tau}{\partial \nu}(x, \zeta)\, ds_\zeta \right| \\ \leq C\epsilon^3 \sup_{\zeta \in \partial B(\eta,\epsilon)} \frac{1}{[|x - \zeta|^2 + 4d(x, \partial\Omega) d(\zeta, \partial\Omega)]^{3/2}} \,. \tag{2.24}$$

For the second term on the right-hand-side of (2.21), it can be easily verified that

$$\left| \nabla \int_{\partial B(\eta,\epsilon)} [\bar{\phi}(\zeta) - \bar{\phi}(\eta) - \zeta \cdot \nabla \bar{\phi}(\eta)] \frac{\partial}{\partial \nu} \frac{1}{4\pi |x - \zeta|}\, ds_\zeta \right| \leq C \frac{\epsilon^4}{|x - \zeta|^3} \tag{2.25}$$

Finally, we turn to estimate the third term on the right-hand-side of (2.21). Let

$$\bar{u}_1(\eta) = \fint_{B(\eta,\epsilon)} u_1(x, \eta)\, dx$$

Clearly,

$$\Big|\nabla \int_{\partial B(\eta,\epsilon)} u_1(\zeta,\eta)\frac{\partial G}{\partial \nu}(x,\zeta)\,ds_\zeta\Big| = \Big|\nabla \int_{\partial B(\eta,\epsilon)} \big(u_1(\zeta,\eta)-\bar u_1(\eta)\big)\frac{\partial G}{\partial \nu}(x,\zeta)\,ds_\zeta\Big|$$
$$\le \|u_1-\bar u_1\|_{L^2(\partial B(\eta,\epsilon))}\|D^2G\|_{L^2(\partial B(\eta,\epsilon))}\,. \quad (2.26)$$

We now argue as in [1] that

$$\|u_1-\bar u_1\|_{L^2(\partial B(\eta,\epsilon))} \le C\epsilon^{1/2}\|\nabla u_1\|_{L^2(B(\eta,\epsilon))}$$

and substitute the above into (2.26) to obtain

$$\Big|\nabla \int_{\partial B(\eta,\epsilon)} u_1(\zeta,\eta)\frac{\partial G}{\partial \nu}(x,\zeta)\,ds_\zeta\Big| \le C\epsilon^{1/2}\|\nabla u_1\|_{L^2(B(\eta,\epsilon))}\|D^2G\|_{L^2(\partial B(\eta,\epsilon))}\,. \quad (2.27)$$

Next, we recall from [1] that

$$\|D^2G\|(x,\xi) \le \frac{C(\Omega)}{|x-\xi|^3}\,.$$

Substituting the above together with (2.14) yields

$$\Big|\nabla \int_{\partial B(\eta,\epsilon)} u_1(\zeta,\eta)\frac{\partial G}{\partial \nu}(x,\zeta)\,ds_\zeta\Big| \le C\frac{\epsilon^{3/2}}{d(x,\partial B(\eta,\epsilon))^3}\Big(\epsilon^{5/2}+\frac{\epsilon^3}{d(\eta,\partial\Omega)^{3/2}}\Big)\,. \quad (2.28)$$

Consider first the case where $|\xi-\eta|\ge 3\epsilon$. Substituting (2.28) together with (2.15), (2.24), (2.25), and (2.21) yields

$$|\nabla u_1(x,\eta)| \le C\epsilon^3\Big\{\frac{1}{|\xi-\eta|^3}\Big[\epsilon+\frac{\epsilon^{3/2}}{d(\eta,\partial\Omega)^{3/2}}\Big]+\frac{1}{[|\xi-\eta|^2+4d(\xi,\partial\Omega)d(\eta,\partial\Omega)]^{3/2}}\Big\}\,,$$

from which (2.19) easily follows for $|\xi-\eta|\ge 3\epsilon$. For $|\xi-\eta|<3\epsilon$ (2.19) follows immediately from (2.14) since

$$\frac{1}{[|\xi-\eta|^2+4d(\xi,\partial\Omega)d(\eta,\partial\Omega)]^{3/2}} \le \frac{1}{8\epsilon^{3/2}d(\eta,\partial\Omega)^{3/2}}\,.$$

□

We next prove the following auxiliary estimate

Lemma 2.5. *Let $h:\mathbb{R}_+\to\mathbb{R}_+$ denote a bounded, measurable function satisfying $h(r)=0$ for all $r<2\epsilon$. Then, for every $\eta_1\in\Omega_\epsilon$ we have*

$$\Big\|\int_{\Omega_\epsilon}\nabla\phi_1(x,\eta_2)h(|\eta_2-\eta_1|)\,d\eta_2\Big\|_{L^2(B(\eta_1,\epsilon)} \le C\epsilon^{9/2}(1+|\ln d(\eta_1,\partial\Omega_\epsilon)|)\,. \quad (2.29)$$

Proof. Step 1: Prove that

$$\Big\| \int_{\Omega_\epsilon} \nabla\phi_0(x,\eta_2) h(|\eta_2-\eta_1|)\, d\eta_2 \Big\|_{L^2(B(\eta_1,\epsilon)} \leq C\epsilon^{9/2}(1+|\ln d(\eta_1,\partial\Omega_\epsilon)|)\,. \tag{2.30}$$

Clearly, by (2.11), when $|x-\eta_2|>\epsilon$ we have that

$$\nabla\phi_0(x,\eta_2) = \frac{\epsilon^3}{|x-\eta_2|^3}\nabla\bar{\phi}(\eta_2)\cdot\Big(I - 3\frac{(x-\eta_2)(x-\eta_2)}{|x-\eta_2|^2}\Big)\,,$$

where I is the 3×3 identity matrix. We next use the fact that $\bar{\phi}$ is bounded in $C^2(\bar{\Omega})$ to obtain that for all $|\eta_2-\eta_1|\geq 2\epsilon$,

$$\Big|\nabla\phi_0(x,\eta_2) - E\Big| \leq C\frac{\epsilon^3}{|\eta_1-\eta_2|^3}\Big[|\eta_2-\eta_1| + \frac{\epsilon}{|\eta_2-\eta_1|}\Big]\,,$$

where

$$E(\eta_2,\eta_1) = \frac{\epsilon^3}{|\eta_1-\eta_2|^3}\nabla\bar{\phi}(\eta_1)\cdot(I-3e_2e_2)\,,$$

and

$$e_2 = \frac{(\eta_1-\eta_2)}{|\eta_1-\eta_2|}\,.$$

Hence, for $D=\operatorname{Diam}\Omega$,

$$\int_{\Omega_\epsilon} |\nabla\phi_0(x,\eta_2) - E|\, h(|\eta_2-\eta_1|)\, d\eta_2 \leq C\epsilon^3\int_{2\epsilon}^D \Big[1+\frac{\epsilon}{r^2}\Big]\,dr \leq C\epsilon^3\,. \tag{2.31}$$

To complete the proof of (2.30) we thus compute

$$\int_{\Omega_\epsilon} E\, h(|\eta_2-\eta_1|)\, d\eta_2 = \int_{B(\eta_1,d(\eta_1,\partial\Omega_\epsilon)} E\, h(|\eta_2-\eta_1|)\, d\eta_2 + \int_{\Omega_\epsilon\setminus B(\eta_1,d(\eta_1,\partial\Omega_\epsilon))} E\, h(|\eta_2-\eta_1|)\, d\eta_2\,. \tag{2.32}$$

As

$$\int_{\partial B(\eta_1,r)} (I-3e_2e_2)\, ds = 0\,,$$

for all $r>0$, we have for the first term on the right-hand-side of (2.32)

$$\int_{B(\eta_1,d(\eta_1,\partial\Omega_\epsilon)} E\, h(|\eta_2-\eta_1|)\, d\eta_2 = 0\,. \tag{2.33}$$

For the second term we have

$$\Big|\int_{\Omega_\epsilon\setminus B(\eta_1,d(\eta_1,\partial\Omega_\epsilon))} E\, h(|\eta_2-\eta_1|)\, d\eta_2\Big| \leq C\epsilon^3\int_{d(\eta_1,\partial\Omega_\epsilon)}^D \frac{1}{r}\,dr \leq C\epsilon^3\ln\frac{1}{d(\eta_1,\partial\Omega_\epsilon)}\,.$$

Combining the above with (2.33), (2.32), and (2.31) yields

$$\Big|\int_{\Omega_\epsilon}\nabla\phi_0(x,\eta_2)\,h(|\eta_2-\eta_1|)\,d\eta_2\Big|\leq C\epsilon^3\ln\frac{1}{d(\eta_1,\partial\Omega_\epsilon)}\,,$$

from which (2.30) readily follows.

Step 2: Prove that

$$\Big\|\int_{\Omega_\epsilon}\nabla u_1(x,\eta_2)h(|\eta_2-\eta_1|)\,d\eta_2\|_{L^2(B(\eta_1,\epsilon)}\leq C\epsilon^{9/2}(1+|\ln d(\eta_1,\partial\Omega)|) \tag{2.34}$$

By Minkowsky inequality we have

$$\Big\|\int_{\Omega_\epsilon}\nabla u_1(\cdot,\eta_2)h(|\eta_2-\eta_1|)\,d\eta_2\|_{L^2(B(\eta_1,\epsilon)}\leq\int_{\Omega_\epsilon}\|\nabla u_1(\cdot,\eta_2)\|_{L^2(B(\eta_1,\epsilon)}h(|\eta_2-\eta_1|)\,d\eta_2\,.$$

Using (2.19) we then obtain

$$\Big\|\int_{\Omega_\epsilon}\nabla u_1(\cdot,\eta_2)h(|\eta_2-\eta_1|)\,d\eta_2\Big\|_{L^2(B(\eta_1,\epsilon)}\leq C\epsilon^{9/2}\times$$

$$\int_{\Omega_\epsilon}\Big\{\frac{1}{|\eta_1-\eta_2|^3}\Big[\epsilon+\frac{\epsilon^{3/2}}{d(\eta_2,\partial\Omega)^{3/2}}\Big]+\frac{1}{[|\eta_1-\eta_2|^2+4d(\eta_1,\partial\Omega)d(\eta_2,\partial\Omega)]^{3/2}}\Big\}\,h(|\eta_2-\eta_1|)d\eta_2\,. \tag{2.35}$$

We next estimate the contribution of each term in the integrand on the right-hand-side of (2.35). For the first term we have

$$\int_{\Omega_\epsilon}\frac{\epsilon}{|\eta_1-\eta_2|^3}h(|\eta_2-\eta_1|)\,d\eta_2\leq C\epsilon\ln\epsilon^{-1}\,. \tag{2.36}$$

Let $\delta>0$ be independent of ϵ. Clearly,

$$\int_{\Omega_\delta}\frac{1}{|\eta_1-\eta_2|^3}\frac{\epsilon^{3/2}}{d(\eta_2,\partial\Omega)^{3/2}}h(|\eta_2-\eta_1|)\,d\eta_2\leq C(\delta)\epsilon^{3/2}\ln\epsilon^{-1} \tag{2.37}$$

Let

$$\mathcal{D}=\{(x,y,z)\in\mathbb{R}^3\,|\,\epsilon\leq z\leq d\cap x^2+y^2+[z-d(\eta_1,\partial\Omega)]^2\geq 4\epsilon^2\}$$

By choosing a curvilinear coordinates (which can be set for sufficiently small δ), one can show that

$$\int_{\Omega_\epsilon\setminus\Omega_\delta}\frac{1}{|\eta_1-\eta_2|^3}\frac{\epsilon^{3/2}}{d(\eta_2,\partial\Omega)^{3/2}}h(|\eta_2-\eta_1|)\,d\eta_2\leq$$

$$C\epsilon^{3/2}\int_{\mathcal{D}}\frac{z^{-3/2}\,dx\,dy\,dz}{[x^2+y^2+(z-d(\eta_1,\partial\Omega))^2]^{3/2}}\,.$$

Moving to the cylindrical coordinate $r = (x^2 + y^2)^{1/2}$, and then substituting $s = r^2 + (z - d(\eta_1, \partial\Omega))^2$ yields

$$\int_{\Omega_\epsilon \setminus \Omega_\delta} \frac{1}{|\eta_1 - \eta_2|^3} \frac{\epsilon^{3/2}}{d(\eta_2, \partial\Omega)^{3/2}} h(|\eta_2 - \eta_1|) d\eta_2 \, d\eta_2 \leq$$

$$C \int_\epsilon^\infty \frac{\epsilon^{3/2} dz}{z^{3/2}} \int_{\max(4\epsilon^2, (z - d(\eta_1, \partial\Omega))^2)}^\infty \frac{ds}{s^{3/2}} \leq C \,.$$

Combining the above with (2.37) gives

$$\int_{\Omega_\epsilon} \frac{1}{|\eta_1 - \eta_2|^3} \frac{\epsilon^{3/2}}{d(\eta_2, \partial\Omega)^{3/2}} h(|\eta_2 - \eta_1|) \, d\eta_2 \leq C \tag{2.38}$$

Finally, for the third term we have

$$\int_{\Omega_\delta} \frac{1}{[|\eta_1 - \eta_2|^2 + 4d(\eta_1, \partial\Omega) d(\eta_2, \partial\Omega)]^{3/2}} \Big\} h(|\eta_2 - \eta_1|) d\eta_2 \leq \int_{\Omega_\delta} \frac{1}{[|\eta_1 - \eta_2|^2 + 4d(\eta_1, \partial\Omega)\delta]^{3/2}} \Big\} h(|\eta_2 - \eta$$

and since for all $\eta_2 \in \Omega_\delta$ we have

$$|\eta_1 - \eta_2|^2 + 4d(\eta_1, \partial\Omega)\delta \geq [\delta - d(\eta_1, \partial\Omega)]^2 + 4d(\eta_1, \partial\Omega)\delta = [\delta + d(\eta_1, \partial\Omega)]^2 \,,$$

we readily obtain that

$$\int_{\Omega_\delta} \frac{1}{[|\eta_1 - \eta_2|^2 + 4d(\eta_1, \partial\Omega) d(\eta_2, \partial\Omega)]^{3/2}} \Big\} h(|\eta_2 - \eta_1|) d\eta_2 \leq C(\delta) \,. \tag{2.39}$$

As above we obtain once again that

$$\int_{\Omega_\epsilon \setminus \Omega_\delta} \frac{1}{[|\eta_1 - \eta_2|^2 + 4d(\eta_1, \partial\Omega) d(\eta_2, \partial\Omega)]^{3/2}} \Big\} h(|\eta_2 - \eta_1|) d\eta_2 \leq$$

$$C \int_{\mathcal{D}} \frac{dx \, dy \, dz}{[x^2 + y^2 + (z - d(\eta_1, \partial\Omega))^2 + 4z d(\eta_1, \partial\Omega)]^{3/2}} \leq$$

$$C \int_\epsilon^D dz \int_{(z + d(\eta_1, \partial\Omega))^2}^\infty \frac{ds}{s^{3/2}} \leq C(1 + |\ln d(\eta_1, \partial\Omega)|) \,.$$

Combining the above with (2.38) and (2.36), yields (2.34), and by (2.12) also (2.29). $\square$

3 Asymptotic independence

Let $u, v \in L^2(\Omega, \mathbb{R}^3)$ and let $S \subset \Omega$. Set

$$\langle u, v \rangle_S = \int_S u \cdot v \, dx \,. \tag{3.1}$$

In this section we employ (1.7) to derive an improved bound on $\mathbb{E}_f\big(\langle\nabla\phi_1(\cdot,\eta_2),\nabla\phi_1(\cdot,\eta_3)\rangle_{B(\eta_1,\epsilon)}\big)$. In [1] we have obtained

$$|\mathbb{E}_f\big(\langle\nabla\phi_1(\cdot,\eta_2),\nabla\phi_1(\cdot,\eta_3)\rangle_{B(\eta_1,\epsilon)}\big)| \leq C\epsilon^9 \ln^2\frac{1}{\epsilon}\,.$$

Using (1.7) we shall be able to drop the logarithmic term from the right-hand-side.

We begin by repeating (1.7) once again,

$$f_3(\eta_2,\eta_3/\eta_1) = \tilde{f}_2(\eta_2/\eta_1)\tilde{f}_2(\eta_3/\eta_1) + g(\eta_1,\eta_2,\eta_3)\,, \tag{3.2a}$$

in which g satisfies

$$g(\eta_1,\eta_2,\eta_3) = \begin{cases} -\tilde{f}_2(\eta_2/\eta_1)\tilde{f}_2(\eta_3/\eta_1) & |\eta_2-\eta_3|\leq 2\epsilon \\ 0 & |\eta_2-\eta_1|\leq 2\epsilon \\ 0 & |\eta_3-\eta_1|\leq 2\epsilon\,. \end{cases} \tag{3.2b}$$

Furthermore

$$|g| \leq C_\alpha\frac{\epsilon^\alpha}{|\eta_2-\eta_3|^\alpha}\,, \tag{3.2c}$$

for some $\alpha>0$. Finally, we have that

$$\tilde{f}_2(\eta_2/\eta_1) = \tilde{C}f_2(\eta_2/\eta_1)\,, \tag{3.2d}$$

where by (3.2c) the normalization constant $\tilde{C}$ must satisfy (assuming $\alpha\neq 3$)

$$|\tilde{C}-1| \leq C\epsilon^{\min(\alpha,3)}\,. \tag{3.2e}$$

We assume further that

$$\tilde{f}_2(\eta_2/\eta_1) = h_1(|\eta_2-\eta_1|)h_2(\eta_2)\,, \tag{3.2f}$$

where $h_1:\mathbb{R}_+\to\mathbb{R}_+$ is bounded, measurable and satisfies $h_1(x)=0$ for all $x\leq 2\epsilon$, and $h_2\in Lip(\Omega_\epsilon,\mathbb{R}_+)$ has a global Lipschitz norm L which is independent of N and ϵ, and satisfies $h_2(x)=0$ for all $x\in\Omega\setminus\Omega_\epsilon$.

We continue by proving the following auxiliary lemma

Lemma 3.1. *Let $(\eta_1,\eta_2)\in\Omega^2$, $\epsilon>0$, and $\alpha>0$. Then, there exists $C>0$, independent of η_1, η_2, and ϵ, such that*

$$\int_{\Omega\setminus[B(\eta_1,2\epsilon)\cup B(\eta_2,2\epsilon)}\frac{\epsilon^\alpha}{|\eta_3-\eta_1|^3|\eta_2-\eta_3|^\alpha}\,d\eta_3 \leq C\frac{\epsilon^\alpha}{|\eta_2-\eta_1|^\alpha}\Big[1+\ln\frac{|\eta_2-\eta_1|}{\epsilon}\Big]\,. \tag{3.3}$$

Proof. Let

$$\Omega_1(\eta_1,\eta_2)=\{\,\eta_3\in\Omega\,|\,(\eta_3-\eta_1)\cdot(\eta_1-\eta_2)\geq -|\eta_1-\eta_2|^2/4\,\}\,.$$

It can be easily verified that for every $\eta_3\in\Omega_1$ we have

$$|\eta_3-\eta_2|^2\geq|\eta_3-\eta_1|^2+\frac{1}{2}|\eta_1-\eta_2|^2\,.$$

Hence (recall that $D=\mathrm{Diam}\,\Omega$)

$$\begin{aligned}&\int_{\Omega_1\backslash B(\eta_1,2\epsilon)}\frac{\epsilon^\alpha}{|\eta_3-\eta_1|^3|\eta_2-\eta_3|^\alpha}\,d\eta_3\leq\\ &\qquad\int_{\Omega\backslash B(\eta_1,2\epsilon)}\frac{\epsilon^\alpha}{|\eta_3-\eta_1|^3\Big[|\eta_1-\eta_3|^2+\frac{1}{2}|\eta_2-\eta_1|^2\Big]^{\alpha/2}}\,d\eta_3\leq\\ &\qquad\qquad C\int_\epsilon^D\frac{\epsilon^\alpha\,dr}{r^{1+\alpha}+r|\eta_2-\eta_1|^\alpha}\leq\int_\epsilon^{|\eta_1-\eta_2|}\frac{\epsilon^\alpha\,dr}{r|\eta_2-\eta_1|^\alpha}+\int_{|\eta_1-\eta_2|}^D\frac{\epsilon^\alpha\,dr}{r^{1+\alpha}}\,,\end{aligned}$$

from which we obtain that

$$\int_{\Omega_1\backslash B(\eta_1,2\epsilon)}\frac{\epsilon^\alpha}{|\eta_3-\eta_1|^3|\eta_2-\eta_3|^\alpha}\,d\eta_3\leq C\frac{\epsilon^\alpha}{|\eta_2-\eta_1|^\alpha}\Big[1+\ln\frac{|\eta_2-\eta_1|}{\epsilon}\Big]\,. \tag{3.4}$$

Next, define

$$\Omega_2(\eta_1,\eta_2)=\{\,\eta_3\in\Omega\,|\,(\eta_3-\eta_2)\cdot(\eta_2-\eta_1)\geq -|\eta_1-\eta_2|^2/4\,\}\,.$$

In a similar manner to (3.4) we then show

$$\begin{aligned}&\int_{\Omega_2\backslash B(\eta_2,2\epsilon)}\frac{\epsilon^\alpha}{|\eta_3-\eta_1|^3|\eta_2-\eta_3|^\alpha}\,d\eta_3\leq\\ &\qquad\int_{\Omega\backslash B(\eta_2,2\epsilon)}\frac{\epsilon^\alpha}{|\eta_3-\eta_2|^\alpha\Big[|\eta_3-\eta_2|^2+\frac{1}{2}|\eta_2-\eta_1|^2\Big]^{3/2}}\,d\eta_3\leq\\ &\qquad\qquad C\int_\epsilon^D\frac{\epsilon^\alpha r^2\,dr}{r^{3+\alpha}+r^\alpha|\eta_2-\eta_1|^3}\leq C\frac{\epsilon^\alpha}{|\eta_2-\eta_1|^\alpha}\Big[1+\ln\frac{|\eta_2-\eta_1|}{\epsilon}\Big]\,.\end{aligned} \tag{3.5}$$

Finally, let $\Omega_3=\Omega\setminus(\Omega_1\cup\Omega_2)$. As can be easily verified we have for all $\eta_3\in\Omega_3$

$$\frac{1}{4}|\eta_2-\eta_1|\leq\min(|\eta_3-\eta_1|,|\eta_3-\eta_2|)\ ,.$$

Hence,

$$\int_{\Omega_3}\frac{\epsilon^\alpha}{|\eta_3-\eta_1|^3|\eta_2-\eta_3|^\alpha}\,d\eta_3\leq C\frac{\epsilon^\alpha}{|\eta_2-\eta_1|^\alpha}\int_{\Omega_3}\frac{1}{|\eta_3-\eta_1|^3}\,d\eta_3 \tag{3.6}$$

Let

$$\eta_3^{\|} = (\eta_3 - \eta_1) \cdot \frac{\eta_2 - \eta_1}{|\eta_2 - \eta_1|} \quad ; \quad \eta_3^{\perp} = (\eta_3 - \eta_1) - \eta_3^{\|}$$

Since for all $\eta_3 \in \Omega_3$ we have

$$\frac{1}{4}|\eta_2 - \eta_1| \leq \eta_3^{\|} \leq \frac{3}{4}|\eta_2 - \eta_1| \,,$$

we obtain that

$$\int_{\Omega_3} \frac{1}{|\eta_3 - \eta_1|^3}\, d\eta_3 \leq \int_{\Omega_3} \frac{1}{[|\eta_3^{\|}|^2 + |\eta_3^{\perp}|^2]^{3/2}}\, d\eta_3 \leq$$
$$\int_{|\eta_1-\eta_2|/4}^{3|\eta_1-\eta_2|/4} d\eta_3^{\|} \int_{\mathbb{R}^2} \frac{1}{[|\eta_2 - \eta_1|^2 + |\eta_3^{\perp}|^2]^{3/2}}\, d\eta_3^{\perp} \leq$$
$$C|\eta_2 - \eta_1| \int_{\mathbb{R}^2} \frac{1}{[|\eta_2 - \eta_1|^2 + |\eta_3^{\perp}|^2]^{3/2}}\, d\eta_3^{\perp} \leq C \,.$$

Combining the above with (3.6) yields

$$\int_{\Omega_3} \frac{\epsilon^\alpha}{|\eta_3 - \eta_1|^3 |\eta_2 - \eta_3|^\alpha}\, d\eta_3 \leq C \frac{\epsilon^\alpha}{|\eta_2 - \eta_1|^\alpha} \,,$$

which together with (3.5) and (3.4) yields (3.3). □

We now prove

Lemma 3.2. *There exists $C > 0$, independent of N and ϵ, such that*

$$\big|\mathbb{E}_f\big(\langle \nabla\phi_1(\cdot,\eta_2), \nabla\phi_1(\cdot,\eta_3)\rangle_{B(\eta_1,\epsilon)}\big)\big| \leq C\epsilon^9 \,. \tag{3.7}$$

Proof. It can be easily verified that

$$\mathcal{J} = \mathbb{E}_f\big(\langle \nabla\phi_1(\cdot,\eta_2), \nabla\phi_1(\cdot,\eta_3)\rangle_{B(\eta_1,\epsilon)}\big) =$$
$$\int_{\Omega^3} \int_{B(\eta_1,\epsilon)} \nabla\phi_1(x,\eta_2) \cdot \nabla\phi_1(x,\eta_3)\, dx\, f_3(\eta_1,\eta_2,\eta_3)\, d\eta_1 d\eta_2 d\eta_3$$

Equivalently we can write

$$\mathcal{J} = \int_\Omega f_1(\eta_1) \int_{B(\eta_1,\epsilon)} \int_{\Omega^2} \nabla\phi_1(x,\eta_2) \cdot \nabla\phi_1(x,\eta_3)\, f_3(\eta_2,\eta_3/\eta_1) d\eta_2 d\eta_3\, dx\, d\eta_1 \,.$$

We then use (3.2a) to obtain that

$$\mathcal{J} = \int_\Omega f_1(\eta_1) \|\nabla\Phi\|^2_{L^2(B(\eta_1,\epsilon)}\, d\eta_1 +$$
$$\int_\Omega f_1(\eta_1) \int_{\Omega^2} \langle \nabla\phi_1(\cdot,\eta_2), \nabla\phi_1(\cdot,\eta_3)\rangle_{B(\eta_1,\epsilon)}\, g(\eta_1,\eta_2,\eta_3)\, d\eta_2 d\eta_3\, d\eta_1 \,. \tag{3.8}$$

where

$$\Phi(x,\eta_1) = \int_\Omega \phi_1(x,\eta_2)\tilde{f}_2(\eta_2/\eta_1)\,d\eta_2\,.$$

Next, we use (3.2f) to obtain that

$$\nabla\Phi = \int_\Omega \nabla\phi_1(x,\eta_2)h_1(|\eta_2-\eta_1|)h_2(\eta_2)\,d\eta_2\,.$$

Hence,

$$|\nabla\Phi(x,\eta_2)| \leq h_2(\eta_1)\int_{\Omega_\epsilon} \nabla\phi_1(x,\eta_2)h_1(|\eta_2-\eta_1|)\,d\eta_2 + L\int_{\Omega_\epsilon} |\nabla\phi_1(x,\eta_2)|h_1(|\eta_2-\eta_1|)|\eta_2-\eta_1|\,d\eta_2\,. \tag{3.9}$$

For the first term on the right-hand-side we have by (2.29) that

$$\Big\|h_2(\eta_1)\int_{\Omega_\epsilon} \nabla\phi_1(x,\eta_2)h_1(|\eta_2-\eta_1|)\,d\eta_2\Big\|_{L^2(B(\eta_1,\epsilon)} \leq C\epsilon^{9/2}(1+|\ln d(\eta_1,\partial\Omega_\epsilon)|)\,. \tag{3.10}$$

For the second term on the right-hand-side of (3.9) we have by (2.13) and Minkowsky inequality

$$\Big\|L\int_{\Omega_\epsilon} |\nabla\phi_1(x,\eta_2)|h_1(|\eta_2-\eta_1|)|\eta_2-\eta_1|\,d\eta_2\Big\|_{L^2(B(\eta_1,\epsilon)} \leq C\int_{\Omega_\epsilon\setminus B(\eta_1,2\epsilon)} \frac{\epsilon^{9/2}}{|\eta_2-\eta_1|^2}\,d\eta_2 \leq C\epsilon^{9/2}\,.$$

Combining the above with (3.10) then yields

$$\int_\Omega f_1(\eta_1)\|\nabla\Phi\|^2_{L^2(B(\eta_1,\epsilon)}\,d\eta_1 \leq C\epsilon^9\int_{\Omega_\epsilon}(1+|\ln d(\eta_1,\partial\Omega_\epsilon)|)^2\,d\eta_1 \leq C\epsilon^9\,. \tag{3.11}$$

We next estimate the second term on the right-hand-side of (3.8). With the aid of (3.2b,c), and the fact that $g=0$ whenever $\eta_3\in B(\eta_1,\epsilon)$, we conclude that

$$\begin{aligned}
&\int_\Omega f_1(\eta_1)\int_{\Omega^2}\langle\nabla\phi_1(\cdot,\eta_2),\nabla\phi_1(\cdot,\eta_3)\rangle_{B(\eta_1,\epsilon)}\,g(\eta_1,\eta_2,\eta_3)\,d\eta_2 d\eta_3\,d\eta_1 \\
&\qquad\leq C\int_\Omega f_1(\eta_1)\int_{\Omega\setminus B(\eta_1,2\epsilon)}\|\nabla\phi_1(\cdot,\eta_2)\|_{L^2(B(\eta_1,\epsilon))} \\
&\qquad\cdot\Big[\int_{\Omega\setminus[B(\eta_1,2\epsilon)\cup B(\eta_2,2\epsilon)]}\|\nabla\phi_1(\cdot,\eta_3)\|_{L^2(B(\eta_1,\epsilon))}\,\frac{\epsilon^\alpha}{|\eta_2-\eta_3|^\alpha}\,d\eta_3 + \\
&\qquad\qquad C\int_{B(\eta_2,2\epsilon)\setminus B(\eta_1,2\epsilon)}\|\nabla\phi_1(\cdot,\eta_3)\|_{L^2(B(\eta_1,\epsilon))}\,d\eta_3\Big]\,d\eta_2\,d\eta_1\,.
\end{aligned}$$

Next, we use (2.13) and (3.2b) to obtain

$$\int_\Omega f_1(\eta_1)\int_{\Omega^2}\langle\nabla\phi_1(\cdot,\eta_2),\nabla\phi_1(\cdot,\eta_3)\rangle_{B(\eta_1,\epsilon)}\,g(\eta_1,\eta_2,\eta_3)\,d\eta_2 d\eta_3\,d\eta_1 \leq$$
$$C\int_\Omega\int_{\Omega\setminus B(\eta_1,2\epsilon)}\Big[\int_{\Omega\setminus[B(\eta_1,2\epsilon)\cup B(\eta_2,2\epsilon)]}\frac{\epsilon^{9+\alpha}}{|\eta_2-\eta_1|^3|\eta_3-\eta_1|^3|\eta_2-\eta_3|^\alpha}\,d\eta_3+$$
$$\int_{B(\eta_2,2\epsilon)\setminus B(\eta_1,2\epsilon)}\frac{\epsilon^9}{|\eta_2-\eta_1|^3|\eta_3-\eta_1|^3}\,d\eta_3\Big]\,d\eta_2\,d\eta_1\,. \quad (3.12)$$

Clearly, as $|\eta_3-\eta_1|\geq|\eta_2-\eta_1|/2$ for all $\eta_3\in B(\eta_2,2\epsilon)\setminus B(\eta_1,2\epsilon)$,

$$\int_{B(\eta_2,2\epsilon)\setminus B(\eta_1,2\epsilon)}\frac{1}{|\eta_2-\eta_1|^3|\eta_3-\eta_1|^3}\,d\eta_3\leq C\frac{\epsilon^3}{|\eta_2-\eta_1|^6}\,.$$

Substituting the above, together with (3.3) into (3.12) yields

$$\int_\Omega f_1(\eta_1)\int_{\Omega^2}\langle\nabla\phi_1(\cdot,\eta_2),\nabla\phi_1(\cdot,\eta_3)\rangle_{B(\eta_1,\epsilon)}\,g(\eta_1,\eta_2,\eta_3)\,d\eta_2 d\eta_3\,d\eta_1 \leq$$
$$C\int_\Omega\int_{\Omega\setminus B(\eta_1,2\epsilon)}\Big[\frac{\epsilon^{9+\alpha}}{|\eta_2-\eta_1|^{3+\alpha}}\Big(1+\ln\frac{|\eta_2-\eta_1|}{\epsilon}\Big)+\frac{\epsilon^{12}}{|\eta_2-\eta_1|^6}\Big]\,d\eta_2\,d\eta_1\leq C\epsilon^9\,.$$

The above, together with (3.11) yields (3.7). □

4 $W^{1,p}$ estimates

Let ϕ denote a weak solution of (3.2). We first represent ϕ in the following manner

$$\phi(x,\eta_1,\ldots,\eta_N)=\bar{\phi}(x)+\sum_{i=1}^N\phi_1(x,\eta_i)+u\,, \quad (4.1)$$

in which ϕ_1 is defined by (2.10). We first derive a boundary value problem for u. Clearly,

$$a(x,\eta_1,\ldots,\eta_N)=1+\sum_{n=1}^N[a_1(x,\eta_n)-1]\,.$$

Hence, by (2.9), (2.8), and (2.10)

$$-\nabla\cdot(a\nabla\bar{\phi})=-\sum_{n=1}^N\nabla\cdot([a_1(x,\eta_n)-1]\nabla\bar{\phi})$$
$$=-\sum_{n=1}^N\nabla\cdot(a_1(x,\eta_n)\nabla\bar{\phi})=\sum_{n=1}^N\nabla\cdot\big(a_1(x,\eta_n)\nabla\phi_1(x,\eta_n)\big)\,.$$

It follows that

$$-\nabla\cdot\Big(a\Big[\nabla\bar{\phi}+\sum_{i=1}^{N}\nabla\phi_1(x,\eta_i)\Big]\Big)=\sum_{n=1}^{N}\nabla\cdot\big(\big[a-a_1(x,\eta_n)\big]\nabla\phi_1(x,\eta_n)\big)\,, \tag{4.2}$$

and as

$$a-a_1(x,\eta_i)=\sum_{\substack{j=1\\ j\neq i}}^{N}[a_1(x,\eta_j)-1]\,,$$

we obtain that

$$-\nabla\cdot(a\nabla u)=\sum_{\substack{i,j=1\\ j\neq i}}^{N}\nabla\cdot\big([a_1(x,\eta_i)-1]\nabla\phi_1(x,\eta_j)\big) \tag{4.3}$$

We now prove

Proposition 4.1. *Under the assumptions of Theorem 1 there exists $C(p,\sigma,\Omega)>0$ such that for all $1<p<2$ we have*

$$\mathbb{E}_f\Big(\|\nabla u\|_p\Big)\leq C\bar{\beta}^{\frac{p+1}{p}-\frac{1}{2}}\,. \tag{4.4}$$

Proof. Let

$$f=\sum_{\substack{i,j=1\\ j\neq i}}^{N}[a_1(x,\eta_i)-1]\nabla\phi_1(x,\eta_j)\,.$$

By (2.7) we have, for all $1<p<2$,

$$\|\nabla u\|_p\leq C_p\Big[\|a-1\|_{\frac{2p}{2-p}}\|f\|_2+\|f\|_p\Big],.$$

Thus,

$$\mathbb{E}_f(\|\nabla u\|_p)\leq C_p\mathbb{E}_f\Big(|\sigma-1|\bar{\beta}^{\frac{1}{p}-\frac{1}{2}}\|f\|_2+\|f\|_p\Big) \tag{4.5}$$

By Jensen's inequality we have that

$$\mathbb{E}_f\Big(\|f\|_p\Big)\leq\big[\mathbb{E}_f(\|f\|_p^p)\big]^{1/p}\,. \tag{4.6}$$

Since, with probability 1,

$$\Big\|\sum_{\substack{i,j=1\\ j\neq i}}^{N}[a_1(x,\eta_i)-1]\nabla\phi_1(x,\eta_j)\Big\|_p^p=\sum_{i=1}^{N}\Big\|\sum_{\substack{j=1\\ j\neq i}}^{N}[a_1(x,\eta_i)-1]\nabla\phi_1(x,\eta_j)\Big\|_p^p,$$

it follows that

$$\mathbb{E}_f\Big(\|f\|_p^p\Big) \le N|\sigma-1|\mathbb{E}_f\Big(\Big\|\sum_{i=2}^N \nabla\phi_1(x,\eta_i)\Big\|^p_{L^p(B(\eta_1,\epsilon))}\Big)\,. \tag{4.7}$$

We now observe that

$$\Big\|\sum_{i=2}^N \nabla\phi_1(x,\eta_i)\Big\|_{L^p(B(\eta_1,\epsilon))} \le C\epsilon^{\frac{3}{p}(1-p/2)}\Big\|\sum_{i=2}^N \nabla\phi_1(x,\eta_i)\Big\|_{L^2(B(\eta_1,\epsilon)}\,,$$

and hence, by (4.7),

$$\mathbb{E}_f\Big(\|f\|_p^p\Big) \le CN\epsilon^{3(1-p/2)}\mathbb{E}_f\Big(\Big\|\sum_{i=2}^N \nabla\phi_1(x,\eta_i)\Big\|^{p/2}_{L^2(B(\eta_1,\epsilon)}\Big)\,.$$

In view of (4.6) we then have

$$\mathbb{E}_f\Big(\|f\|_p\Big) \le CN^{\frac{1}{p}}\epsilon^{3\left(\frac{1}{p}-\frac{1}{2}\right)}\mathbb{E}_f\Big(\Big\|\sum_{i=2}^N \nabla\phi_1(x,\eta_i)\Big\|^{p/2}_{L^2(B(\eta_1,\epsilon)}\Big)^{\frac{1}{p}}\,.$$

Applying Jensen's inequality once again yields

$$\mathbb{E}_f\Big(\Big\|\sum_{i=2}^N \nabla\phi_1(x,\eta_i)\Big\|^{p/2}_{L^2(B(\eta_1,\epsilon)}\Big)^{\frac{1}{p}} \le \mathbb{E}_f\Big(\Big\|\sum_{i=2}^N \nabla\phi_1(x,\eta_i)\Big\|^{2}_{L^2(B(\eta_1,\epsilon)}\Big)^{\frac{1}{2}}\,,$$

and hence

$$\mathbb{E}_f\Big(\|f\|_p\Big) \le CN^{\frac{1}{p}}\epsilon^{3\left(\frac{1}{p}-\frac{1}{2}\right)}\mathbb{E}_f\Big(\Big\|\sum_{i=2}^N \nabla\phi_1(x,\eta_i)\Big\|^{2}_{L^2(B(\eta_1,\epsilon)}\Big)^{\frac{1}{2}}\,. \tag{4.8}$$

Next we write

$$\Big\|\sum_{i=2}^N \nabla\phi_1(\cdot,\eta_i)\|^2_{L^2(B(\eta_1,\epsilon)} = \sum_{i=2}^N \Big\|\nabla\phi_1(\cdot,\eta_i)\|^2_{L^2(B(\eta_1,\epsilon)} + \sum_{\substack{i,j=2\\ i\neq j}}^N \langle\nabla\phi_1(\cdot,\eta_i),\nabla\phi_1(\cdot,\eta_j)\rangle_{B(\eta_1,\epsilon)}\,,$$

where the definition of $\langle\cdot,\cdot\rangle_{B(\eta_1,\epsilon)}$ is given in (3.1). Consequently, we have

$$\begin{aligned}\mathbb{E}_f\Big(\Big\|\sum_{i=2}^N \nabla\phi_1(\cdot,\eta_i)\|^2_{L^2(B(\eta_1,\epsilon)}\Big) &= (N-1)\mathbb{E}_f\big(\|\nabla\phi_1(\cdot,\eta_2)\|^2_{L^2(B(\eta_1,\epsilon))}\big)\\ &\quad + (N-1)(N-2)\mathbb{E}_f\big(\langle\nabla\phi_1(\cdot,\eta_2),\nabla\phi_1(\cdot,\eta_3)\rangle_{B(\eta_1,\epsilon)}\big)\,.\end{aligned} \tag{4.9}$$

We now estimate each term on the right-hand-side of (4.9). For the first term we use (2.13) to establish that

$$(N-1)\mathbb{E}_f\big(\|\nabla\phi_1(\cdot,\eta_2)\|^2_{L^2(B(\eta_1,\epsilon))}\big) \leq \\ CN\int_{\Omega\times\Omega\setminus B(\eta_1,\epsilon)} \frac{\epsilon^9}{|\eta_2-\eta_1|^6}\,d\eta_2\,d\eta_1 \leq \frac{C}{N}\bar{\beta}^2 . \quad (4.10)$$

For the second term we use (3.7) to obtain that

$$(N-1)(N-2)\mathbb{E}_f\big(\langle\nabla\phi_1(\cdot,\eta_2),\nabla\phi_1(\cdot,\eta_3)\rangle_{B(\eta_1,\epsilon)}\big) \leq \frac{C}{N}\bar{\beta}^3 .$$

Substituting the above together with (4.10) and (4.9) into (4.8) yields

$$\mathbb{E}_f\Big(\|f\|_p\Big) \leq CN^{\frac{1}{p}}\epsilon^{3\left(\frac{1}{p}-\frac{1}{2}\right)}\frac{1}{N^{1/2}}\bar{\beta} \leq C\bar{\beta}^{\frac{p+1}{p}-\frac{1}{2}} . \quad (4.11)$$

As (4.11) is valid for any $1<p\leq 2$ we may conclude that

$$\mathbb{E}_f\Big(\|f\|_2\Big) \leq C\bar{\beta} .$$

Substituting the above together with (4.11) into (4.5) yields (4.4). □

Proof of Theorem 1. We recall from [1] that

$$N\mathbb{E}_f(\phi_1) + \frac{3(\sigma-1)}{2+\sigma}\Delta^{-1}\big(\nabla\cdot(\beta\nabla\bar{\phi})\big) = N\mathbb{E}_f(\phi_{1,2}) , \quad (4.12)$$

where $\phi_{1,2}$ satisfies

$$\begin{cases} -\Delta\phi_{1,2} = \frac{\sigma-1}{2+\sigma}\nabla\cdot\big([a_1(x,\eta)-1][\nabla\bar{\phi}(x)-\nabla\bar{\phi}(\eta)]\big) + \nabla\cdot\big([a_1(x,\eta)-1]\nabla u_1(x,\eta)\big) & \text{in } \Omega , \\ \phi_{1,2}=0 & \text{on } \partial\Omega , \end{cases}$$

and Δ^{-1} denotes the inverse Dirichlet Laplacian. Upon taking the average of the above equation, we use (2.2) to obtain that for any $1<p\leq 2$ we have

$$\|\nabla\mathbb{E}_f(\phi_{1,2})\|_p \leq C\big[\|\mathbb{E}_f([a_1(\cdot,\eta)-1][\nabla\bar{\phi}(\cdot)-\nabla\bar{\phi}(\eta)])\|_p + \|\mathbb{E}_f([a_1(\cdot,\eta)-1]\nabla u_1(\cdot,\eta))\|_p\big] . \quad (4.13)$$

As in [1] we can show that

$$\|\mathbb{E}_f([a_1(\cdot,\eta)-1][\nabla\bar{\phi}(\cdot)-\nabla\bar{\phi}(\eta)])\|_p \leq C\epsilon^4 ,$$

and that

$$\|\mathbb{E}_f([a_1(\cdot,\eta)-1]\nabla u_1(\cdot,\eta))\|_p \leq C\epsilon^{7/2} .$$

Substituting the above into (4.13) and (4.12) then yields, with the aid of Poincare inequality

$$\|N\mathbb{E}_f b(\phi_1) + \frac{3(\sigma-1)}{2+\sigma}\Delta^{-1}\big(\nabla\cdot(\beta\nabla\bar{\phi})\big)\|_{1,p} \leq C\epsilon^{1/2}\bar{\beta}\,. \tag{4.14}$$

By (4.1) and (4.4) in conjunction with Minkowsky inequality we have that

$$\|\mathbb{E}_f(\phi) - \bar{\phi} - N\mathbb{E}(\phi_1)\|_{1,p} \leq C\bar{\beta}^{\frac{p+1}{p}-\frac{1}{2}}\,.$$

With the aid of (4.14) we then obtain

$$\|\mathbb{E}_f(\phi) - \bar{\phi} - \frac{3(\sigma-1)}{2+\sigma}\Delta^{-1}\big(\nabla\cdot(\beta\nabla\bar{\phi})\big)\|_{1,p} \leq C\big(\bar{\beta}^{\frac{p+1}{p}-\frac{1}{2}} + \epsilon^{1/2}\bar{\beta}\big)\,.$$

Hence, by (1.9) we have

$$\|\mathbb{E}_f(\phi) - \bar{\phi} - \frac{3(\sigma-1)}{2+\sigma}\Delta^{-1}\big(\nabla\cdot(\beta\nabla\bar{\phi})\big)\|_{1,p} \leq C\bar{\beta}^{\frac{p+1}{p}-\frac{1}{2}}\,. \tag{4.15}$$

Let ϕ_e denote the solution of (1.13). In [1] we showed that

$$\|\phi_e - \bar{\phi} + \frac{3(\sigma-1)}{2+\sigma}\Delta^{-1}\big(\nabla\cdot(\beta\nabla\bar{\phi})\big)\|_{1,2} \leq C\bar{\beta}^2\,.$$

Hölder inequality readily yields for all $1 < p < 2$

$$\|\phi_e - \bar{\phi} + \frac{3(\sigma-1)}{2+\sigma}\Delta^{-1}\big(\nabla\cdot(\beta\nabla\bar{\phi})\big)\|_{1,p} \leq C\bar{\beta}^2\,.$$

Combining the above with (4.15) yields (1.15). $\square$

Acknowledgments The author was supported by NSF grant DMS-1109030 and by US-Israel BSF grant no. 2010194. He also wishes to thank the Technion's Mathematics department and Professor Itai Shafrir for supporting his stay at the Technion, during the work on this research. Itai shafrir is also acknowledged for some good advice on L^p regularity .

A Proof of (2.23)

Let $\Omega \subset \mathbb{R}^3$ be bounded with smooth boundary $\partial\Omega \in C^{2,\alpha}$ for some $\alpha > 0$. For some $\epsilon > 0$, let Ω_ϵ be defined by (1.11). Let $\delta_0 > 0$ be sufficiently small so that for every $y \in \Omega \setminus \Omega_{\delta_0}$ there exists a unique point $\bar{y} \in \partial\Omega$ so that $d(y, \partial\Omega) = |y - \bar{y}|$. Denote the reflection of y with respect to $\bar{y}$ by y^*. We begin by the following elementary lemma

Lemma A.1. *There exists $C>0$ depending only on Ω such that*

$$\sup_{(x,y)\in\Omega\times(\Omega\setminus\Omega_{\delta_0})}\frac{1}{|x-y^*|}\leq\frac{C}{[|x-\xi|^2+4d(x,\partial\Omega)d(\xi,\partial\Omega)]^{1/2}}\,. \tag{A.1}$$

Proof. Let $y\in\Omega\setminus\Omega_{\delta_0}$. We set the origin of a coordinate system at its projection on $\partial\Omega$, i.e., $\bar{y}=(0,0,0)$. We further set $y=(0,0,d)$ and hence for the reflection point we have $y^*=(0,0,-d)$. Let $x=(x_1,x_2,x_3)\in\Omega\setminus\Omega_{\delta_0}$, and let $r=|x-y|$ and $r^*=|x-y^*|$. We next describe the boundary $\partial\Omega$ by the explicit representation $z=z(x_1,x_2)$, which is possible, in some neighborhood of $\bar{y}$, if we choose δ_0 small enough. By the smoothness of $\partial\Omega$ there exists $C_0(\Omega)$ such that

$$|z|\leq C_0(x_1^2+x_2^2)\quad x_1^2+x_2^2\leq\delta_0^2\,, \tag{A.2}$$

which can again be guaranteed by choosing δ_0 to be sufficiently small.

Suppose first that $x_1^2+x_2^2\leq\delta_0^2$. Clearly,

$$|r^*|^2=r^2+4dx_3\,.$$

Since $d_x=d(x,\partial\Omega)\leq x_3-z$, we have by (A.2) that

$$|r^*|^2=r^2+4dx_3\geq r^2+4d_xd-4d|z|\geq r^2+4d_xd-4\delta_0(x_1^2+x_2^2)\,.$$

Using the fact that $r^2\geq x_1^2+x_2^2$ we easily obtain (A.2) for sufficiently small δ_0.

If $x_1^2+x_2^2\geq\delta_0^2$ then $|r^*|^2\geq\delta_0^2$. Since for some $C(\Omega)$,

$$\sup_{(x,y)\in\Omega^2}|x-y|^2+4d(x,\partial\Omega)d(y,\partial\Omega)\leq C^2\,,$$

we have that

$$|r^*|\geq\frac{\delta_0}{C}[|x-y|^2+4d(x,\partial\Omega)d(y,\partial\Omega)]^{1/2}\,,$$

from which (A.2) easily follows. □

Proof of (2.23). *step 1:* We first prove that

$$|\tau(x,y)|\leq\frac{C(\Omega)}{[|x-y|^2+4d(x,\partial\Omega)d(y,\partial\Omega)]^{1/2}|}\quad\forall(x,y)\in\Omega^2\,. \tag{A.3}$$

Let τ be given by (2.20). In [1] we show that

$$\|\tau(\cdot,y)\|_\infty\leq\frac{1}{4\pi d(y,\partial\Omega)}\,. \tag{A.4}$$

Consequently, there exists $C(\Omega)$ such that for all $(x,y)\in\Omega\times\Omega_{\delta_0}$

$$|\tau(x,y)|\leq\frac{1}{4\pi\delta_0}\leq\frac{C(\Omega)}{[|x-y|^2+4d(x,\partial\Omega)d(y,\partial\Omega)]^{1/2}}\,. \tag{A.5}$$

Suppose that $y \in \Omega \setminus \Omega_{\delta_0}$. Set then

$$\tau(x,y) = h(x,y) - \frac{1}{4\pi|x-y^*|}\,.$$

In [1] we show that

$$\sup_{y\in\Omega\setminus\Omega_{\delta_0}} \|h(\cdot,y)\|_\infty \leq C\,,$$

and hence, for all $y \in \Omega \setminus \Omega_{\delta_0}$ we have

$$h(x,y) \leq C \leq \frac{C}{[|x-y|^2 + 4d(x,\partial\Omega)d(y,\partial\Omega)]^{1/2}}$$

Combining the above with (A.5) yields (A.3).

Step 2: We next show that

$$|\nabla_y \tau(x,y)| \leq \frac{C(\Omega)}{[|x-y|^2 + 4d(x,\partial\Omega)d(y,\partial\Omega)]} \tag{A.6}$$

Suppose first that $y \in \Omega \setminus \Omega_{\delta_0}$. In this case we have by Theorem 6.6 in [6]

$$\|\nabla_x \tau(\cdot,y)\|_\infty \leq \frac{C}{\delta_0^2}\,. \tag{A.7}$$

Next, let $y \in \Omega_{\delta_0}$. In this case we distinguish again between three different cases

1. $2d(x,\partial\Omega) \geq \delta_0$
2. $\delta_0 \geq 2d(x,\partial\Omega) > |x-y^*|\,,$
3. $2d(x,\partial\Omega) \leq \min(\delta_0, |x-y^*|)\,.$

In case 1 we prove in the same manner as in the proof of (A.7) that for some $C(\delta_0,\Omega)$ we have

$$|\nabla_x \tau(x,y)| \leq \frac{C}{\delta_0^2} \quad \forall x \in \Omega \setminus \Omega_{\delta_0/2}\,. \tag{A.8}$$

In case 2 we use the fact that τ is harmonic in $B(x,|x-y^*|/2)$ to obtain with the aid of Theorem 4.8 that

$$|\nabla_x \tau(x,y)| \leq \frac{C}{|x-y^*|}\|\tau(\cdot,y)\|_{L^\infty(B(x,|x-y^*|/2))} \leq \frac{C}{|x-y^*|^2}\,. \tag{A.9}$$

In case 3 we denote by $\bar{x}$ the projection of x on $\partial\Omega$. Set further $\delta_1 = 2d(x,\partial\Omega) < \delta_0$. Let $D = B(\bar{x},\delta_1)\cap\Omega$. Then, since τ is harmonic in D we obtain by Lemma 6.5 in [6] that there exists $C>0$, independent of ξ, such that

$$|\nabla_x \tau(x,y)| \leq \frac{C}{d(x,\partial D\setminus\partial\Omega)}\|\tau(\cdot,y)\|_{L^\infty(D)} \leq \frac{C}{|x-y^*|^2}\,.$$

Combining the above, (A.9), and (A.7) yields

$$\sup_{(x,y)\in\Omega\times\Omega} \big| |x-y^*|^2 \nabla_x \tau(x,y) \big| \leq C\,.$$

The symmetry of τ then yields (A.6).

Step 3: We complete the proof of (2.23) as in [1]. □

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