# The loss of stability of surface superconductivity Y. ALMOG \*

#### Abstract

The Ginzburg-Landau equations in a half-plane are considered in the large  $\kappa$  limit. We look at the reduced set of equations obtained in that limit. It is proved that the one-dimensional solution presented by Pan [22] undergoes a bifurcation for an infinite number of applied magnetic field values which are lower than  $H_{C_2}$ . We also prove that each bifurcating mode is energetically preferable to the one-dimensional surface superconductivity solution, and thus, prove that the surface superconductivity becomes unstable for applied fields which are lower than  $H_{C_2}$ 

## 1 Introduction

Consider a planar superconducting body which is placed at a sufficiently low temperature (below the critical one) under the action of an applied magnetic field . Its energy is given by the Ginzburg-Landau energy functional which can be represented in the following dimen-

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sionless form [10]

$$E = \int_{\Omega} \left( -|\Psi|^2 + \frac{|\Psi|^4}{2} + |h - h_{ex}|^2 + \left| \frac{i}{\kappa} \nabla \Psi + A\Psi \right|^2 \right) dx_1 dx_2$$
(1.1)

in which  $\Psi$  is the (complex) superconducting order parameter, such that  $|\Psi|$  varies from  $|\Psi| = 0$  (when the material is at a normal state) to  $|\Psi| = 1$  (for the purely superconducting state). The magnetic vector potential is denoted by A (the magnetic field is, then, given by  $h = \nabla \times A$ ),  $h_{ex}$  is the constant applied magnetic field, and  $\kappa$  is the Ginzburg-Landau parameter which is a material property. Superconductors for which  $\kappa < 1/\sqrt{2}$  are called type I superconductors, and those for which  $\kappa > 1/\sqrt{2}$  are called type II.  $\Omega$  is a connected domain of superconductor, whose Gibbs free energy is given by E. Note that E is invariant to the gauge transformation

$$\Psi \to e^{i\kappa\eta}\Psi \qquad ; \qquad A \to A + \nabla\eta \;.$$
 (1.2)

where  $\eta$  is any smooth function.

For sufficiently large magnetic fields it is well known, both from experimental observations [21] and both from theoretical predictions [15], that superconductivity is destroyed and the material must be in the normal state. If the applied magnetic field is then decreased there is a critical field where the material enters the superconducting phase once again. This field is called "the onset field" and is denoted by  $H_{C_3}$ .

It is well-known that at the bifurcation from the normal state, superconductivity remains concentrated near the boundary. Alternatively we can say that  $\Psi$  decays exponentially fast away from the boundaries as either  $\kappa$  or the size of  $\Omega$  tend to infinity, which is the reason why the phenomenon has been termed surface superconductivity. This result has first been obtained for a half-plane [24], then also for disks [6], and for general smooth domains in  $\mathbb{R}^2$ [7, 20, 13, 17]. It was extended later to weakly non-linear cases in the large  $\kappa$  limit [18].

In the absence of boundaries the critical field at which superconductivity nucleates is denoted by  $H_{C_2}$  and is smaller than  $H_{C_3}$  ( $H_{C_3} \approx 1.7\kappa$  whereas  $H_{C_2} = \kappa$ ). Furthermore, the bifurcating modes are periodic lattices, named after Abrikosov [2, 9, 4] which have been observed experimentally [14]. It has been conjectured, therefore, by Rubinstein [23] that superconductivity remains concentrated near the boundary for  $H_{C_2} < h_{ex} < H_{C_3}$ . When  $h_{ex} \approx H_{C_2}$  (either for  $\kappa$  large or for large domains) a bifurcation of Abrikosov's lattices far away from the wall was conjectured [23].

Recently, it has been proved both in the large  $\kappa$  limit [22, 3], and in the large domain limit [5] that as long as  $H_{C_2} < h_{ex} < H_{C_3}$  superconductivity remains concentrated near the boundaries. From a different direction, Sandier and Serfaty [25] showed for the global minimizer of (1.1) that as  $h_{ex} \to H_{C_2}$  from below and  $\kappa \to \infty$ , superconductivity vanishes in the domain's interior, away from the boundaries.

Despite the above-mentioned progress the transition from the surface superconductivity solution to the mixed state, where Abrikosov's lattices appear in the bulk of the material, has not been clarified yet. In particular, if the applied magnetic field is decreased below  $H_{C_2}$ it has not been proved yet that:

1. The surface superconductivity solution becomes unstable, i.e., it is not a local minimizer of E for  $h_{ex} < H_{C_2}$ . 2. The bifurcating mode is indeed the global minimizer and has to be periodic.

In the present contribution we prove, in the large  $\kappa$  limit, for a domain wall, that the surface superconductivity solution in a half-plane is not a local minimizer of E for  $h_{ex} < H_{C_2}$ , and hence cannot be stable. To this end we assume, just like Pan [22] did, that the global minimizer is essentially one-dimensional in the boundary layer. In addition to the instability proof, we find the bifurcating modes and show, by an heuristic argument, that when properly superposed, Abrikosov's lattices can be formed. However, since linear superposition of modes is impossible, in view of the equation's non-linearity, further research is necessary in that direction.

The Euler-Lagrange equations associated with (1.1), known as the steady state Ginzburg-Landau equations, are given in the form

$$\left(\frac{i}{\kappa}\nabla + A\right)^2 \Psi = \Psi \left(1 - |\Psi|^2\right) \quad , \tag{1.3a}$$

$$-\nabla \times \nabla \times A = \frac{i}{2\kappa} \left( \Psi^* \nabla \Psi - \Psi \nabla \Psi^* \right) + |\Psi|^2 A \quad , \tag{1.3b}$$

and the natural boundary conditions by

$$\left(\frac{i}{\kappa}\nabla + A\right)\Psi\cdot\hat{n} = 0$$
;  $h = h_{ex}$ . (1.4a,b)

In [22, 3] it is proved that as  $\kappa \to \infty$ ,  $h_{ex} - \kappa \gg 1/\kappa$  we have, near the boundary

$$\Psi(x_0 + \xi/\kappa) \xrightarrow[\kappa \to \infty]{} \psi(\xi)$$
 pointwise

where  $x_0 \in \partial \Omega, \, \xi \in \mathbb{R}^2_+$ ,

$$\mathbb{R}^2_+ = \{ (x_1, x_2) ; x_1 > 0 \}$$

and  $\psi$  must satisfy

$$\left(i\nabla + x_1\hat{i}_2\right)^2\psi = \lambda\psi\left(1 - |\psi|^2\right) \quad \text{in } \mathbb{R}^2_+ \tag{1.5a}$$

$$\frac{\partial \psi}{\partial x_1} = 0 \quad \text{on } \partial \mathbb{R}^2_+ \tag{1.5b}$$

where

$$\lambda = \frac{\kappa}{h_{ex}}$$

Let

$$\mathcal{H} = \left\{ u \in H^1(z, \infty) \mid xu \in L^2(z, \infty) \right\},\tag{1.6}$$

where z is a real number, and let

$$\beta(z) = \inf_{\phi \in \mathcal{H}} \frac{\int_{z}^{\infty} |\phi'|^2 + x^2 |\phi|^2}{\int_{z}^{\infty} |\phi|^2}.$$
(1.7)

The dependence of  $\beta$  on z has been studied in [8] and [19] afterward. In particular, it has been proved that there exist  $z_1(\lambda)$  and  $z_2(\lambda)$  such that  $\lambda > \beta(z)$  if and only if  $z_1 < z < z_2$ , and that  $z_2(1) = 0$  and  $z_1(1) = -\infty$  It is also proved in [20] that

$$\beta_0 = \inf_{z \in \mathbb{R}} \beta(z) = \lim_{\kappa \to \infty} \frac{\kappa}{H_{C_3}} \approx 0.59.$$

The same result was also proved in [13].

Pan [22] conjectured that any bounded solution of (1.5a) for  $\beta_0 < \lambda \leq 1$  must be in the form

$$\psi = e^{i(\omega_0 x_2 + c)} f(x_1) \tag{1.8}$$

where  $\omega_0$  is a real number and  $f(x_1, \lambda)$  satisfies in  $\mathbb{R}_+$ 

$$-f'' + (x - \omega_0)^2 f = \lambda f(1 - f^2) \quad ; \quad f'(0) = 0.$$
(1.9)

In [22] it is proved that if  $\beta(-\omega_0) < \lambda \leq 1$  and

$$-z_2 < \omega_0 < -z_1$$

then there exists a solution for (1.9). Furthermore, it is proved in [22] the

$$f(x) \sim x^{-\frac{1-\lambda}{2}} e^{-\frac{1}{2}x^2} \text{ as } x \to \infty$$
 (1.10)

The discussion in [22] was limited to the case  $\lambda \leq 1$ , since this is the regime where the surface superconductivity solution is expected to be the global minimizer of E. Nevertheless, it is not difficult to show that the above existence result and (1.10) still hold when  $\lambda > 1$  for any  $\omega_0 \geq 0$ . We bring the proof in appendix A.

Weaker conjectures can be made instead of assuming that (1.8) is the unique class of bounded solutions of (1.5a). Consider the energy functional

$$\mathcal{E}(\psi) = \int_{\mathbb{R}^2_+} |\left(i\nabla + x_1\hat{i}_2\right)\psi|^2 + \lambda\left(\frac{1}{2}|\psi|^4 - |\psi|^2\right),\tag{1.11}$$

where  $\hat{i}_2$  is a unit vector in the  $x_2$  direction, and let

$$H^1_{mag}(\Omega) = \left\{ \phi \in L^2(\Omega) \mid \partial_{x_1} \phi \in L^2 , \ \partial_{x_2} \phi + ix_1 \phi \in L^2 \right\}$$
(1.12)

It is well-known [22] that when  $\lambda > \beta_0$  we have

$$\inf_{\phi \in H^1_{mag}(R^2_+)} \mathcal{E}(\phi) = -\infty$$

We therefore modify the definition of  $H^1_{mag}$  so it would guarantee the existence of a global minimizer to  $\mathcal{E}$  in the modified space. We thus apply the transformation

$$x_1 \to x_1 - \omega_0$$
 ;  $\psi \to e^{-i\omega_0 x_2} \psi$ 

to obtain

$$\mathcal{E}(\psi) = \int_{-\omega_0}^{\infty} dx_1 \int_{\mathbb{R}} dx_2 \left\{ |\left(i\nabla + x_1 \hat{i}_2\right)\psi|^2 + \lambda \left(\frac{1}{2}|\psi|^4 - |\psi|^2\right) \right\},$$
(1.13)

and define the space

$$\mathcal{P}_{L}^{\omega_{0}} = \left\{ \phi \in H_{mag}^{1}([-\omega_{0}, \infty) \times \mathbb{R}) \mid \phi(x_{1}, x_{2} + L) = \phi(x_{1}, x_{2}) \right\}.$$
 (1.14)

We can now conjecture, just like Pan [22] did, that

$$\psi = f(x_1, \lambda)$$

is the global minimizer of  $\mathcal{E}$  in  $\mathcal{P}_L^{\omega_0}$ , for every L > 0 and  $\omega_0 \ge 0$ .

We note that Pan [22] studied the same problem for  $\lambda > 1$  and found that the global minimizer of (1.11) in  $\mathcal{P}_L^{\omega_0}$  decays exponentially fast away from the wall. Moreover, it is proved in [22] that the global minimizer of (1.1) in a smooth bounded domain must tend, as  $\kappa \to \infty$ , to a periodic solution whose period is of  $O(\kappa)$ .

Periodic solutions have already been studied in the absence of boundaries [2, 9, 4]. Periodicity was imposed in those works in both the  $x_1$  and the  $x_2$  directions. In this work we add the effect of a planar wall: we impose periodicity only in the direction which is parallel to the wall, whereas away from the wall we expect the solution to decay. This problem, which is still much simpler than the determination of the global minimizer of (1.1), is much closer to real situations than the problem in  $\mathbb{R}^2$  [2, 9, 4].

The present contribution can be summarized by the following theorem

**Theorem 1.1** There exists  $n_0 \in \mathbb{N}$ , which may depend on  $\omega = 2\pi/L$  and  $\omega_0$ , and a sequence  $\{\lambda_n\}_{n=n_0}^{\infty}$ , such that

- 1. There exists a solution to (1.5a) in  $\mathcal{P}_L^{\omega_0}$  which bifurcates from  $\psi = f(x_1, \lambda)$ , in some right semi-neighborhood of  $\lambda_n$  for every  $n \ge n_0$ .
- 2. For every  $n \ge n_0$

$$1 + C_1 \exp\left\{\frac{-(n\omega + \omega_0)^2}{2}\right\} < \lambda_n < 1 + C_2 \exp\left\{\frac{-(n\omega + \omega_0)^2}{2}\right\},$$
 (1.15)

where  $C_1$  and  $C_2$  are independent of n.

3. Denote the bifurcating solution in  $\mathfrak{P}_L^{\omega_0}$  by  $\psi_n(x_1,\lambda)$ . Then

$$\mathcal{E}(\psi_n,\lambda) < \mathcal{E}(f,\lambda)$$

in some right semi-neighborhood of  $\lambda_n$  for every  $n \geq n_0$ .

In the next section we discuss the linearized equation and prove (1.15). Statements 1 and 3 are proved in §3. Finally, in §4 we briefly summarized the results obtained in §2 and §3 and list some related open problems.

# 2 Linear analysis

Consider the problem

$$\left(i\nabla + x_1\hat{i}_2\right)^2\psi = \lambda\psi\left(1 - |\psi|^2\right) \quad (x_1, x_2) \in (-\omega_0, \infty) \times \mathbb{R}$$
(2.1a)

$$\psi_{x_1}(-\omega_0, x_2) = 0$$
;  $\psi(x_1, x_2 + L) = \psi(x_1, x_2)$  (2.1b)

Let

$$u = \psi(x_1, x_2) - f(x_1, \lambda)$$
(2.2)

wherein f satisfies (1.9). Denote by X the space

$$\mathbb{X} = \left\{ u \in C^2 \left\{ \left[ -\omega_0, \infty \right) \times \mathbb{R} \right\} \bigcap \mathcal{P}_L^{\omega_0} \mid u_{x_1}(-\omega_0, x_2) = 0 \right\}$$

with the  $C^2$  norm. Let  $F : \mathbb{R}^+ \times \mathbb{X} \to C\{[-\omega_0, \infty) \times \mathbb{R}\}$  be the operator

$$F(\lambda, u) = \left(i\nabla + x_1\hat{i}_2\right)^2 u - \lambda \left[u - f^2(2u + \bar{u}) - f\left(2|u|^2 + u^2\right) - |u|^2 u\right].$$
 (2.3)

Clearly, if  $u \in \mathbb{X}$  satisfies  $F(u, \lambda) = 0$  for some  $\lambda > \beta_0$ , then  $\psi = u + f$  is a solution of (2.1). Furthermore, since  $F(\lambda, 0) \equiv 0$  for all  $\lambda > \beta_0$  we can consider the linear bifurcation of nontrivial solutions of  $F(u, \lambda) = 0$  from  $u \equiv 0$ . Let  $F_u$  denote the Fréchet derivative of F. Then, the linearized form of  $F(u, \lambda) = 0$  near  $u \equiv 0$  is

$$F_u(0,\lambda)\phi = 0$$

or,

$$\left(i\nabla + x_1\hat{i}_2\right)^2\phi - \lambda\left[\phi - f^2(2\phi + \bar{\phi})\right] = 0.$$
(2.4)

Our first result proves the existence of non-trivial solutions in X for (2.4) and gives the corresponding critical values of  $\lambda$ .

**Theorem 2.1** There exists  $n_0(\omega_0, \omega) \in \mathbb{N}$  and a sequence  $\{\lambda_n\}_{n=n_0}^{\infty}$ , such that when  $\lambda = \lambda_n$ non trivial solutions of (2.4) exist. Furthermore, for all  $n \ge n_0 \lambda_n$  satisfies (1.15). *Proof:* Since we look for periodic solutions we multiply (2.4) by  $e^{-in\omega x_2}$  where  $n \in \mathbb{N}$  and integrate with respect to  $x_2$  over  $[-\pi/\omega, \pi/\omega]$  to obtain

$$-\hat{\phi}_n'' + \left[ (x - n\omega)^2 - \lambda \right] \hat{\phi}_n + \lambda f^2 \left( 2\hat{\phi}_n + \overline{\hat{\phi}_{-n}} \right) = 0$$
(2.5a)

$$-\hat{\phi}_{-n}^{\prime\prime} + \left[ (x+n\omega)^2 - \lambda \right] \hat{\phi}_{-n} + \lambda f^2 \left( 2\hat{\phi}_{-n} + \overline{\hat{\phi}_n} \right) = 0$$
(2.5b)

$$\hat{\phi}'_{n}(-\omega_{0}) = \hat{\phi}'_{-n}(-\omega_{0}) = 0$$
(2.5c)

where

$$\hat{\phi}_n(x_1) = \int_{-\pi/\omega}^{\pi/\omega} \phi(x_1, x_2) e^{-in\omega x_2} dx_2.$$
(2.6)

To prove the lower bound in (1.15) we need the following perturbation lemma.

**Lemma 2.2** Let  $\mathcal{H}(\xi)$  be defined by (1.6), and let

$$\alpha(\xi,g) = \inf_{\substack{\phi \in \mathcal{H} \\ \|\phi\|_{L^{2}(-\xi,\infty)} = 1}} \int_{-\xi}^{\infty} |\phi'|^{2} + (x^{2} + g(x+\xi)) |\phi|^{2} dx,$$

where  $g:[0,\infty) \to \mathbb{R}$  is continuous and decays as  $x \to \infty$ . Then,

$$\alpha = 1 + \int_{-\xi}^{\infty} gv^2 + \delta(\xi).$$
(2.7a)

In which v is the quasi-mode

$$v = c_{\xi} \chi(x+\xi) e^{-x^2/2}$$
 (2.7b)

whose  $L^2(-\xi,\infty)$  norm is unity,  $\chi$  is a  $C^\infty$  cutoff function satisfying

$$\chi = \begin{cases} 0 & 0 \le x < \frac{1}{2} \\ 1 & 1 < x \end{cases}$$
(2.7c)

and, for sufficiently large  $\xi$ ,

$$\delta(\xi) \le 2 \int_{-\xi}^{\infty} g^2 v^2 + C e^{-\xi^2}$$
 (2.7d)

where C is independent of  $\xi$ .

*Proof:* Denote by  $\mathcal{P}$  the operator

$$\mathcal{P} =: -\frac{d^2}{dx^2} + x^2 + g,$$

and let

$$\nu = 1 + \int_{-\xi}^{\infty} g v^2.$$

Then,

$$(\mathcal{P} - \nu)v = c_{\xi} \left[ -\chi'' + 2x\chi' + g\chi \right] e^{-x^2/2} - (\nu - 1)v,$$

and hence,

$$\left| \int_{-\xi}^{\infty} v(\mathfrak{P} - \nu) v \right| \le \int_{-\xi}^{\infty} \left| -\frac{\chi''}{\chi} + 2x \frac{\chi'}{\chi} \right| v^2 \le C e^{-\xi^2}.$$
(2.8)

Let  $\{\mu_j\}_{j=0}^{\infty}$  denote the eigenvalues and  $\{u_j\}_{j=0}^{\infty}$  the corresponding eigenmodes, whose  $L^2$  norm is unity, of the following problem

$$\mathcal{P}u_j = \mu_j u_j \qquad x > \xi$$
  
 $u'_j(-\xi) = 0.$ 

It is well known [11], that  $\mu_j \uparrow \infty$  and that  $\{u_j\}_{j=0}^{\infty}$  are square integrable and orthogonal. Let

$$\tilde{v} = v - a_0 u_0 \tag{2.9a}$$

where

$$a_0 = \int_{-\xi}^{\infty} v u_0 \tag{2.9b}$$

Substituting (2.9) in (2.8) we obtain (note that  $\mu_0 = \alpha$ )

$$a_0^2 |\alpha - \nu| \le \int_{-\xi}^{\infty} |\tilde{v}(\mathcal{P} - \nu)\tilde{v}| + Ce^{-\xi^2}$$
 (2.10)

To estimate the first term on the right-hand side of (2.10)we make use of the following inequality

$$\int_{-\xi}^{\infty} \left| (\mathfrak{P} - \nu) \tilde{v} \right|^2 \le \int_{-\xi}^{\infty} \left| (\mathfrak{P} - \nu) v \right|^2 = \int_{-\xi}^{\infty} \left| -\frac{\chi''}{\chi} + 2x \frac{\chi'}{\chi} + g - (\nu - 1) \right|^2 v^2 + \le \int_{-\xi}^{\infty} g^2 v^2 + C e^{-\xi^2}. \quad (2.11)$$

Since the distance of  $\nu$  from the spectrum of  $\mathfrak{P}$  in  $\mathfrak{H} \setminus \operatorname{Span}(u_0)$  is  $|\mu_1 - \nu|$  we have

$$(\mu_1 - \nu)^2 \int_{-\xi}^{\infty} |\tilde{v}|^2 \le \int_{-\xi}^{\infty} |(\mathcal{P} - \nu)\tilde{v}|^2 \le \int_{-\xi}^{\infty} |(\mathcal{P} - \nu)v|^2$$
(2.12)

It is not difficult to show, using standard arguments from semi-classical analysis (cf. for instance theorem 3.4.1 in [16]), that

$$\mu_1 \xrightarrow[\xi \to \infty]{} 3. \tag{2.13}$$

Hence, for sufficiently large  $\xi$ ,

$$\int_{-\xi}^{\infty} |\tilde{v}(\mathcal{P}-\nu)\tilde{v}| \le \left[\int_{-\xi}^{\infty} |\tilde{v}|^2 \int_{-\xi}^{\infty} |(\mathcal{P}-\nu)\tilde{v}|^2\right]^{1/2} \le \int_{-\xi}^{\infty} |(\mathcal{P}-\nu)v|^2$$

Substituting (2.11) in the above inequality yields

$$a_0^2 |\alpha - \nu| \le \int_{-\xi}^{\infty} g^2 v^2 + C e^{-\xi^2}$$
(2.14)

By (2.12) and (2.13) we have

$$a_0^2 = 1 - \int_{-\xi}^{\infty} |\tilde{v}|^2 \ge 1 - \int_{-\xi}^{\infty} g^2 v^2 - C e^{-\xi^2}$$

from which (2.7) can be easily obtained.

We now continue the proof of Theorem 2.1. Let

$$\alpha_n(\lambda) =: \alpha(-n\omega - \omega_0, \lambda f^2).$$
(2.15)

Since, by (1.10)

$$\int_{-n\omega-\omega_0}^{\infty} f^2(x+n\omega+\omega_0)e^{-x^2} \ge C(n\omega+\omega_0)^{\lambda-1} \exp\left\{-\frac{1}{2}\left(n\omega+\omega_0\right)^2\right\},\,$$

we have, by (2.7),

$$\alpha_n \ge 1 + C(n\omega + \omega_0)^{\lambda - 1} \exp\left\{-\frac{1}{2}\left(n\omega + \omega_0\right)^2\right\}.$$
(2.16)

We now define the functional

$$\mathcal{J}(\chi_n, \chi_{-n}) =: \int_{-\omega_0}^{\infty} |\chi'_n|^2 + (x - n\omega)^2 |\chi_n|^2 + |\chi'_{-n}|^2 + (x + n\omega)^2 |\chi_{-n}|^2 - \lambda \left[ |\chi_n|^2 + |\chi_{-n}|^2 - f^2 \left( |\chi_n|^2 + |\chi_n + \bar{\chi}_{-n}|^2 + |\chi_{-n}|^2 \right) \right] \quad (2.17)$$

Let  $(\phi_n, \phi_{-n}) \in \mathcal{H} \times \mathcal{H}$  satisfy (2.5a,b) and

$$\int_{-\omega_0}^{\infty} \left( |\phi_n|^2 + |\phi_{-n}|^2 \right) = 1.$$

Multiplying (2.5a) by  $\bar{\phi}_n$  and the complex conjugate of (2.5b) by  $\phi_{-n}$  and integrating their sum over  $[-\omega_0, \infty)$  we obtain

$$\mathcal{J}(\phi_n, \phi_{-n}) = 0.$$

However, from the definition of  $\alpha_n$  it follows that

$$\int_{-\omega_0}^{\infty} |\phi_n'|^2 + (x - n\omega)^2 |\phi_n|^2 - \lambda \left(1 - f^2\right) |\phi_n|^2 \ge (\alpha_n - \lambda) \int_{-\omega_0}^{\infty} |\phi_n|^2.$$

Furthermore, for sufficiently large n, we have  $(x + n\omega)^2 > (x - n\omega)^2$  for every  $x \in [-\omega_0, \infty)$ , and hence

$$\mathcal{J}(\phi_n, \phi_{-n}) \ge (\alpha_n - \lambda) \tag{2.18}$$

Consequently, the value of  $\lambda$  for which the minimal value of  $\mathcal{J}$  vanishes, must be greater than  $\alpha_n$ . Therefore, by (2.16) the lower bound in (1.15) is proved.

To prove the upper bound we need, once again, to prove an auxiliary result:

### Lemma 2.3 Let

$$\gamma_n(\lambda) = \inf_{\substack{(\chi_n, \chi_{-n}) \in \mathcal{H} \times \mathcal{H} \\ \|\chi_n\|_{L^2}^2 + \|\chi_{-n}\|_{L^2}^2 = 1}} \mathcal{J}(\chi_n, \chi_{-n}).$$

Then,

- 1. For every  $\lambda \geq 1$ , there exists a minimizer in  $\mathcal{H} \times \mathcal{H}$ .
- 2.  $\gamma_n$  is a continuous function of  $\lambda$ .

*Proof:* Since the proof is rather standard, we bring here only the main details and very briefly. Let  $\{\phi_n^m, \phi_{-n}^m\}_{m=1}^{\infty}$  denote a minimizing sequence satisfying  $\|\phi_n^m\|^2 + \|\phi_{-n}^m\|^2 = 1$  for all m. Obviously,

$$\int_{a}^{\infty} |\phi_{n}^{m}|^{2} + |\phi_{-n}^{m}|^{2} \le \frac{C}{a^{2}}$$

otherwise  $\limsup_{m\to\infty} \mathcal{J}(\phi_n^m, \phi_{-n}^m) = \infty$ . It is easy to show that the minimizing sequence is bounded in  $H^1 \times H^1$ , and hence, there exists a subsequence which converges weakly to  $(\phi_n, \phi_{-n})$ . Clearly,

$$1 \ge \int_{-\omega_0}^a |\phi_n|^2 + |\phi_{-n}|^2 \ge 1 - \frac{C}{a^2},$$

and hence  $\|\phi_n\|^2 + \|\phi_{-n}\|^2 = 1$ . To complete the proof of existence we need yet to show that *E* is weakly lower semicontinuous. This, however is a very simple task. For instance,

$$\int_{-\omega_0}^{\infty} (\phi_n^m)' \overline{\phi_n'} \to \int_{-\omega_0}^{\infty} |\phi_n'|^2$$

in view of the weak convergence. Applying the Cauchy-Schwarz inequality we obtain

$$\liminf_{m \to \infty} \|(\phi_n^m)'\| \ge \|\phi_n'\|.$$

Similar treatment can be given to the rest of the terms in (2.17).

The proof that  $\gamma_n(\lambda)$  is continuous is completely straightforward.

We now calculate  $\mathcal{J}(w_n, 0)$  where  $w_n = \exp\{-(x - n\omega)^2/2\}$ . It is not difficult to show that

$$\mathcal{J}(w_n, 0) \le (1 - \lambda)\sqrt{\pi} + \int_{-\omega_0}^{\infty} f^2 w_n^2 + C e^{-(\omega_0 + n\omega)^2} \le (1 - \lambda)\sqrt{\pi} + C e^{-(\omega_0 + n\omega)^2/2}.$$

Hence, there exists C > 0 such that when

$$\lambda > 1 + Ce^{-(\omega_0 + n\omega)^2/2}$$

we have  $\mathcal{J}(w_n, 0) < 0$ , and therefore,  $\gamma_n(\lambda) < 0$ . Since, in view of (2.18), for sufficiently large  $n, \gamma_n(\lambda) > 0$  whenever  $\lambda < \alpha_n$ , and since  $\gamma_n(\lambda)$  must be continuous, there exists  $\lambda_n$ satisfying (1.15) and  $\gamma_n(\lambda_n) = 0$ . By lemma 2.3 there exists a minimizer which must satisfy (2.5), which completes the proof of the theorem.  $\Box$  We note that the above theorem proves, only for sufficiently large n, that bifurcating modes can exist and that  $\lambda_n > 1$ . Nevertheless, it seems plausible to conjecture that the bifurcation may take place only for  $\lambda > 1$ . Furthermore, it appears reasonable to believe that  $\lambda_n$  is monotone decreasing, from which the previous conjecture readily follows.

It still remains necessary to find the dimension of the space of solutions of (2.4) for  $\lambda = \lambda_n$ . Consider then, (2.5), once again. Let  $\phi_n^r = \Re \phi_n$ , and  $\phi_n^i = \Im \phi_n$ . Then, the real part satisfies

$$-(\phi_n^r)'' + \left[ (x - n\omega)^2 - \lambda \right] \phi_n^r + \lambda f^2 \left( 2\phi_n^r + \phi_{-n}^r \right) = 0$$
 (2.19a)

$$-(\phi_{-n}^{r})'' + [(x+n\omega)^{2} - \lambda] \phi_{-n}^{r} + \lambda f^{2} (2\phi_{-n}^{r} + \phi_{n}^{r}) = 0$$
(2.19b)

$$(\phi_n^r)'(-\omega_0) = (\phi_{-n}^r)'(-\omega_0) = 0$$
 (2.19c)

whereas the imaginary part satisfies

$$-(\phi_n^i)'' + [(x - n\omega)^2 - \lambda] \phi_n^i + \lambda f^2 (2\phi_n^i - \phi_{-n}^i) = 0$$
 (2.20a)

$$-\left(\phi_{-n}^{i}\right)'' + \left[(x+n\omega)^{2} - \lambda\right]\phi_{-n}^{i} + \lambda f^{2}\left(2\phi_{-n}^{i} - \phi_{n}^{i}\right) = 0$$
(2.20b)

$$(\phi_n^i)'(-\omega_0) = (\phi_{-n}^i)'(-\omega_0) = 0.$$
 (2.20c)

Consequently, if  $(\phi_n^r, \phi_{-n}^r)$  is a solution of (2.19), then  $(\phi_n^r, -\phi_{-n}^r)$  is a solution of (2.20). By (2.6) we have

$$\phi = \phi_n e^{in\omega x_2} + \phi_{-n} e^{-in\omega x_2}$$

Substituting in the above a linear combination of the two independent modes  $(\phi_n^r, \phi_{-n}^r)$  and  $(i\phi_n^r, -i\phi_{-n}^r)$  we obtain

$$\phi = C\phi_n^r e^{in\omega x_2} + \bar{C}\phi_{-n}^r e^{-in\omega x_2},$$

where  $C \in \mathbb{C}$  is an arbitrary constant. We can now represent  $\phi$ , upon substituting  $C = |C|e^{-ix_2^0}$ , in the following form,

$$\phi = |C| \left[ \phi_n^r e^{in\omega(x_2 - x_2^0)} + \phi_{-n}^r e^{-in\omega(x_2 - x_2^0)} \right]$$

Consequently, the additional mode stands for translations in the  $x_2$  direction and is, therefore, of very limited interest. Furthermore, since  $\lambda_n$  must be of even multiplicity in X, it is not possible to apply the Crandall-Rabinowitz theorem [12]. Thus, it is desirable to confine the discussion to an appropriate *real* subspace of X. We thus define

$$\mathbb{X}^* = \{ u \in \mathbb{X} \mid \bar{u}(x_1, x_2) = u(x_1, L - x_2) \}.$$

In this space, we have  $\phi_n = \phi_n^r$  for all n, and hence we need only to show that the solution space of (2.19) is one-dimensional.

**Lemma 2.4**  $\lambda_n$  is a simple eigenvalue of (2.19).

*Proof:* Let  $(\phi_n, \phi_{-n})$  and  $(\tilde{\phi}_n, \tilde{\phi}_{-n})$  be two different solutions of (2.19). We show that they must be linearly dependent. To this end we first multiply (2.19a) by  $\tilde{\phi}_n$  to obtain

$$\int_{-\omega_0}^{\infty} f^2 \left( \phi_n \tilde{\phi}_{-n} - \tilde{\phi}_n \phi_{-n} \right) = 0$$

Hence,

$$\exists x_0 \in (-\omega_0, \infty) \qquad \left[\phi_n \tilde{\phi}_{-n} - \tilde{\phi}_n \phi_{-n}\right]_{x=x_0} = 0$$

Consequently,

$$\exists C \in \mathbb{R} \text{ such that } \begin{bmatrix} \chi_n \\ \chi_{-n} \end{bmatrix} = \begin{bmatrix} \phi_n \\ \phi_{-n} \end{bmatrix} + C \begin{bmatrix} \tilde{\phi}_n \\ \tilde{\phi}_{-n} \end{bmatrix}$$
(2.21)

vanishes at  $x = x_0$ . Since  $(\chi_n, \chi_{-n})$  is a solution of (2.19) we must have  $\mathcal{J}(\chi_n, \chi_{-n}) = 0$ . Let then,

$$\tilde{\chi}_n = \begin{cases} \chi_n & x_0 \le x \\ -\chi_n & -\omega_0 \le x < x_0 \end{cases}$$

Clearly,  $\mathcal{J}(\tilde{\chi}_n, \tilde{\chi}_{-n}) = 0$ , and hence,  $(\tilde{\chi}_n, \tilde{\chi}_{-n})$  is a minimizer, which must have a continuous derivative at  $x_0$ . Consequently,  $\chi'_n(x_0) = \chi'_{-n}(x_0) = 0$  from which we conclude that  $\chi_n \equiv \chi_{-n} \equiv 0$ .  $\Box$ 

# 3 Weakly non-linear analysis

In the previous section, we showed that the linearized equation (2.4) has non-trivial solutions for a sequence of eigenvalues satisfying (1.15). However, our goal is to prove that each of these eigenvalues is a bifurcation point for the non-linear equation

$$F(\lambda, u) = 0 \tag{3.1}$$

where F is defined in (2.3).

In this section we prove the existence of a bifurcating branch at  $(0, \lambda_n)$ , for sufficiently large n. Furthermore, we prove that the bifurcation is supercritical and prove that the bifurcating branch is energetically lower than  $u \equiv 0$ , representing the one-dimensional solution (1.8).

## 3.1 Existence of the bifurcation

**Theorem 3.1** Equation (3.1) has a bifurcation point at  $(0, \lambda_n)$  in  $\overset{*}{\mathbb{X}}$ .

*Proof:* We use Theorem 1.7 in [12] to prove the existence of bifurcation. In view of the results of the previous section it remains to show that

$$F_{\lambda u}\phi \notin R\left(F_u(0,\lambda_n)\right)$$

where  $\phi$  spans the solution space of (2.4) in  $\overset{*}{\mathbb{X}}$  at  $\lambda = \lambda_n$ . Alternatively, we can write

$$\Re\left\{\int_{-\omega_0}^{\infty} \bar{\phi} F_{\lambda u} \phi\right\} = 0 \tag{3.2}$$

The above condition may be applied also by applying to (3.1) the Taylor expansion

$$u = \epsilon u^{(0)} + \epsilon^2 \tilde{u} \tag{3.3a}$$

$$\lambda = \lambda^{(0)} + \epsilon \tilde{\lambda} \tag{3.3b}$$

In the above  $\lambda^{(0)} = \lambda_n$  satisfies (1.15), and  $u^{(0)}$  is a solution of (2.4). Theorem 1.18 in [12] guarantees that

$$\tilde{u} = u^{(1)} + \epsilon u^{(2)} + O(\epsilon^2)$$
(3.4a)

$$\tilde{\lambda} = \lambda^{(1)} + \epsilon \lambda^{(2)} + O(\epsilon^2)$$
(3.4b)

This Taylor expansion, in powers of  $\epsilon$ , would be useful while investigating whether the bifurcation is subcritical or supercritical and while estimating the energy of the bifurcating branch near the bifurcation point.

The  $O(\epsilon^2)$  equation is given by

$$\left(i\nabla + x_1\hat{i}_2\right)^2 u^{(1)} - \lambda^{(0)} \left[u^{(1)} - (f(x,\lambda^{(0)}))^2 (2u^{(1)} + \bar{u}^{(1)})\right] = \lambda^{(1)}g_\lambda + \lambda^{(0)}f(x,\lambda^{(0)}) \left[2|u^{(0)}|^2 + (u^{(0)})^2\right]$$
(3.5a)

where

$$g_{\lambda} = \left. \frac{\partial}{\partial \lambda} \left\{ \lambda \left[ u^{(0)} - (f(x,\lambda))^2 (2u^{(0)} + \bar{u}^{(0)}) \right] \right\} \right|_{\lambda = \lambda^{(0)}}$$
(3.5b)

which is exactly equation 1.20 in [12] applied to our particular case. Multiplying (3.5a) by  $\bar{u}^{(0)}$  we obtain after some manipulation that

$$\lambda^{(1)}I_{\lambda} = \lambda^{(0)} \Re \left\{ \int f |u^{(0)}|^2 (u^{(0)} + 2\bar{u}^{(0)}) \right\}$$
(3.6a)

where

$$I_{\lambda} = \frac{\partial}{\partial\lambda} \left\{ \lambda \int |u^{(0)}|^2 - \frac{1}{2} f^2(x, \lambda^{(0)}) \left| u^{(0)} + \bar{u}^{(0)} \right|^2 - f^2 |u^{(0)}|^2 \right\} \Big|_{\lambda = \lambda^{(0)}}$$
(3.6b)

Condition (3.2) is a solvability condition of (3.5). By (3.6) it can be expressed in the form  $I_{\lambda} \neq 0$ .

In the previous section we showed that when  $\lambda^{(0)} = \lambda_n$  we have

$$u^{(0)} = \phi_n e^{i\omega nx_2} + \phi_{-n} e^{-i\omega nx_2}, \qquad (3.7)$$

where  $(\phi_n, \phi_{-n})$  is a solution of (3.6). Hence,

$$I_{\lambda} = \frac{2\pi}{\omega} \int_{-\omega_0}^{\infty} |\phi_n'|^2 + (x - n\omega)^2 |\phi_n|^2 + |\phi_{-n}'|^2 + (x + n\omega)^2 |\phi_{-n}|^2 - 2\lambda_n^2 f \frac{\partial f}{\partial \lambda} \left( |\phi_n|^2 + |\phi_n + \bar{\phi}_{-n}|^2 + |\phi_{-n}|^2 \right)$$
(3.8)

In the following, we prove that  $I_{\lambda} > 0$ . To this end we need first the following lemma.

#### Lemma 3.2 Let

$$\tilde{\beta}_n = \inf_{\phi \in \mathcal{H}} \frac{\int_{-\omega_0}^{\infty} |\phi'|^2 + (x - n\omega)^2 |\phi|^2 - 2\lambda_n^2 f \frac{\partial f}{\partial \lambda} |\phi|^2}{\int_{-\omega_0}^{\infty} |\phi|^2}.$$
(3.9)

Then,  $\lim \tilde{\beta}_n = 1$ .

*Proof:* We first prove that

$$\left\|\frac{\partial f}{\partial \lambda}\right\|_{\infty} \le C.$$

The equation satisfied by  $\partial f / \partial \lambda = f_{\lambda}$  is

$$-f_{\lambda}'' + \left[x^2 - \lambda + 3\lambda f^2\right] f_{\lambda} = f(1 - f^2) \quad ; \quad f_{\lambda}'(0) = 0$$

Clearly, there exists  $x_0$  such that

$$x > x_0 \Rightarrow x^2 - \lambda + 3\lambda f^2 > 1$$

Suppose now, for a contradiction, that at some  $x_1 > x_0$ , for some  $\lambda = \lambda_0$  we have  $f_{\lambda}(x_1, \lambda_0) > 1$ 1 and  $f'_{\lambda}(x_1, \lambda_0) > 0$ . Then, since for  $x > x_0$   $f_{\lambda}$  cannot have a maximum greater than 1 we must have  $f_{\lambda}(x, \lambda_0) > 1$  for all  $x > x_1$ . Since both  $f_{\lambda}$  and  $f'_{\lambda}$  are continuous in  $\lambda$  there must be a neighborhood  $(\lambda_0 - \epsilon, \lambda_0 + \epsilon)$  where  $f_{\lambda}(x_1, \lambda) > 1$  and  $f'_{\lambda}(x_1, \lambda) > 0$ . Consequently,

$$x \ge x_1$$
;  $\lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon) \Rightarrow f_\lambda(x, \lambda) > 1$ ,

and hence,

$$f(x, \lambda_0 + \epsilon) - f(x, \lambda_0 - \epsilon) \ge 2\epsilon$$

for all  $x \ge x_1$ , contradicting (1.10). Thus, for  $x > x_0$  we have

$$f_{\lambda}(x,\lambda) > 1 \Rightarrow f'_{\lambda}(x,\lambda) \le 0$$

from which we can conclude that, for  $x > x_0$ ,

$$f_{\lambda}(x,\lambda) \le \max(f_{\lambda}(x_0,\lambda),1) \le C(\lambda)$$

where C is independent of x. In a similar manner we can obtain a lower bound for  $f_{\lambda}$ , and hence,

$$\left| f \frac{\partial f}{\partial \lambda} \right| \le C x^{(\lambda - 1)/2} \exp\{-x^2/2\}.$$
(3.10)

The lemma now follows from (2.7) with  $g = -2\lambda_n^2 f \partial f / \partial \lambda$  and  $\xi = -n\omega - \omega_0$ .

We now return to the proof of Theorem 3.1. From (3.8) it easily follows that

$$I_{\lambda} \geq \frac{4\pi}{\omega} \tilde{\beta}_n,$$

and hence, for sufficiently large n,  $I_{\lambda}$  must be positive, which proves our theorem.

## 3.2 Nature of the bifurcation.

In the following we show that in some neighborhood of  $(0, \lambda_n)$  in  $X^* \times \mathbb{R}$  we must have  $\lambda > \lambda_n$ along the bifurcating branch. Alternatively, we can state that the bifurcation is supercritical. From a physical point of view we can say that if we decrease the applied magnetic field (and consequently increase  $\lambda$ ) below the critical field which corresponds to  $\lambda_n$  then the bifurcating branch continues to develop, i.e., ||u|| increases. Consider then, once again, (3.6). Using (3.7) it is not difficult to show that  $\lambda^{(1)} = 0$ , which is a natural result inasmuch as we do not expect the sign of  $\lambda - \lambda^{(0)}$  to depend on the sign of  $\epsilon$ . Hence,

$$\left(i\nabla + x_1\hat{i}_2\right)^2 u^{(1)} - \lambda^{(0)} \left[u^{(1)} - f^2(2u^{(1)} + \bar{u}^{(1)})\right] = \lambda^{(0)} f\left[2|u^{(0)}|^2 + (u^{(0)})^2\right].$$
(3.11)

The next order equation is given by

$$\left(i\nabla + x_1\hat{i}_2\right)^2 u^{(2)} - \lambda^{(0)} \left[u^{(2)} - f^2(2u^{(2)} + \overline{u^{(2)}})\right] = \lambda^{(2)}g_\lambda - \lambda^{(0)} \left\{ |u^{(0)}|^2 u^{(0)} + 2f \left[u^{(0)}\overline{u^{(1)}} + u^{(1)}\overline{u^{(0)}} + u^{(1)}u^{(0)}\right] \right\}.$$
 (3.12)

The above equation no longer follows directly from Theorem 1.18 in [12]. Nevertheless, it can be easily obtained, using the implicit function Theorem, in the same way it is used in the proof of equation 1.20 in [12].

Multiplying (3.12) by  $\overline{u^{(0)}}$  and integrating by parts we obtain

$$\lambda^{(2)}I_{\lambda} = \lambda^{(0)} \int |u^{(0)}|^4 + 2f|u^{(0)}|^2(u^{(1)} + \overline{u^{(1)}}) + 2f\overline{u^{(0)}})^2 u^{(1)}.$$
(3.13)

We now multiply (3.11) by  $u^{(1)}$  to obtain

$$\int |(i\nabla + x_1\hat{i}_2)u^{(1)}|^2 - \lambda^{(0)} \left[ |u^{(1)}|^2 - 2f^2|u^{(1)}|^2 - \frac{1}{2}f^2 \left( \overline{u^{(1)}}^2 + (u^{(1)})^2 \right) \right] = \lambda^{(0)} \int 2f|u^{(0)}|^2 \left( u^{(1)} + \overline{u^{(1)}} \right) + \frac{1}{2}f \left[ \overline{u^{(0)}}^2 u^{(1)} + \overline{u^{(1)}} (u^{(0)})^2 \right]. \quad (3.14)$$

Hence,

$$\lambda^{(2)}I_{\lambda} = \lambda^{(0)} \int |u^{(0)}|^4 + 2 \int |(i\nabla + x_1\hat{i}_2)u^{(1)}|^2 - \lambda^{(0)} \left[ \left| u^{(1)} \right|^2 - f^2 |u^{(1)}| \right|^2 - \frac{1}{2} f^2 \left| \overline{u^{(1)}} + u^{(1)} \right|^2 \right]$$
(3.15)

By (2.17) and (3.7) we obtain, that if we substitute  $u^{(0)}$  instead of  $u^{(1)}$  in the second integral on the right hand side of (3.15) it must vanish identically. Furthermore, except for a finite number of *n* values, (3.7) must span the solution space of (2.4) in X\* when  $\lambda^{(0)} = \lambda_n$ . Hence,  $u^{(0)}$  must serve as the non-trivial global minimizer of the second integral on the right hand side of (3.15). Consequently,

$$\int |(i\nabla + x_1\hat{i}_2)u^{(1)}|^2 - \lambda^{(0)} \left[ |u^{(1)}|^2 - f^2|u^{(1)}|^2 - \frac{1}{2}f^2 \left| \overline{u^{(1)}} + u^{(1)} \right|^2 \right] \ge 0$$
(3.16)

and hence,

$$\lambda^{(2)} \ge \frac{\lambda^{(0)}}{I_{\lambda}} \int |u^{(0)}|^4 > 0.$$
(3.17)

This proves our assertion, namely, that  $\lambda > \lambda^{(0)}$  along the bifurcating branch, or, that the bifurcation is supercritical.

#### 3.2.1 Energy

In this subsection we prove that (1.8) is not a local minimizer of (1.13) for  $\lambda > 1$ . To this end we show that for every  $n \ge n_0$ , there exist a right neighborhood of  $\lambda_n$  in  $\mathbb{R}$  such that  $\mathcal{E}(f+u,\lambda) < \mathcal{E}(f,\lambda)$ . Let

$$\mathcal{E}_0(\lambda) = \mathcal{E}(f,\lambda) = \frac{2\pi}{\omega} \int_{-\omega_0}^{\infty} |f'| + x^2 f^2 + \lambda \left[\frac{1}{2}f^4 - f^2\right]$$

Let further

$$\begin{split} \Delta \mathcal{E}(u,\lambda) &= \mathcal{E}(f+u) - \mathcal{E}_0(\lambda) = 2\Re \left\{ \int (i\nabla + x_1 \hat{i}_2) f \cdot (-i\nabla + x_1 \hat{i}_2) \bar{u} \right\} + \int \left| (i\nabla + x_1 \hat{i}_2) u \right|^2 + \\ &+ 2\lambda \Re \left\{ \int |f|^2 \left[ f \bar{u} + |u|^2 \right] + \frac{1}{2} f^2 \bar{u}^2 + |u|^2 \left[ f \bar{u} + \frac{1}{4} |u|^2 \right] - f \bar{u} - \frac{1}{2} |u|^2 \right\} \end{split}$$

Using (1.9) and integration by parts yields

$$\Delta \mathcal{E}(u,\lambda) = \int \left| (i\nabla + x_1 \hat{i}_2) u \right|^2 + \lambda \int 2 |f|^2 |u|^2 + 2 |u|^2 \Re (f\bar{u}) + \lambda \int \frac{1}{2} |u|^4 - |u|^2 + \frac{1}{2} \left| f\bar{u} + \bar{\psi}_0 u \right|^2 \quad (3.18)$$

We now multiply the (3.1) by  $\bar{u}$  and integrate to obtain

$$\int \left| (i\nabla + x_1 \hat{i}_2) u \right|^2 = -\lambda \int 2 |f|^2 |u|^2 + |u|^2 \left( 2f\bar{u} + u\bar{\psi}_0 \right) + f^2 \bar{u}^2 + |u|^4 - |u|^2.$$

Combining the above with (3.18) we obtain

$$\Delta \mathcal{E}(u,\lambda) = -\frac{1}{2}\lambda \int \left[ |f+u|^2 - |f|^2 \right] |u|^2.$$
(3.19)

We now expand  $\Delta \mathcal{E}$  in powers of  $\epsilon$ . By (3.4) we have

$$\Delta \mathcal{E} = \epsilon^3 \Delta \mathcal{E}^{(0)} + \epsilon^4 \Delta \mathcal{E}^{(1)} + O(\epsilon^5)$$

Substituting (3.4) in (3.19) yields

$$\Delta \mathcal{E}^{(0)} = -\lambda^{(0)} \int |u^{(0)}|^2 \Re \left( f \overline{u^{(0)}} \right) = 0.$$

Once again, this result corresponds to the natural expectation that the sign of  $\Delta E$  does not depend on the sign of  $\epsilon$ . The next order term is expressible in the form

$$\Delta \mathcal{E}^{(1)} = -\lambda^{(0)} \int \frac{1}{2} \left| u^{(0)} \right|^4 + \left| u^{(0)} \right|^2 \Re \left( f \overline{u^{(1)}} \right) + 2\Re \left( f \overline{u^{(0)}} \right) \Re \left( u^{(0)} \overline{u^{(1)}} \right)$$
(3.20)

Using (3.14) we obtain

$$\Delta \mathcal{E}^{(1)} = -\lambda^{(0)} \int \frac{1}{2} |u^{(0)}|^4 - \left\{ \int |(i\nabla + x_1\hat{i}_2)u^{(1)}|^2 - \lambda^{(0)} \left[ |u^{(1)}|^2 - f^2 |u^{(1)}|^2 - \frac{1}{2} f^2 \left| \overline{u^{(1)}} + u^{(1)} \right|^2 \right] \right\}$$

and by (3.16) we have

$$\Delta \mathcal{E}^{(1)} \le -\frac{1}{2} \lambda^{(0)} \int \left| u^{(0)} \right|^4 < 0$$

which proves our assertion, and hence completes the proof of theorem 1.1.

# 4 Concluding remarks

In §2 we proved the existence of a set of critical values  $\{\lambda_n\}_{n=n_0}^{\infty}$  for which non-trivial solutions of (2.4) exist. We also show that  $\lambda_n \downarrow 1$  exponentially fast according to (1.15). However, there might exist, finitely many, additional values of  $\lambda$  for which non-trivial solutions of (2.4) can exist. It would be reasonable to conjecture that  $\{\lambda_n\}_{n=1}^{\infty}$  is monotone decreasing, yet, this hypothesis is proved only for large n. In fact, it is not proved yet that  $\lambda_n > 1$  for all n.

One can formulate the above conjecture in the following alternative manner: Let  $\omega = 1$ and

$$\gamma(\lambda,\alpha) = \inf_{\substack{(\phi_{\alpha},\phi_{-\alpha})\in\mathcal{H}\times\mathcal{H}\\ \|\phi_{\alpha}\|_{\ell_{2}}^{2}+\|\phi_{-\alpha}\|_{\ell_{2}}^{2}=1}} \mathcal{J}(\phi_{\alpha},\phi_{-\alpha}),$$

where  $\mathcal{J}$  is defined in (2.17). We look for values of  $\lambda$  and  $\alpha$  for which  $\gamma = 0$ . For sufficiently large  $\alpha$  it is proved in §2 that there exists  $\alpha_0 > 0$  and a function  $\lambda(\alpha) : [\alpha_0, \infty) \to \mathbb{R}$  such that  $\gamma(\lambda(\alpha), \alpha) = 0$ , and such that  $\lambda(\alpha) \downarrow 1$  as  $\alpha \to \infty$ . If one can show that  $\lambda(\alpha)$  can be continued into  $\mathbb{R}_+$  such that  $\lambda(\alpha)$  is monotone decreasing, the the above conjecture is proved.

In  $\S 3$  we proved:

- 1. Existence of the bifurcation points.
- 2. Super-criticality of the bifurcation.
- 3. That the bifurcating solution is energetically preferable to the one-dimensional surface superconductivity solution.

Statements 1 and 2 were proved only for sufficiently large n. For n which is not large, the existence of non-trivial solutions of (2.4) does not guarantee  $I_{\lambda} > 0$ , and hence the bifurcation points do not necessarily exist. In fact, even if the bifurcation from  $(\lambda_n, 0)$  exists, it is not clear that it must be supercritical (if  $I_{\lambda} < 0$  then a subcritical bifurcation exists).

In contrast, statement 3 is correct whenever a bifurcating solution exists. It is correct even for n which is not necessarily large, and even in the unlikely situation that the bifurcation takes place at  $\lambda < 1$ . The surface superconductivity one-dimensional solution becomes therefore locally unstable at each bifurcation point.

Finally, we note that if it was possible to linearly superpose the bifurcating modes then the resulting combination would have the form

$$\psi = f + \sum_{n = -\infty}^{\infty} C_n \phi_n e^{i\omega nx_2}$$

and since

$$\phi_n \sim e^{-\frac{1}{2}(x_1 - n\omega)^2}$$

as  $n \to \infty$  for  $x_1 \sim O(n)$  we have

$$\psi \sim e^{i\omega P x_2} \sum_{n=-\infty}^{\infty} C_{n+P} e^{i\omega n x_2} e^{-\frac{1}{2}[x_1 - (n+P)\omega]^2}$$
 (4.1)

for  $P \gg 1$  and  $x \sim O(P)$ . The above formula thus approximates  $\psi$  far away from the wall at  $x_1 = 0$ . If

$$\exists N: \ C_{n+N} = C_n \ \forall n$$

Then the right-hand-side of (4.1) is periodic, or an Abrikosov lattice [2, 9, 4].

Clearly, it is impossible to linearly superpose modes since the equations are non-linear and since the bifurcations take place at different values of  $\lambda$ . Nevertheless, if  $0 < \lambda - 1 \ll 1$ then  $0 < \lambda - \lambda_n < \lambda - 1$  for almost every n. Hence, one might expect that the effect of non-linearity tends to 0 as  $\lambda \to 1$ , and thus, that the solution far away from the wall can be approximated by an Abrikosov lattice.

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## A The one-dimensional solution

**Theorem A.1** Let  $\mathcal{H}$  and  $\beta(z)$  be, respectively defined by (1.6) and (1.7). Let  $\lambda > \beta(z)$ . Then, there exists a positive solution in  $\mathcal{H}$  to the equation

$$-f'' + x^2 f = \lambda f(1 - f^2) \quad ; \quad f'(z) = 0 \; . \tag{A.1}$$

Moreover,

$$f(x) \sim x^{-\frac{1-\lambda}{2}} e^{-\frac{1}{2}x^2} \quad as \ x \to \infty \tag{A.2}$$

*Proof:* Let

$$\epsilon_z(\phi) = \int_z^\infty (\phi')^2 + x^2 \phi^2 - \lambda \left[\phi^2 - \frac{1}{2}\phi^4\right] dx.$$

We first prove the existence of a minimizer  $\mathcal{H}$ . Clearly, there exists  $x_0 > z$  such that for  $x \ge x_0$  we have  $x^2 \ge \lambda + 1$ . Then, since

$$\phi^2 - \frac{1}{2}\phi^4 \le \frac{1}{2}$$

we have for all  $\phi \in \mathcal{H}$ 

$$\epsilon_z(\phi) \ge -\frac{\lambda}{2}(x_0 - z).$$

Since  $\epsilon_z$  is semibounded there is a minimizing sequence  $\{\phi_n\}_{n=1}^{\infty}$  in  $\mathcal{H}$ . As

$$\frac{1}{x_0 - z} \left( \int_z^{x_0} \phi_n^2 \right)^2 \le \int_z^{x_0} \phi_n^4 \le 2 \int_z^{x_0} \phi_n^2 + C$$

where C is independent of n, we obtain that  $\|\phi_n\|_{L^2[z,x_0]} \leq C$ . Recalling that  $x^2 - \lambda \geq 1$ in  $[x_0,\infty)$  yields  $\|\phi_n\|_{L^2(z,\infty)} \leq C$  and hence  $\|\phi_n\|_{H^1(z,\infty)} \leq C$ . Thus, there is a subsequence which converges weakly in  $H^1(z,\infty)$  to a limit, which we denote by f. We skip the proof of lower semicontinuity - some of the details can be found in the proof of lemma 2.4.

Since f is a minimizer of  $\epsilon_z$  in  $\mathcal{H}$  it must satisfy (A.1). Suppose for a contradiction that it changes sign at  $x = x_1$ . Then let

$$g = \begin{cases} f(x) & x \le x_1 \\ \\ -f(x) & x > x_1 \end{cases}$$

Clearly,  $\epsilon_z(f) = \epsilon_z(g)$ , and thus, g must be a minimizer and therefore, a solution of (A.1). Thus, either  $f \equiv 0$ , or f does not change its sign (both f and -f are minimizers). We now prove that f is non-trivial. Let  $u_z$  be the minimizer of the fraction on the right-hand-side of (1.7) such that  $||u_z||_2 = 1$ . Then

$$\epsilon_z(cu_z) = -c^2[\lambda - \beta(z)] + c^4 \int_z^\infty |u_z|^4$$

Therefore, for sufficiently small c the minimizer must be non-trivial.

It remains necessary, yet, to prove (A.2). We first prove that  $f \to 0$  as  $x \to \infty$ . Suppose first, for a contradiction, that for some  $x_2 > x_0$  we have  $f'(x_2) > 0$ . Then, since f cannot have a maximum for  $x > x_0$ , f must be greater than  $f(x_2)$ , contradicting  $f \in \mathcal{H}$ . Thus,  $f' \leq 0$  for all  $x > x_0$  from which we easily conclude that  $f \to 0$  at infinity.

Let w(x,t), where  $t > x_0$ , denote the decaying solution of

$$-w'' + [x^2 - \lambda]w = 0 \quad ; \quad x > tw(t, t) = 1$$

By the maximum principle we must have that f < f(t)w for all  $x > x_0$ . Let f(x) = f(t)v(x,t)w(x,t). Substituting in (A.1) we obtain

$$v' = -\frac{\lambda}{f(t)w^2} \int_x^\infty f^3(s)w(s,t)ds \ge -\lambda f^2(t) \int_x^\infty w^2(s,t)ds$$

The properties of w have been obtain in [19] but can be also found in chapter 19 of [1]. From both references we find that as  $t \to \infty, x \to \infty$ 

$$w \sim \left(\frac{x}{t}\right)^{\frac{\lambda-1}{2}} e^{-\frac{1}{2}(x^2-t^2)} \quad \forall x \ge t$$

and hence, for all x > t

$$-2\lambda f^{2}(t)\frac{x^{\lambda-2}}{t^{\lambda-1}}e^{t^{2}-x^{2}} \le v' < 0$$

Since v is decreasing it must converge to a limit as  $x \to \infty$ . Integrating the above inequality by parts we obtain

$$v_{\infty}(t) = \lim_{x \to \infty} v(x, t) \ge 1 - \lambda \frac{f^2(t)}{t}$$

For sufficiently large t we therefore have  $v_{\infty}(t) > 0$ , proving (A.2).

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