Non-linear surface superconductivity in three dimensions in the large κ limit

Y. ALMOG *

Abstract

The Ginzburg-Landau model of superconductivity is considered in three dimensions. We show, for smooth bounded domains, that the superconductivity order parameter decays exponentially fast away from the boundary as the Ginzburg-Landau parameter κ tends to infinity. We prove this result for applied magnetic fields satisfying $h_{ex} - \kappa \gg \kappa^{1/2}$. Additionally, we prove that for applied fields greater than H_{C_2}, the only solution in $\mathbb{R}^3$ satisfying a certain decay condition is the normal state. Finally we prove that bulk superconductivity decreases to zero as $h_{ex} \uparrow H_{C_2}$, and thus extend (though in a weaker sense), the results in [?] to three-dimensional settings.

*Faculty of Mathematics, Technion - Israel Institute of Technology, Haifa 32000, Israel

1 Introduction

Consider a superconducting body which is placed in a sufficiently low temperature (below the critical one) under the action of an external magnetic field . Its energy is given by the Ginzburg-Landau energy functional which can be represented in the following dimensionless form [?]

$$E = \int_{\Omega} -|\Psi|^2 + \frac{|\Psi|^4}{2} + |h - h_{ex}|^2 + \left| \frac{i}{\kappa} \nabla \Psi + A\Psi \right|^2 \tag{1.1}$$

in which Ψ is the (complex) superconducting order parameter, such that $|\Psi|$ varies from $|\Psi| = 0$ (when the material is at the normal state) to $|\Psi| = 1$ (for the purely superconducting state). The magnetic vector potential is denoted by A (the magnetic field is, then, given by $h = \nabla \times A$), h_{ex} is the applied magnetic field, which is assumed constant in this work, and κ is the Ginzburg-Landau parameter which is a material property. Superconductors for which $\kappa < 1/\sqrt{2}$ are termed type I superconductors, and those for which $\kappa > 1/\sqrt{2}$ are termed type II. The domain Ω is the domain of superconductor whose Gibbs free energy is given by E. Note that E is invariant to the gauge transformation

$$\Psi \to e^{i\kappa\eta}\psi \qquad ; \qquad A \to A + \nabla\eta \; . \tag{1.2}$$

It is known both from experiments [?] and rigorous analysis [?] that for a sufficiently strong magnetic field the normal state ($\psi \equiv 0$, $h = h_{ex}$) would prevail. If the field is then decreased, there is a critical field, depending on the sample's geometry, where the material would enter the superconducting state. For samples with boundaries, this field is known as the onset field and has been termed H_{C_3}.

The simplest case in which the bifurcation from the normal state to the superconducting one was calculated is the case of a domain-wall [?]. The analysis in this case is one dimensional: the linearized Ginzburg-Landau equations were solved on $\mathbb{R}_+$. Even in this simple case the onset field is substantially larger than the bifurcation field on $\mathbb{R}$ [?], which has been termed H_{C_2}, Thus, $H_{C_3} \approx 1.7\kappa$, whereas $H_{C_2} = \kappa$. Furthermore, it was found that superconductivity is concentrated at the onset near the wall, i.e. ψ decays exponentially fast away from the boundary, which is why this phenomenon has been termed surface superconductivity.

The situation is no different in two dimensions: it was proved in [?] and [?] that the bifurcating mode $\mathbb{R}^2_+$ is one-dimensional and that the value of H_{C_3} is exactly the same as in the one-dimensional case (similarly, $H_{C_2} = \kappa$ in $\mathbb{R}^2$ where the bifurcation takes place in the form of periodic solutions). It was later proved for general two-dimensional domains with smooth boundaries [?, ?], that as the domain's scale tends to infinity the onset field tends to de-Gennes' value, and that ψ decays exponentially fast away from the boundary. For boundaries which include wedges the onset field will be larger than de-Gennes' value [?, ?, ?, ?].

Three recent contributions [?, ?, ?] study the behaviour of the global minimizer of the energy functional (??) for external fields satisfying $\kappa = H_{C_2} < h_{ex} < H_{C_3}$. In [?] the limit $\kappa \to \infty$ is considered: it is demonstrated that ψ decays, in L^2 sense, exponentially fast away form the boundary. The results are valid whenever $h_{ex} - \kappa \gg 1$ as $\kappa \to \infty$. In addition the energy of the global minimizer is shown to be evenly distributed along the boundary. In [?]

the exponential rate of decay is demonstrated for $h_{ex} - \kappa \gg 1/\kappa$. In [?] the large domain limit is considered: it is demonstrated for the global minimizer that both ψ and h tend, in C^α sense, exponentially fast away from the boundary, to the normal state. The results are valid whenever $h_{ex} - \kappa \sim O(1)$ as the domain's size tends to infinity.

In contrast with the significant progress in the study of surface superconductivity in two dimensions, very few works address this problem in three dimensions. Lu and Pan obtained in [?] several important results in the limit $\kappa \to \infty$. Some of these results, for the case of constant applied field, are listed below:

1. $H_{C_3} \approx 1.7\kappa$.

2. Let $\partial\Omega_h$ denote the portion of $\partial\Omega$ whose normal is orthogonal to the applied field h_{ex}. Then, as $h_{ex} \uparrow H_{C_3}$,

   (a) $\|\psi\|_{L^\infty(\Omega)} \to 0$

   (b) $\|h - h_{ex}\|_{L^\infty(\Omega)} \to 0$

   (c) $\frac{\psi(x)}{\|\psi\|_{L^\infty(\Omega)}} \to 0$ on $\bar{\Omega} \setminus \partial\Omega_h$.

In a previous contribution [?] the same authors obtained similar results for cylindrical domains.

In the present contribution we focus on the limit $\kappa \to \infty$ for general smooth three dimensional domains. We prove that any solution of the Ginzburg-Landau equations (??) (which are the Euler-Lagrange equations associated with the energy functional (??)) together with the natural boundary conditions (??), tends to the normal state ($\psi \equiv 0$, $h = h_{ex}$)

exponentially fast away from the boundary as long as $h_{ex} - \kappa \gg \kappa^{1/2}$. This improves the results in [**?**] in three different aspects:

1. While the results in [**?**] are valid only as $h_{ex} \sim H_{C_3}$, or near the linear bifurcation, the present results remains valid deep into the non-linear regime, as long as $h_{ex} - \kappa \gg \kappa^{1/2}$.

2. While the rate at which the solution attends the normal state is not clear in [**?**], we prove exponentially fast convergence.

3. The results in [**?**] are valid only for the global minimizer of (**??**), whereas here the estimates are valid for any solution of the Euler-Lagrange equations and the natural boundary conditions.

In contrast, however, the information on the distribution of ψ over $\partial\Omega$, provided in [**?**] near the nucleation, has not been obtained here.

In addition to the above results concerning surface superconductivity we briefly discuss the decrease of bulk superconductivity as $h_{ex} \uparrow H_{C_2}$. In a recent contribution, Sandier and Serfaty [**?**], proved that in the above limit, in two dimensions,

$$\frac{1}{|B_r|} \int_{B_r} |\psi|^4 \to 0$$

for any ball $B_r \subset \Omega$. We show here, for three-dimensional domains, that

$$\frac{1}{|\Omega|} \int_{\Omega} |\psi|^4 \to 0$$

in the same limit. While the above result is weaker than the one obtained in [**?**], it still shows that bulk superconductivity decays, in three dimensions, as $h_{ex} \uparrow H_{C_2}$.

Finally, we prove that no solution, satisfying the decay condition (**??**c) of the Ginzburg-Landau equation, but the normal state, can exist in $\mathbb{R}^3$ as long as $h_{ex} > \kappa$. In [**?**] we showed the same result in $\mathbb{R}^2$, for a larger class of bounded solutions (for which $h \to h_{ex}$ as $x \to \infty$). Thus, the additional dimension requires an additional decay condition.

In the next section we prove the above non-existence result in $\mathbb{R}^3$. In § 3 we prove exponentially fast convergence of any solution of the Ginzburg-Landau equations and the natural boundary conditions to the normal state away from the boundary, as long as $h_{ex} - \kappa \gg \kappa^{1/2}$. In addition, we show in this section that bulk superconductivity decreases as $h_{ex} \uparrow H_{C_2}$. Finally, in § 4 we briefly discuss a few key points which are insufficiently emphasized in the previous sections.

2 Non-existence in $\mathbb{R}^3$

In this section we prove the following result:

Theorem 2.1 *Let $\psi : \mathbb{R}^3 \to \mathbb{C}$ and $A : \mathbb{R}^3 \to \mathbb{R}^3$ satisfy the equations*

$$\left(\frac{i}{\kappa}\nabla + A\right)^2 \Psi = \Psi\left(1 - |\Psi|^2\right) \quad x \in \mathbb{R}^3 \tag{2.1a}$$

$$-\nabla \times (\nabla \times A) = \frac{i}{2\kappa}\left(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*\right) + |\Psi|^2 A \quad x \in \mathbb{R}^3, \tag{2.1b}$$

together with the rate of decay condition

$$\liminf_{R\to\infty} \frac{1}{R}\int_{R<|x|<\beta R} |\psi|^4 + |h - h_{ex}|^2 = 0 \tag{2.1c}$$

where the applied magnetic field, h_{ex}, is a constant and $\hat{k}$ is a unit vector in the z direction, and $\beta > 1$. (For reasons of convenience we pick the Coulomb gauge in sequel, i.e., $\nabla \cdot A = 0$.)

Then if either

1. $h_{ex} > \kappa$ *and* $\kappa > 1/\sqrt{2}$,

   or

2. $h_{ex} < \frac{1}{2\kappa}$ *and* $\kappa < 1/\sqrt{2}$,

then, $\psi \equiv 0$, $h \equiv h_{ex}$ *is the unique solution of* (**??**).

Proof: Let $\psi = \rho e^{i\phi}$. Multiplying (**??**a) by $e^{-i\phi}$ and taking the real part we obtain

$$\frac{1}{\kappa^2}\nabla^2\rho + \rho(1-\rho^2) = \rho\left|\frac{1}{\kappa}\nabla\phi - A\right|^2 = \frac{|\nabla \times h|^2}{\rho^3}, \tag{2.2a}$$

whenever $\rho \neq 0$. [To obtain the second equality in (**??**) we used (**??**b)]. Taking the curl of (**??**b) [by standard elliptic estimates any solution of (**??**) must be C^∞ on any smooth compact subset of $\mathbb{R}^3$] yields

$$\nabla^2 h - \rho^2 h = \frac{2}{\rho}\nabla\rho \times (\nabla \times h) \, , \tag{2.2b}$$

whenever $\rho \neq 0$. Consider first the case $h_{ex} > \kappa$, $\kappa > 1/\sqrt{2}$. Define

$$v_\alpha = h - \kappa + \frac{1}{2\kappa}\rho^2\hat{\alpha} \quad ; \quad u_\alpha = v_\alpha \cdot \hat{\alpha} \tag{2.3}$$

where $\hat{\alpha}$ is any constant unit vector. Combining (**??**a) and (**??**b) we obtain after some manipulation

$$\nabla^2 u_\alpha - \rho^2 u_\alpha = \kappa\left|\frac{\nabla \times v_\alpha}{\rho}\right|^2 + \frac{1}{\kappa}\,|\hat{\alpha}\cdot\nabla\rho|^2 + \left(\kappa - \frac{1}{2\kappa}\right)\rho^4 \tag{2.4}$$

whenever $\rho \neq 0$. Note that when $\hat{\alpha} = \hat{k}$, $h = h(x,y)\hat{k}$, and $\rho = \rho(x,y)$ we obtain the same equations as in the two-dimensional case.

It is possible to derive a similar equation which is valid when $\rho = 0$ as well. As

$$\nabla^2 h - \rho^2 h = \nabla(\rho^2) \times A + \frac{i}{\kappa}\nabla\psi^* \times \nabla\psi \,, \tag{2.5a}$$

$$(\nabla^2 - \rho^2)\left(\frac{1}{2\kappa}\rho^2 - \kappa\right) = \kappa\left|\left(\frac{i}{\kappa}\nabla + A\right)\psi\right|^2 + \left(\kappa - \frac{1}{2\kappa}\right)\rho^4, \tag{2.5b}$$

we have,

$$\begin{aligned}
&\frac{\kappa}{2}\left|\left(\frac{i}{\kappa}\nabla + A\right)\psi + i\hat{\alpha} \times \left(\frac{i}{\kappa}\nabla + A\right)\psi\right|^2 = \\
&\quad = \kappa\left|\left(\frac{i}{\kappa}\nabla + A\right)\psi\right|^2 + \hat{\alpha}\cdot\left[\nabla(\rho^2) \times A + \frac{i}{\kappa}\nabla\psi^* \times \nabla\psi\right] - \frac{\kappa}{2}\left|\left(\frac{i}{\kappa}\nabla + A\right)\psi \cdot \hat{\alpha}\right|^2
\end{aligned} \tag{2.6}$$

Combining (**??**) and (**??**) yields, then

$$\begin{aligned}
\nabla^2 u_\alpha - \rho^2 u_\alpha = {} & \frac{\kappa}{2}\left|\left(\frac{i}{\kappa}\nabla + A\right)\psi + i\hat{\alpha} \times \left(\frac{i}{\kappa}\nabla + A\right)\psi\right|^2 + \\
& + \frac{\kappa}{2}\left|\left(\frac{i}{\kappa}\nabla + A\right)\psi \cdot \hat{\alpha}\right|^2 + \left(\kappa - \frac{1}{2\kappa}\right)\rho^4,
\end{aligned} \tag{2.7}$$

and comparing with (**??**) we obtain

$$\begin{aligned}
J_\alpha^2 \stackrel{def}{=} {} & \frac{\kappa}{2}\left|\left(\frac{i}{\kappa}\nabla + A\right)\psi + i\hat{\alpha} \times \left(\frac{i}{\kappa}\nabla + A\right)\psi\right|^2 + \frac{1}{2}\left|\left(\frac{i}{\kappa}\nabla + A\right)\psi \cdot \hat{\alpha}\right|^2 = \\
& = \left|\frac{\nabla \times v_\alpha}{\rho}\right|^2 + \frac{1}{\kappa^2}\left|\hat{\alpha}\cdot\nabla\rho\right|^2
\end{aligned} \tag{2.8}$$

whenever $\rho \neq 0$.

We now pick $\hat{\alpha}$ to be parallel with h_{ex} and denote u_α by u in that direction. Let

$$u^+ = \begin{cases} u & u > 0 \\ & \\ 0 & u \leq 0 \end{cases}.$$

Multiplying (**??**) by u^+ and integrating over $B(0,r)$ we obtain

$$\int_{\partial B(0,r)} u^+ \frac{\partial u^+}{\partial r} \geq F(r) \tag{2.9a}$$

where

$$F(r) = \int_{B(0,r)} \kappa u^+ J_\alpha^2 + \rho^2 \left(u^+\right)^2 + \left|\nabla u^+\right|^2 \ . \tag{2.9b}$$

multiplying (**??**a) by $1/r^2$ and integrating between R_0 and R_1 yields

$$\frac{1}{2}\int_{S_2} \left(u^+\right)^2 \Big|_{R_0}^{R_1} \geq \int_{R_0}^{R} \frac{F(r)}{r^2} dr \tag{2.10}$$

Hence,

$$\int_{R_0}^{R_1} \frac{F(r)}{r^2} dr \leq -\frac{1}{2}\int_{S_2} \left[(h_{ex}-\kappa)^2 - \left(u^+\right)^2\right]\Big|_{r=R_0}^{R_1}$$

Yet,

$$\left|\int_{S_2} \left[(h_{ex}-\kappa)^2 - \left(u^+\right)^2\right]\right|_{r=R_i} \leq C\left\{\int_{S_2} \left|h_{ex}^2 - h^2\right| + \int_{S_2} |h_{ex}-h| + \int_{S_2} \rho^4\right\}_{r=R_i} \qquad i=0,1.$$

However,

$$\begin{aligned}\int_{S_2} \left|h_{ex}^2 - h^2\right|_{r=R_i} &\leq \left\{\int_{S_2} |h_{ex}+h|^2\right\}_{r=R_i}^{1/2} \left\{\int_{S_2} |h_{ex}-h|^2\right\}_{r=R_i}^{1/2} \leq \\ &\leq \left\{2\int_{S_2} |h_{ex}-h|^2 + h_{ex}^2\right\}_{r=R_i}^{1/2} \left\{\int_{S_2} |h_{ex}-h|^2\right\}_{r=R_i}^{1/2} \qquad i=0,1,\end{aligned}$$

and

$$\int_{S_2} |h_{ex}-h| \leq C\left\{\int_{S_2} |h_{ex}-h|^2\right\}^{1/2}.$$

Thus,

$$\int_{R_0}^{R_1} \frac{F(r)}{r^2} dr \leq C\sum_{i=0,1}\left[\left\{\int_{S_2} |h_{ex}-h|^2\right\}^{1/2} + \int_{S_2} |h_{ex}-h|^2 + \rho^4\right]_{R=R_i} . \tag{2.11}$$

In view of (??c) there exists a sequence $\{R_k\}_{k=0}^{\infty}$ such that $R_k \uparrow \infty$ and

$$R_k^2 \int_{S_2} |h_{ex} - h|^2 + \rho^4 \big|_{r=R_k} \to 0 \ .$$

Hence, by (??)

$$\int_{R_k}^{R_j} \frac{F(r)}{r^2} dr \le \frac{\epsilon_k}{R_k} + \frac{\epsilon_j}{R_j} \quad j > k \ \epsilon_k \to 0,$$

from which we conclude that

$$\int_{R_k}^{\infty} \frac{F(r)}{r^2} dr \le \frac{\epsilon_k}{R_k}. \tag{2.12}$$

Suppose, for a contradiction, that $\exists r_0 > 0$ such that $F(r_0) > 0$. Since $F(r)$ is monotone increasing we have

$$\int_{R_k}^{\infty} \frac{F(r)}{r^2} dr \ge \frac{F(r_0)}{R_k}$$

which contradicts (??), and hence $F(r) \equiv 0$. Consequently, we must have $u^+ = C$ and if $C > 0$ we must also have $\rho = 0$. By (??c) we must have, however, $u = h_{ex} - \kappa > 0$ from which our theorem can readily be proved for type II superconductors.

To prove that the same result holds whenever $\kappa < 1/\sqrt{2}$ and $h_{ex} > \frac{1}{2\kappa}$ we define

$$w_\alpha = h - \frac{1}{2\kappa}(1 - \rho^2)\hat{\alpha} \tag{2.13}$$

to obtain

$$\left(\nabla^2 - \rho^2\right) w_\alpha \cdot \hat{\alpha} = \kappa J_\alpha^2 + \left(\frac{1}{2\kappa} - \kappa\right)(\rho^2 - \rho^4),$$

and proceed in the same manner as in the previous case.

□

3 Surface superconductivity as $\kappa \to \infty$

In this section we consider the Ginzburg-Landau equations (**??**) in a bounded domain $\Omega \subset \mathbb{R}^3$ whose boundary is at least $C^{2,\alpha}$ for some $0 < \alpha < 1$. Let (ψ, A) satisfy (**??**) and the boundary conditions

$$\left(\frac{i}{\kappa}\nabla + A\right)\psi \cdot \hat{n} = 0 \quad ; \quad (h - h_{ex}) \times \hat{n} = 0 \; . \tag{??a,b}$$

We show here that as long as the applied magnetic field h_{ex} is sufficiently greater than κ, ψ decays exponentially fast away from the boundaries. Thus, we prove the following result

Theorem 3.1 *Let (ψ, A) satisfy (**??**) and (**??**). Let $\lambda = \sqrt{(h_{ex} - \kappa)\kappa}$. If $h_{ex} - \kappa \gg \kappa^{1/2}$ as $\kappa \to \infty$, then there exist $\beta, C > 0$, independent of κ and λ, such that*

$$\|\psi\|_{L^2[B(x,1/\lambda)]} \leq \frac{C}{\lambda^{3/2}} e^{-\beta\lambda d(x,\partial\Omega} \tag{3.2}$$

For every $x \in \Omega$ satisfying $d(x, \partial\Omega) > 1/\lambda$

We note that the above theorem is valid not only for the global minimizer of (**??**) but for any solution of (**??**) and (**??**). To prove the theorem we need first the following lemma.

Lemma 3.2 *Let (ψ, A) denote any solution of (**??**) and (**??**). Then if $h_{ex} \geq \kappa$*

$$\int_\Omega \rho^4 \leq \frac{C}{\kappa} \tag{3.3a}$$

$$\int_\Omega |h - h_{ex}|^2 \leq \frac{C}{\kappa} \tag{3.3b}$$

where C is independent of κ. Furthermore, let $\hat{A}$ denote the vector field satisfying

$$\nabla \times \hat{A} = h_{ex} \quad ; \quad \nabla \cdot \hat{A} = 0 \quad ; \quad \hat{A} \cdot \hat{n}\Big|_{\partial\Omega} = 0. \tag{3.4}$$

Then,

$$\left\| A - \hat{A} \right\|_{L^\infty(\Omega)} \le \frac{C}{\kappa^{1/2}}. \tag{3.5}$$

Proof: Multiplying (**??**a) by $\rho^2 \bar{\psi}$ and integrating over Ω we obtain, after some manipulation

$$\int_\Omega \rho^2 \left| \left(\frac{i}{\kappa} \nabla + A \right) \Psi \right|^2 + \frac{1}{\kappa^2} \int_\Omega \rho^2 |\nabla \rho|^2 = \int_\Omega \rho^4 (1 - \rho^2) \tag{3.6}$$

By (**??**b) we then have

$$\int_\Omega |\nabla \times h|^2 \le \int_\Omega \rho^2 \left| \left(\frac{i}{\kappa} \nabla + A \right) \Psi \right|^2 \le \int_\Omega \rho^4 \tag{3.7}$$

Note that $\nabla \cdot h = 0$ and that $(h - h_{ex}) \times \hat{n}|_{\partial\Omega} = 0$, and hence [**?**],

$$\int_\Omega |h - h_{ex}|^2 + \int_\Omega |\nabla h|^2 \le C \int_\Omega |\nabla \times h|^2 \tag{3.8}$$

where $C = C(\Omega)$ is independent of κ. Consequently,

$$\int_\Omega |\nabla h|^2 + \int_\Omega |h - h_{ex}|^2 \le C \int_\Omega \rho^4. \tag{3.9}$$

Let Ω_δ denote the following subdomain of Ω

$$\Omega_\delta = \{ x \in \Omega \mid d(x, \partial\Omega) \ge \delta \},$$

where $\delta = \delta(\kappa)$. For sufficiently small δ $\partial\Omega_\delta$ is as smooth as $\partial\Omega$ (at least $C^{2,\alpha}$ in our case). Integration of (**??**) over Ω_δ yields

$$\int_{\Omega_\delta} \rho^2 u + \left(\kappa - \frac{1}{2\kappa} \right) \int_{\Omega_\delta} \rho^4 \le \int_{\partial\Omega_\delta} \frac{\partial u}{\partial n},$$

where $u = u_\alpha$ and $\hat{\alpha} = \hat{i}_3$. Hence,

$$\int_{\Omega_\delta} \rho^2 (h_{ex} - \kappa) + \int_{\Omega_\delta} \rho^2 (h_3 - h_{ex}) + \kappa \int_{\Omega_\delta} \rho^4 \le \int_{\partial\Omega_\delta} \frac{\partial u}{\partial n}. \tag{3.10}$$

However, by (**??**)

$$\left|\int_{\Omega_\delta} \rho^2(h_3 - h_{ex})\right| \leq \frac{1}{2}\left[\int_\Omega \rho^4 + |h - h_{ex}|^2\right] \leq C\int_\Omega \rho^4.$$

Consequently, since $\|\rho\|_\infty < 1$ [**?**],

$$(\kappa - C)\int_{\Omega_\delta} \rho^4 \leq \int_{\partial\Omega_\delta} \frac{\partial u}{\partial n} + C\delta$$

From the foregoing discussion we may conclude that for sufficiently large κ

$$\int_{\Omega_\delta} \rho^4 \leq \frac{C}{\kappa}\int_{\partial\Omega_\delta} \frac{\partial u}{\partial n} + C\frac{\delta}{\kappa} \tag{3.11}$$

We now estimate the boundary integral on the right-hand-side of (**??**). Clearly,

$$\int_{\partial\Omega_\delta} \frac{\partial u}{\partial n} \leq \int_{\partial\Omega_\delta} |\nabla h| + \frac{1}{2\kappa}\int_{\partial\Omega_\delta} |\nabla\rho^2|$$

The second term on the right-hand-side can be bounded as follows

$$\frac{1}{2\kappa}\int_{\partial\Omega_\delta} |\nabla\rho^2| \leq C\left[\frac{1}{\kappa^2}\int_{\partial\Omega_\delta} \rho^2|\nabla\rho|^2\right]^{1/2} \leq C\left[\int_{\partial\Omega_\delta} \rho^2\left|\left(\frac{i}{\kappa}\nabla + A\right)\Psi\right|^2\right]^{1/2}.$$

Hence,

$$\int_{\partial\Omega_\delta} \frac{\partial u}{\partial n} \leq C\left[\int_{\partial\Omega_\delta} |\nabla h|^2 + \rho^2\left|\left(\frac{i}{\kappa}\nabla + A\right)\Psi\right|^2\right]^{1/2}$$

However, there exists $1/\kappa < \delta < 2/\kappa$ such that

$$\int_{\partial\Omega_\delta} |\nabla h|^2 + \rho^2\left|\left(\frac{i}{\kappa}\nabla + A\right)\Psi\right|^2 \leq C\kappa\int_\Omega |\nabla h|^2 + \rho^2\left|\left(\frac{i}{\kappa}\nabla + A\right)\Psi\right|^2$$

Utilizing (**??**) and (**??**) we obtain

$$\int_\Omega \rho^4 \leq \frac{C}{\kappa^{1/2}}\left[\int_\Omega \rho^4\right]^{1/2} + \frac{C}{\kappa}$$

yielding

$$\int_\Omega \rho^4 \leq \frac{C}{\kappa},$$

and, by (??),

$$\int_\Omega |h - h_{ex}|^2 \leq \frac{C}{\kappa}.$$

To prove (??) we need (??) and theorem 10.5 in [?]. The boundary value problem satisfied by $A - \hat{A}$ is

$$\nabla^2(A - \hat{A}) = -\nabla \times h \qquad \text{in } \Omega \tag{3.12a}$$

$$(A - \hat{A}) \cdot \hat{n} = 0 \qquad \text{on } \partial\Omega \tag{3.12b}$$

$$\nabla \times (A - \hat{A}) \times \hat{n} = 0 \qquad \text{on } \partial\Omega \tag{3.12c}$$

Since (??) fits into the rather general framework in [?] we may conclude that

$$\|A - \hat{A}\|^2_{H^2(\Omega)} \leq C \left[\int_\Omega |\nabla \times h|^2 + \int_\Omega |A - \hat{A}|^2 \right]. \tag{3.13}$$

However, in view of (??a) and (??b) we have (cf. [?])

$$\int_\Omega |A - \hat{A}|^2 \leq C \int_\Omega |h - h_{ex}|^2 \tag{3.14}$$

where $C = C(\Omega)$ is independent of κ. Combining (??), (??), (??), and (??), we obtain

$$\|A - \hat{A}\|^2_{H^2(\Omega)} \leq \frac{C}{\kappa}$$

Sobolev embedding completes the proof of the lemma.

□

We note that the condition $h_{ex} \geq \kappa$ is crucial only in deriving (??) from (??). If $h_{ex} < \kappa$ we can still prove the following result

Theorem 3.3 *Let (ψ, A) denote any solution of (**??**) and (**??**). Let further $b = h_{ex}/\kappa$. Then, if $b(\kappa) \leq 1$ we have*

$$\int_\Omega \rho^4 \leq \frac{C}{\kappa} + |\Omega|(1-b)^2 \tag{3.15a}$$

$$\int_\Omega |h - h_{ex}|^2 \leq C\left[\frac{1}{\kappa} + (1-b)^2\right] \tag{3.15b}$$

Furthermore, if (ψ, A) is the global minimizer, then

$$-\frac{1}{2}|\Omega|(1-b)^2 - \frac{C}{\kappa} \leq E(\psi, A) \leq 0 \tag{3.16}$$

Proof: From (**??**) we easily obtain

$$\int_\Omega \rho^4 \leq \frac{C}{\kappa}\int_{\partial\Omega_\delta} \frac{\partial u}{\partial n} + C\delta + (1-b)\int_\Omega \rho^2. \tag{3.17}$$

Following the same steps as in the proof of lemma **??** below (**??**), we obtain that for some $1/\kappa < \delta < 2/\kappa$ we have

$$\int_{\partial\Omega_\delta} \frac{\partial u}{\partial n} \leq \frac{C}{\kappa^{1/2}}\left[\int_\Omega \rho^4\right]^{1/2}.$$

Combining the above with (**??**) yields

$$\int_\Omega \rho^4 \leq \frac{C}{\kappa^{1/2}}\left[\int_\Omega \rho^4\right]^{1/2} + \frac{C}{\kappa} + (1-b)|\Omega|^{1/2}\left[\int_\Omega \rho^4\right]^{1/2},$$

from which (**??**a) is easily obtained. To obtain (**??**b) we use (**??**). Finally, since (ψ, A) is a solution of (**??**) and (**??**) we have

$$E(\psi, A) = \int_\Omega |h - h_{ex}|^2 - \frac{1}{2}\int_\Omega \rho^4.$$

The lower bound in (**??**) follows now from (**??**a) and the upper bound from the fact that $\rho \equiv 0$ and $h = h_{ex}$ is a solution of (**??**) and (**??**).

□

Stronger results were obtained in [?] for domains in $\mathbb{R}^2$. We compare theorem **??** to the results in [?] in the last section.

Proof of theorem **??***:* We prove the theorem by invoking blow-up arguments. Let

$$\lambda = \sqrt{k(h_{ex} - \kappa}.$$

We first prove (**??**). Let

$$\Omega(\lambda, k, s) = \left\{ x \in \Omega |\; d(x, \partial\Omega) \geq k\frac{s}{\lambda} \right\}.$$

We prove (**??**) by showing that

$$\exists \lambda_0, s_0 : \sup_{x \in \Omega(\lambda,k,s)} \|\psi(\kappa, \lambda)\|_{L^2[B(x,1/\lambda)]} \leq \frac{1}{2} \sup_{x \in \Omega(\lambda,k+1,s)} \|\psi(\kappa, \lambda)\|_{L^2[B(x,1/\lambda)]} \; \forall s > s_0 \; \lambda > \lambda_0, \; k \in \mathbb{N} \tag{3.18}$$

Suppose, for a contradiction, that (**??**) does not hold. Then, sequences $\{\lambda_j\}_{j=1}^{\infty}$,$\{\kappa_j\}_{j=1}^{\infty}$, $\{s_j\}_{j=1}^{\infty}$, and $\{k_j\}_{j=1}^{\infty}$ exist such that $\lambda_j \uparrow \infty$, $\kappa_j \uparrow \infty$, $s_j \uparrow \infty$, $k_j \in \mathbb{N}$, and

$$\sup_{x \in \Omega(\lambda_j,k_j+1,s_j)} \|\psi(\kappa_j, \lambda_j)\|_{L^2[B(x,1/\lambda_j)]} \geq \frac{1}{2} \sup_{x \in \Omega(\lambda_j,k_j,s_j)} \|\psi(\kappa_j, \lambda_j)\|_{L^2[B(x,1/\lambda_j)]} \overset{def}{=} \frac{1}{2} m_j \tag{3.19}$$

Let

$$\tilde{\psi}_j \overset{def}{=} \frac{\psi(\kappa_j, \lambda_j)}{m_j}.$$

By (**??**) there exists $x_j \in \Omega(\lambda_j, k_j + 1, s_j)$ such that $\|\tilde{\psi}_j\|_{L^2[B(x_j,1/\lambda_j)]} \geq \frac{1}{2}$. Furthermore, since $B(x_j, 1/\lambda_j) \in \Omega(\lambda_j, k_j, s_j)$ we have

$$\frac{1}{2} \leq \left\| \tilde{\psi}_j \right\|_{L^2[B(x_j 1/\lambda_j)]} \leq 1.$$

Define

$$f_j = \tilde{\psi}_j \left(x_j + \frac{x}{\lambda_j} \right).$$

In view of the above we have

$$\frac{1}{4}\lambda_j^3 \le \int_{B(0,1)} |f_j|^2 \le \lambda_j^3. \tag{3.20}$$

It is easy to show that

$$\left(\frac{i\lambda_j}{\kappa_j} \nabla + \tilde{A}_j \right)^2 f_j = f_j \left(1 - m_j^2 \, |f_j|^2 \right) \quad x \in B(0, s_j) \tag{3.21a}$$

wherein

$$A_j = A_j(x_j + x/\lambda_j). \tag{3.21b}$$

we now define a cut-off function

$$\eta_r = \begin{cases} 1 & \text{in } B(0,r) \\ 0 & \text{in } \mathbb{R}^3 / B(0,2r) \end{cases}$$

$$|\nabla \eta_r| \le C \quad \text{in } \mathbb{R}^3$$

Multiplying (**??**a) by η_r^2, and integrating over $B(0,2r)$ we obtain, for all $r \le \frac{s_j}{2}$ (cf. [**?**]), that

$$\int_{B(0,2r)} \left| \left(\frac{i\lambda_j}{\kappa_j} \nabla + A_j \right) (\eta_r f_j) \right|^2 = \int_{B(0,2r)} \eta_r^2 |f_j|^2 \left(1 - m_j^2 \, |f_j|^2 \right) + \frac{\lambda_j^2}{\kappa_j^2} \, |\nabla \eta_r|^2 \, |f_j|^2 \tag{3.22}$$

Let $\hat{A}_j = \hat{A}(x_j + x/\lambda_j)$, where $\hat{A}$ is defined in (**??**) . Then,

$$\int_{B(0,2r)} \left| \left(\frac{i\lambda_j}{\kappa_j} \nabla + A_j \right) (\eta_r f_j) \right|^2 = \int_{B(0,2r)} \left| \left(\frac{i\lambda_j}{\kappa_j} \nabla + \hat{A}_j \right) (\eta_r f_j) \right|^2 +$$

$$+ \int_{B(0,2r)} \frac{\lambda_j}{\kappa_j} (A_j - \hat{A}_j) \eta_r^2 \left[i \left(\bar{f}_j \nabla f_j - f_j \nabla \bar{f}_j \right) + 2 \frac{\kappa_j}{\lambda_j} |f_j|^2 \hat{A}_j \right] +$$

$$+ \int_{B(0,2r)} \left| A_j - \hat{A} \right|^2 \eta_r^2 |f_j|^2 .$$

Clearly,

$$\eta_r^2 \left[i \frac{\lambda_j}{\kappa_j} \left(\bar{f}_j \nabla f_j - f_j \nabla \bar{f}_j \right) + 2 |f_j|^2 \hat{A}_j \right] = 2\Re \left\{ \eta_r \bar{f}_j \left(\frac{i\lambda_j}{\kappa_j} \nabla + \hat{A}_j \right) (\eta_r f_j) \right\} ,$$

and hence

$$\int_{B(0,2r)} \left| \left(\frac{i\lambda_j}{\kappa_j} \nabla + A_j \right) (\eta_r f_j) \right|^2 \geq$$

$$\geq \left\{ \left[\int_{B(0,2r)} \left| \left(\frac{i\lambda_j}{\kappa_j} \nabla + \hat{A}_j \right) (\eta_r f_j) \right|^2 \right]^{1/2} - \left[\int_{B(0,2r)} \left| A_j - \hat{A} \right|^2 \eta_r^2 |f_j|^2 \right]^{1/2} \right\}^2 .$$

By (**??**) we have

$$\int_{B(0,2r)} \left| \left(\frac{i\lambda_j}{\kappa_j} \nabla + A_j \right) (\eta_r f_j) \right|^2 \geq$$

$$\geq \left\{ \left[\int_{B(0,2r)} \left| \left(\frac{i\lambda_j}{\kappa_j} \nabla + \hat{A}_j \right) (\eta_r f_j) \right|^2 \right]^{1/2} - \frac{C}{\kappa_j^{1/2}} \left[\int_{B(0,2r)} \eta_r^2 |f_j|^2 \right]^{1/2} \right\}^2 \tag{3.23}$$

However, in [**?**] it was shown that

$$\int_{\mathbb{R}^3} \left| \left(\frac{i\lambda_j}{\kappa_j} \nabla + \hat{A}_j \right) (\eta_r f_j) \right|^2 \geq \left(1 + \frac{\lambda_j^2}{\kappa_j^2} \right) \int_{\mathbb{R}^3} \eta_r^2 |f_j|^2$$

Combining the above with (**??**) and (**??**) we obtain

$$\left[\left(1 + \frac{\lambda_j^2}{\kappa_j^2} \right)^{1/2} - \frac{C}{\kappa_j^{1/2}} \right]^2 \int_{B(0,2r)} \eta_r^2 |f_j|^2 \leq \int_{B(0,2r)} \eta_r^2 |f_j|^2 + \frac{\lambda_j^2}{\kappa_j^2} \int_{B(0,2r)} |\nabla \eta_r|^2 f_j^2 .$$

Hence,

$$\left[1 - \frac{C}{\kappa_j^{1/2}} \frac{\kappa_j^2}{\lambda_j^2}\right] \int_{B(0,2r)} \eta_r^2 |f_j|^2 \leq \int_{B(0,2r)} |\nabla \eta_r|^2 f_j^2.$$

Thus, as long as $\lambda_j^2 \gg \kappa_j^{3/2}$ (or $h_{ex} - \kappa \gg \kappa^{1/2}$) we have, for sufficiently large j

$$\int_{B(0,2r)} \eta_r^2 |f_j|^2 \leq 2 \int_{B(0,2r)} |\nabla \eta_r|^2 f_j^2.$$

Consequently,

$$\int_{B(0,r)} |f_j|^2 \leq \frac{C}{r^2} \int_{B(0,2r)} |f_j|^2 \leq \frac{C}{r^4} \int_{B(0,4r)} |f_j|^2.$$

By (**??**) we have, however,

$$\int_{B(0,4r)} |f_j|^2 \leq C(r\lambda_j)^3,$$

and hence,

$$\frac{1}{4}\lambda_j^3 \leq \int_{B(0,r)} |f_j|^2 \leq \frac{C}{r}\lambda_j^3,$$

a contradiction. Hence, ψ must satisfy (**??**).

□

4 Concluding remarks

In the following we briefly summarize the main results of this work and address some key points insufficiently emphasized within the previous section:

1. In § 2 we proved that no solution which satisfies (**??**c) can exist in $\mathbb{R}^3$ as long as $h_{ex} > \kappa$. We note that in [**?**] it was shown that no *bounded* solution can exist in $\mathbb{R}^2$ for $h_{ex} > \kappa$ (as long as $h \to h_{ex}$ as $|x| \to \infty$. It is not clear whether (**??**c) is

really necessary so that theorem **??** would hold. Nevertheless, the necessity of this decay condition in the proof of the theorem results from the additional dimension: in (**??**) we integrate $F(r)/r^2$ instead of $F(r)/r$. The fact that such an additional decay condition is necessary in $\mathbb{R}^3$ and not in $\mathbb{R}^2$ is not unusual (cf. [?] for instance).

2. In § 3 we proved exponentially fast decay of ψ away from the boundary whenever $h_{ex} - \kappa \gg \kappa^{1/2}$. This result is much weaker than the result in [?] where exponential rate of decay is proved, in two dimensions, whenever $h_{ex} - \kappa \gg 1/\kappa$. This relative weakness of the present results stems from the difficulty to provide good estimates for $|\nabla h|$ in three dimensions: while in two-dimensional settings $|\nabla \times h| = |\nabla h|$, we need more complicated relationships, like (**??**), between $|\nabla \times h|$ and $|\nabla h|$ in three dimensions. We note that (**??**) is valid only because of (**??**b), and hence, we cannot apply it to subdomains of Ω. For the same reason it appears difficult to prove exponentially fast convergence, away from the boundaries, of h to a constant, in contrast to the situation in two dimensions. The large domain limit, which has been addressed in [?] in two dimensions appears to be also difficult for the same reason.

3. We note that while the decay of ψ is proved in theorem **??** only in L^2 sense, it is possible to show, in the same manner shown in [?] that

$$|\psi| \leq C \left(\frac{\kappa}{\lambda}\right)^{3/2} e^{-\beta\lambda d(x,\partial\Omega)}$$

It is also possible to obtain similar estimates for the derivatives of ψ.

4. In theorem **??** we prove that $\|\psi\|_{L^4(\Omega)}$ decays as both $h_{ex} \to \kappa$ and $\kappa \to \infty$. Much

stronger results were obtained in [?] for two-dimensional settings:

(a) The upper bound of $E(\psi, A)$ in $H^1(\Omega) \times H^1(\Omega)$ is better than the rather trivial upper bound in (??).

(b) In [?] it is proved that

$$\frac{1}{|B_r|} \int_{B_r} \rho^4 \to 0$$

as both $h_{ex} \to \kappa$ and $\kappa \to \infty$, for any ball $B_r \subset \Omega$, whereas here the above result is proved only for the whole domain Ω.

It seems possible to obtain better estimates than (??) using the same techniques as in [?]. In contrast, it appears more difficult to obtain (??) for any $B_r \subset \Omega$ since the estimates of $|\nabla h|$ and $|\nabla \rho|$ obtained in [?] are not available in three dimensions yet.

Acknowledgements

This research was supported by the fund for the promotion of research at the Technion.

References

[1] A. Aftalion and W. C. Troy, *On the solution of the one-dimensional ginzburg-landau equations of superconductivity*, Phys. D, 132 (1999), pp. 214–232.

[2] Y. Almog, *Existence and non-existence of solutions to the ginzburg-landau equations in a semi-infinite superconducting film.* Preprint available at *http://www.math.technion.ac.il/∼almog/ipub.html.*

[3] C. Bolley and B. Helffer, *Rigorous results on ginzburg-landau models in a film submitted to exterior parallel magnetic field i*, Nonlinear Studies, 3 (1996), pp. 1–29.

[4] M. H. Protter and H. F. Weinberger, *Maximum principles in differential equations*, Prentice-Hall, 1967.