The onset of superconductivity in semi-infinite strips

Y. ALMOG *

Abstract

The existence of a superconducting branch bifurcating from the normal state is proved in semi-infinite strips. It is proved that the critical magnetic field at which bifurcation takes place, or the onset field, for a semi-infinite strip, is greater than the onset field for an infinite strip with the same width. In addition we find the loci of the vortices far away from the corners and show convergence of the bifurcating modes in long rectangles to those in the semi-infinite strip with the same width.

I Introduction

If a superconducting body is placed in a sufficiently strong magnetic field, the body will revert to the normal state. This result which is known from experiments [1], was recently proved for bounded domains in $\mathbb{R}^2$ and $\mathbb{R}^3$ [2]. If the field is now decreased, there will be a point, depending on the sample's geometry, at which the material would become superconducting once again.

*Faculty of Mathematics, Technion - Israel Institute of Technology, Haifa 32000, Israel

Apart from infinite domains with no boundaries, the simplest case in which the bifurcation from the normal state to the superconducting one was calculated is the case of a half-plane [3]. Even in this simple case the onset field is substantially larger than the onset field on the real line [4]. Furthermore, it was found by Saint-James and de-Gennes [3] that superconductivity is concentrated in this case near the boundary. This phenomenon, which appears in the presence of boundaries have been termed, therefore, surface superconductivity. As surface superconductivity appears at much higher magnetic fields than those at which superconductivity appears in the bulk of the material, the bifurcation point is often called the onset of superconductivity. We shall refer to the magnetic field at which the bifurcation takes place as the onset field, and to the bifurcating mode as the onset mode.

The significance of Saint-James and de Gennes' solution [3] extends far beyond the simple, one-dimensional example of a half-plane. It was proved, first for films [5], then for discs [6], and finally for general two-dimensional domains with smooth boundaries [7, 8], that as the domain's scale tends to infinity the onset field tends to de-Gennes' value. If the boundaries include wedges the onset will be larger than de-Gennes' value [9, 10, 11].

Most of the above works obtain the onset field and the behavior of the superconducting order parameter near the boundaries. Very few works obtain, however, the structure of the vortices deep inside the domain's interior. Two exceptions are the works of Bauman *et al* [6] who found the solution inside a circular disk, and the works of Kulik [12] and Boeck & Chapman [13] who found the structure of vortices in an infinite strip.

In the present contribution we focus on semi-infinite strips and long rectangles. We leave to future research the numerical calculation of the onset field, which is expected to tend, in the wide semi-strip limit, to the onset field for a rectangular wedge. Instead, we shall be interested in the superconducting order parameter far away from the corners: its exponential rate of decay, and the structure of the vortices. We also prove the existence of the onset field and onset modes for a semi-infinite strip, and show that the onset field is greater for a semi-infinite strip than for an infinite strip with the same width.

The Ginzburg-Landau energy functional may be represented in the following dimensionless form [14]

$$E = \int \left(-|\psi|^2 + \frac{|\psi|^4}{2} + |\boldsymbol{H}|^2 + \left| \frac{1}{\kappa} \boldsymbol{\nabla} \psi - i\boldsymbol{A}\psi \right|^2 \right) dxdy \tag{1.1}$$

in which Ψ is the (complex) superconducting order parameter, such that $|\Psi|$ varies from $|\Psi| = 0$ (when the material is at a normal state) to $|\Psi| = 1$ (for the purely superconducting state). The magnetic vector potential is denoted by $\boldsymbol{A}$ (the magnetic field is, then, given by $\boldsymbol{H} = \boldsymbol{\nabla} \times \boldsymbol{A}$), and κ is the Ginzburg-Landau parameter which is a material property. Superconductors for which $\kappa < 1/\sqrt{2}$ are termed type I superconductors, and those for which $\kappa > 1/\sqrt{2}$ have been termed type II. Note that E is invariant to the gauge transformation

$$\Psi \to e^{i\kappa\theta}\psi \qquad ; \qquad \boldsymbol{A} \to \boldsymbol{A} + \boldsymbol{\nabla}\theta \tag{1.2}$$

We look for local minimizers of E in the semi-infinite strip $\mathcal{S} = \left\{(x,y) \in \mathbb{R}^2 | -l \leq x \leq l \;;\; 0 \leq y\right\}$, in the the case where the applied magnetic field is constant and perpendicular to the plane. The Euler-Lagrange equations associated with E (the steady state Ginzburg-Landau equations) are given by

$$\left(\frac{i}{\kappa}\boldsymbol{\nabla} + \boldsymbol{A}\right)^2 \Psi = \Psi\left(1 - |\Psi|^2\right) \quad , \tag{1.3a}$$

$$-\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) = \frac{i}{2\kappa}\left(\Psi^* \boldsymbol{\nabla}\Psi - \Psi\boldsymbol{\nabla}\Psi^*\right) + |\Psi|^2 \boldsymbol{A} \quad , \tag{1.3b}$$

The natural boundary conditions satisfied on $\partial\mathcal{S}$ for this problem are

$$\left(\frac{i}{\kappa}\boldsymbol{\nabla} + \boldsymbol{A}\right)\psi \cdot \hat{\boldsymbol{n}} = 0 \tag{1.4a}$$

$$\boldsymbol{H} = h\hat{\boldsymbol{z}} \tag{1.4b}$$

As the Ginzburg-Landau equations are gauge-invariant, we may choose the gauge (following [15, 16]) $\boldsymbol{A} = (0, A(x,y), 0)$. Thus, $\boldsymbol{H} = (0, 0, H(x,y))$ and $H = \partial A/\partial x$. We then linearize (1.3) near the normal state $\Psi \equiv 0$, $A = hx$, to which end we assume the asymptotic expansion

$$\Psi = \epsilon^{1/2}\psi \tag{1.5a}$$

$$A = hx + \epsilon a \tag{1.5b}$$

$$h = h^{(0)} + \epsilon h^{(1)} + \dots \tag{1.5c}$$

$$a = a^{(0)} + \epsilon a^{(1)} + \dots \tag{1.5d}$$

$$\psi = \psi^{(0)} + \epsilon\psi^{(1)} + \dots \tag{1.5e}$$

Applying the transformations

$$x^1 = \kappa x \quad ; \quad y^1 = \kappa y \quad ; \quad h^1 = \frac{h^{(0)}}{\kappa}$$

the linearized form of (1.3a) becomes (we omit the superscripts 1 and (0) in the following)

$$-\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)+2ihx\frac{\partial\psi}{\partial y}=\psi-h^2x^2\psi \tag{1.6}$$

The boundary condition (1.4a) for the specific gauge we have chosen becomes after linearization

$$\psi_x(\pm l,y)=0 \qquad ; \qquad i\psi_y(x,0)+hx\psi(x,0)=0\ . \tag{1.7a,b}$$

As some of the results we obtain in the next two sections, are based on the linear bifurcation analysis of the Ginzburg-Landau equations in the infinite strip $[-l,l]\times\mathbb{R}$, we briefly review some of the results in [5]. In this case the conditions which should be satisfied on the boundary of the film (strip) are given by (1.7a). Some of the solutions of (1.6) together with (1.7a) are expressible in the form

$$\psi(x,y)=F(x)e^{-i\omega y} \quad , \tag{1.8}$$

wherein F satisfies

$$F''-\left[(hx-\omega)^2-1\right]F=0 \quad ; \quad F'(\pm l)=0 \tag{1.9a,b}$$

The general solution of (1.9a) is

$$F(x)=C_1U(\xi)+C_2U(-\xi) \tag{1.10}$$

where

$$\xi(x,\omega)=\sqrt{\frac{2}{h}}(hx-\omega)\ , \tag{1.11}$$

$U(x)\equiv U(a,x)$ is a parabolic cylinder function (or Whitaker's function), and $a=-1/2h$.

Non trivial solutions to (1.9) exist *iff*

$$U'\big(\xi(-l,\omega)\big)U'\big(-\xi(l,\omega)\big) - U'\big(\xi(l,\omega)\big)U'\big(-\xi(-l,\omega)\big) = 0 \tag{1.12}$$

for some real value of ω. The above equation, thus, implicitly defines a function $h = h(l,\omega)$ (cf. [5]). Furthermore, let $h_{1D}(l)$ be the infimum of the set of h values in $\mathbb{R}^+$ s.t. the normal solution ($\psi \equiv 0$, $H = h$) is a local minimizer of E. Then,

$$h_{1D}(l) = \sup_{\omega\in\mathbb{R}} h(l,\omega)\ . \tag{1.13}$$

In the next section we derive the asymptotic behaviour of solutions of (1.6) and (1.7) for $y \gg 1$. We first prove the following result:

Theorem 1. *Any solution of* (1.6) *and* (1.7) *for which* $h > h_{1D}$ *satisfies* $\forall\epsilon > 0$

$$\lim_{y\to\infty} |\psi(x,y)|e^{(\Im\omega_0-\epsilon)y} = 0$$

where ω_0 *is the root of* (1.12) *with the smallest positive imaginary part (*$\Im\omega_0 > 0$*).*

Note that the roots of (1.12) depend on the value of h_S the onset field of (1.6) and (1.7). Once h_S is found, the exponential rate of decay of ψ can be easily deduced from (1.12).

In the second part of § 2 we find, after making two mild assumptions on the roots of (1.12), the exact asymptotic form of ψ for $y \gg 1$. In § 3 we prove compactness of the set of solutions of (1.7) in the rectangles $R_L = [-l,l]\times[0,L]$ for $L \geq L_0 > 0$. The compactness proof as well as the proof of Theorem 1 assume the inequality $h_S > h_{1D}$. In § 4 we prove this inequality by using a variational approach, concluding the proof of the following theorem

Theorem 2. *The onset field of* (1.6) *and* (1.7) $h_{\mathcal{S}}$ *exists and is greater than the one-dimensional onset field* h_{1D}.

From the results in § 3 we may then deduce the existence of at least one onset mode in $\mathcal{S}$.

We conclude § 4 by proving that the onset field in R_L, h_L tends to $h_{\mathcal{S}}$ as $L \to \infty$. Under the assumptions made in § 2 on the roots of (1.12) we also show that h_L is strictly greater than $h_{\mathcal{S}}$.

II Asymptotic behaviour of solutions

In the following we obtain the leading order behaviour of solutions of (1.6) in the semi-infinite strip $\mathcal{S}$ satisfying (1.7) on its boundary $\partial\mathcal{S}$. To this end we first take the Fourier transform of (1.6) in the y direction, i.e., we multiply it by $\exp\{-i\omega y\}$ and integrate between 0 and ∞. Integration by parts then yields

$$-\hat{\psi}'' + \left[(hx-\omega)^2 - 1\right]\hat{\psi} = i(hx-\omega)\psi(x,0) \tag{2.1a}$$

$$\hat{\psi}'(\pm l) = 0 \; , \tag{2.1b}$$

where

$$\hat{\psi}(x,\omega) = \int_0^\infty e^{-i\omega y}\psi(x,y)dy \; . \tag{2.2}$$

The solution of (2.1a) can be written in the form

$$\hat{\psi}(x,\omega) = i\int_{-l}^{l}(hs-\omega)G(x,s,\omega)\psi(s,0)ds \; . \tag{2.3}$$

In which Green's function G is given by

$$G(x,s,\omega)=m(h)\begin{cases}\dfrac{[U(\eta)U'(-\xi(l))+U(-\eta)U'(\xi(l))]\,[U'(-\xi(-l))U(\xi)+U'(\xi(-l))U(-\xi)]}{U'(\xi(-l))U'(-\xi(l))-U'(\xi(l))U'(-\xi(-l))} & x<s\\ \dfrac{[U(\eta)U'(-\xi(-l))+U(-\eta)U'(\xi(-l))]\,[U'(-\xi(l))U(\xi)+U'(\xi(l))U(-\xi)]}{U'(\xi(-l))U'(-\xi(l))-U'(\xi(l))U'(-\xi(-l))} & x>s\end{cases} \tag{2.4}$$

wherein

$$m(h)=\frac{\Gamma(1/2+a)}{2\sqrt{\pi h}}\,, \tag{2.5}$$

$\xi=\xi(x,\omega)$ is given by (1.11), and $\eta=\xi(s,\omega)$. We note that (2.3) is valid only when G exists for all ω, i.e., when the denominator in (2.4) does not vanish. The largest value of h at which the denominator vanishes, or at which (1.12) is satisfied, for some $\omega=\omega_0$ is exactly the onset field of the one-dimensional problem, or h_{1D}. Thus, the analysis in the rest of this section is valid only for h values which are greater than h_{1D}. In § 4 we prove that the onset field for the semi-infinite strip $\mathcal{S}$, $h_{\mathcal{S}}$, is indeed greater than h_{1D}.

In view of (2.2) the inverse Fourier transform of $\hat{\psi}$ is

$$\frac{1}{2\pi}\text{P.V.}\int_{-\infty}^{\infty}e^{i\omega y}\hat{\psi}(x,\omega)d\omega=\begin{cases}\psi(x,y) & y>0\\ \frac{1}{2}\psi(x,0) & y=0\\ 0 & y<0\end{cases}. \tag{2.6}$$

As stated earlier we seek an approximation of $\psi(x,y)$ for $y\gg 1$. We obtain such an approximation by using complex plane methods when evaluating the Fourier integral in (2.6). Figure 1 displays the path, denoted in the following by $\mathcal{C}$, along which the integration is carried in the complex plane. We now calculate the integral

$$\int e^{i\omega y}\hat{\psi}(x,\omega)d\omega \tag{2.7}$$

along the various parts of $\mathcal{C}$. We note that that the Fourier transform of ψ, $\hat{\psi}(x,\omega)$, need not exist for all $\Im\omega > 0$. Yet, its analytic continuation, defined by (2.3), does exist in $\mathcal{C}$, except perhaps for a finite number of poles, where the denominator in (2.4) vanishes. For real values of ω, for which (2.2) is valid, it can be shown, via integration by parts that

$$\hat{\psi}(x,\omega) \sim i\frac{\psi(x,0)}{\omega} + O\left(\frac{1}{\omega^2}\right) . \tag{2.8}$$

It is easy to show that (2.8) remains valid on $\mathcal{C}_+$ and $\mathcal{C}_-$ as long as $\hat{\psi}(x,\omega)$ is analytic in the strip $0 \leq \Im\omega \leq \beta$. Furthermore, (2.8) remains valid even if $\hat{\psi}(x,\omega)$ possesses a finite number of poles in $\mathcal{C}$. To prove it we just need to subtract the singular terms (which are of the form $C_k(\omega-\omega_k)^{-n_k}$). The remainder would then be analytic in $\mathcal{C}$, and one can substitute its inverse Fourier transform into (2.2), and obtain (2.8) once again via integration by parts.

By (2.8), the integrals along the segments $\mathcal{C}_+$ and $\mathcal{C}_-$ decay in the limit $R \to \infty$. The integral on the interval $[-R,R]$ tends, by definition, to (2.6). To estimate the integral along $\mathcal{C}_\beta$ we utilize the asymptotic approximations (cf. for instance [17])

$$U(a,z) = \begin{cases} e^{-z^2/4} z^{-a-1/2}\left[1+\frac{C_1(z)}{|z|^2}\right] & 0 \leq \arg z < \frac{3\pi}{4} \\ \frac{\sqrt{2\pi}}{\Gamma(1/2+a)} e^{z^2/4}(-z)^{a-1/2}\left[1+\frac{C_3(z)}{|z|^2}\right] & \frac{3\pi}{4} < \arg z \leq \pi \end{cases} \tag{2.9a}$$

$$U'(a,z) = \begin{cases} -\frac{1}{2} e^{-z^2/4} z^{-a+1/2}\left[1+\frac{C_2(z)}{|z|^2}\right] & 0 \leq \arg z < \frac{3\pi}{4} \\ -\frac{\sqrt{\pi}}{2\Gamma(1/2+a)} e^{z^2/4}(-z)^{a-1/2}\left[1+\frac{C_4(z)}{|z|^2}\right] & \frac{3\pi}{4} < \arg z \leq \pi \ , \end{cases} \tag{2.9b}$$

in which $C_i(z)$, $(i=1,2)$ and their derivatives are bounded in any sector $0 \leq \arg z < 3\pi/4-\epsilon$ for positive ϵ, and similarly, $C_i(z)$, $(i=3,4)$ are bounded in $3\pi/4-\epsilon < \arg z \leq$

π. Similar expansions are satisfied in the lower half plane. For $\frac{3\pi}{4} - \epsilon < \arg z < \frac{3\pi}{4} - \epsilon$ we have

$$U(a,z) \sim \left[e^{-z^2/4} z^{-a-1/2} + \frac{\sqrt{2\pi}}{\Gamma(1/2+a)} e^{z^2/4} (-z)^{a-1/2} \right] \left[1 + O\left(\frac{1}{|z|^2} \right) \right] \tag{2.10a}$$

$$U'(a,z) \sim \left[-\frac{1}{2} e^{-z^2/4} z^{-a+1/2} - \frac{\sqrt{\pi}}{2\Gamma(1/2+a)} e^{z^2/4} (-z)^{a-1/2} \right] \left[1 + O\left(\frac{1}{|z|^2} \right) \right] \tag{2.10b}$$

From the above approximations it can easily be deduced that

$$G(x,s,R+i\beta) \sim G(x,s,R) \left[1 + O\left(\frac{\beta}{R} \right) \right] ,$$

and hence, in view of (2.3) that

$$\lim_{R\to\infty} \hat{\psi}(x,R+i\beta) \sim \hat{\psi}(x,R) \left[1 + O\left(\frac{\beta}{R} \right) \right] . \tag{2.11}$$

(Note that the above approximation can be obtained directly from (2.8) as well.) Consequently,

$$\left| \int_{\mathcal{C}_\beta} e^{i\omega y} \hat{\psi}(x,\omega) d\omega \right| < C(\beta) e^{-\beta y} \tag{2.12}$$

We conclude that if (1.12) is not satisfied inside $\mathcal{C}$, then

$$|\psi(x,y)| < C(\beta) e^{-\beta y} .$$

For sufficiently large β, however, (1.12) is satisfied inside the contour $\mathcal{C}$ on a finite set of points. We demonstrate this fact by calculating the phase change of the left-hand-side of (1.12) along asymptotically large circles centered on the origin. Utilizing (2.9) and the identity

$$U(a,x) = \frac{\Gamma(\frac{1}{2}-a)}{\sqrt{2\pi}} \left[e^{-i\pi(a/2+1/4)} U(-a,ix) + e^{i\pi(a/2+1/4)} U(-a,-ix) \right]$$

we find

$$\log\left\{U'(\xi(-l))U'(-\xi(l))-U'(\xi(l))U'(-\xi(-l))\right\}\sim\begin{cases}2l\omega & |\arg\omega|<\pi/2\\ -2l\omega & \pi/2<|\arg\omega|<\pi\ .\end{cases} \tag{2.13}$$

For $\pi/2-\epsilon<|\arg\omega|<\pi/2+\epsilon$ the derivative of the left-hand-side is bounded as $R\to\infty$ except perhaps for a countable set of points where (1.12) is satisfied. Consequently,

$$\Im\left[\int_{|\omega|=R}\frac{d}{d\omega}\log\left\{U'(\xi(-l))U'(-\xi(l))-U'(\xi(l))U'(-\xi(-l))\right\}d\omega\right]=(8l-C\epsilon)R\ ,$$

where C is independent of R. As ϵ can be set to be arbitrarily small, the phase change of the left-hand-side in (1.12) along the circles $|\omega|=R$ tends to ∞ as $R\to\infty$. Therefore, by the argument principle (1.12) is satisfied countably many times in $\mathbb{C}$

Suppose, then, that (1.12) is satisfied on the set $\left\{\omega_k^{\pm}=\pm\alpha_k+i\gamma_k\right\}_{k=1}^N$ $(\alpha_k\geq 0)$. We note that if (1.12) is satisfied at $\omega=\alpha+i\gamma$, it must be satisfied at $\omega=-\alpha+i\gamma$ as well, since the equation is invariant to the transformation $\omega\to-\omega^*$. Yet, α may be equal to zero. Such indeed is the situation in the one dimensional case, for which it was proved [5] that for sufficiently small l only symmetric solutions can exist (and thus $\omega=0$ in (1.8)). We arrange the set of zeros such that $\gamma_{k+1}\geq\gamma_k$. Then, since

$$\psi(x,y)\sim 2\pi i\sum_{k=1}^N\left[\text{Res}\left\{\omega_k^+,\hat{\psi}(x,\omega)e^{i\omega y}\right\}+\text{Res}\left\{\omega_k^-,\hat{\psi}(x,\omega)e^{i\omega y}\right\}\right]+O(e^{-\beta y})\ , \tag{2.14}$$

we have

$$\psi(x,y)\sim O(e^{-\gamma_0\,y})\ , \tag{2.15}$$

which proves theorem 1.

Note that (2.14) does not guarantee that $\lim_{y\to\infty}\psi e^{(\gamma_0+\epsilon)y} = \infty$ as the relevant residues may vanish. To have the above asymptotic behaviour we must have

$$\int_{-l}^{l}(hx-\omega_0^{\pm})F_0^{\pm}(x)\psi(x,0)dx \neq 0 \tag{2.16}$$

where $F_0^+ : [-l,l] \to \mathbb{C}$ is any solution of (1.9) with $h = h_S$ and $\omega = \omega_0^+$ (F_0^- is just the same for $\omega = \omega_0^-$. As (2.16) seems quite plausible, we shall assume it through the rest of this section.

It is possible to obtain, however, not only the exponential rate of decay of ψ as $y \to \infty$, but also the exact form of the leading order term in (2.14). We assume, to this end, that $\omega_0^{\pm}$ are simple poles of $\hat{\psi}(x,\omega)$, and that no more than two poles for which $\Im\omega = \gamma_0$ coexist, i.e.,

$$\gamma_0 < \gamma_1 \tag{2.17}$$

Utilizing (1.12), we obtain from (2.14), (2.3), and (2.4)

$$\psi(x,y) \sim Ce^{-\gamma_0\, y}\left\{\left(\mathcal{L}_{\omega_0^+}f\right)g(x,\omega_0^+)e^{i\alpha_0 y} + \left(\overline{\mathcal{L}_{\omega_0^+}\bar{f}}\right)\bar{g}(-x,\omega_0^+)e^{-i\alpha_0 y}\right\} \tag{2.18a}$$

wherein

$$g(x,\omega_0^+) = U(\xi(x,\omega_0^+))U'(\xi(l,-\omega_0^+)) + U(-\xi(x,\omega_0^+))U'(-\xi(l,-\omega_0^+)) \tag{2.18b}$$

$\mathcal{L}_{\omega_0^+} : C[-l,l] \to \mathbb{C}$ is the functional

$$\mathcal{L}_{\omega_0^+}f = \int_{-l}^{l} g(s,\omega_0^+)\left(hs-\omega_0^+\right)f(s)ds \tag{2.18c}$$

and

$$\tilde{f}(s) = \bar{f}(-s) \tag{2.18d}$$

where $f(s) = \psi(s,0)$. If (2.17) is violated (which we believe to be very unlikely) it would be necessary to introduce additional terms in (2.18a) reflecting the additional poles on $\Im\omega = \gamma_0$. It is not difficult to show, using (2.9) and (2.10) that the number of such poles is finite. If one of the poles is not simple the corresponding term would be different than (2.18a) but the exact term can still be obtained. The latter case seems to be improbable as well.

Let $\psi(x,y)$ be a solution of (1.6) and (1.7), then

$$\phi(x,y) = \big(a\psi(x,y) + b\bar{\psi}(-x,y)\big) \tag{2.19}$$

where $|a| = |b|$, is a solution as well. Substituting (2.18) into (2.19) it is not difficult to show that

$$\phi(x,y) \sim e^{-\gamma_0 y}\left\{C_1 g(x,\omega_0^+)e^{i\alpha_0 y} + C_2\bar{g}(-x,\omega_0^+)e^{-i\alpha_0 y}\right\} \;, \tag{2.20}$$

where $|C_1| = |C_2|$. In this case, if vortices exist, they must be located periodically along the line $x = 0$ with spacing π/α_0 between them. To prove their existence it suffices to follow the variation of $\arg\phi$ around the boundary of the rectangle $[-l,l]\times[y, y+\pi/\alpha_0]$, which is exactly 2π , for sufficiently large y, by (2.20).

Except for the exponential rate of decay, (2.20) bears striking similarities to the classical periodic solution in an infinite strip [12, 13], i.e.,

$$\psi(x,y) = aF(x)e^{-icy} + bF(-x)e^{icy} \;, \tag{2.21}$$

where F satisfies (1.9), and c is the positive root of (1.12) at $h = h_{1D}$. In fact (2.18b) is a solution of (1.9) for $\omega = \omega_0^+$ and $h = h_{\mathcal{S}}$. Kulik [12] has demonstrated, by obtaining the solvability condition for the next order balance in the expansion (1.5), that two different types of solutions of the form (2.21) can exist. The first one, for which $|a| = |b|$ is the symmetric state, and the other one for which either a or b vanish, is known as the boundary state. Boeck and Chapman [13] performed weakly non-linear stability analysis of both the boundary and the symmetric states. For sufficiently large κ their results suggest that the symmetric state is the only stable solution.

In view of the above results it appears reasonable to believe that the symmetric solution in a semi-infinite strip would be stable at least in some finite domain in the (κ, l) plane. The equivalent of the boundary state in an infinite strip [13] can be obtained by picking

$$a = \overline{\mathcal{L}_{\omega_0^+} f} \qquad b = -\overline{\mathcal{L}_{\omega_0^+} \tilde{f}} \tag{2.22a,b}$$

in (2.19) to obtain

$$\psi \sim Ce^{-\gamma_0 y}\left[\left|\mathcal{L}_{\omega_0^+} f\right|^2 - \left|\mathcal{L}_{\omega_0^+} \tilde{f}\right|^2\right] g(x, \omega_0^+) e^{i\alpha_0 y} \, . \tag{2.23}$$

If $\left|\mathcal{L}_{\omega_0^+} f\right| = \left|\mathcal{L}_{\omega_0^+} \tilde{f}\right|$ we have $\bar{\psi}(-x, y) \sim C\psi(x, y)$ as $y \to \infty$ where $|C| = 1$, in which case only the equivalent of the symmetric state in an infinite strip exists. Equation (2.18) suggests that the number of independent non-trivial solutions of (1.6) and (1.7) at $h = h_{\mathcal{S}}$ is two. If, however, $\left|\mathcal{L}_{\omega_0^+} f\right| = \left|\mathcal{L}_{\omega_0^+} \tilde{f}\right|$ only the symmetric mode seems to exist. Further research is necessary in this direction.

III The solution in long rectangles

In this section we prove the following result

Lemma 1. *Let $\psi^L(x,y,h_L)$ denote any non-trivial solution of* (1.6) *in the rectangle $R_L = [-l,l] \times [0,L]$, satisfying the boundary conditions*

$$\psi_x^L(\pm l, y) = 0 \qquad ; \qquad i\psi_y^L(x,0) + h_L x \psi^L(x,0) = i\psi_y^L(x,L) + h_L x \psi^L(x,L) = 0 \; . \tag{3.1a,b}$$

Let $f_L(x) = \psi^L(x,0)$ and let $\|f_L(x)\|_\infty = 1$. Let h_L denote the maximal value of h for which non-trivial solutions to (1.6) *together with* (3.1) *exist. Suppose that $\exists L_0$ such that*

$$L \geq L_0 \Rightarrow h_{1D} < \tilde{h} \leq h_L < h_1 \tag{3.2}$$

Then,

a) *The set $\{f_L(x)\}_{L \geq L_0}$ is compact in $C[-l,l]$.*

b) *Let $\{f_{L_K}(x)\}_{k=0}^{\infty}$, where $L_k \uparrow \infty$, be convergent in $C[-l,l]$. Then, $\psi_{L_k} \xrightarrow{k\to\infty} \psi$ pointwise in $\mathcal{S}$. Furthermore, ψ is a solution of* (1.6) *together with* (1.7).

The assumed inequality (3.2) will be proved in lemma 3 in the next section. The existence of the critical fields h_L and their corresponding modes ψ^L will be proved in the next section as well. To prove lemma 1 we need first the following auxiliary result

Lemma 2. *Denote by $\mathcal{D}$ the domain $[-l,l] \times [-l,l] \times \mathbb{R} \setminus [-N,N] \times [h_0,h_1]$, where*

$N > h_1 l$. *Then,* $\forall (x,s,\omega,h) \in \mathcal{D}$ *we have*

$$G(x,s,\omega,h) = \frac{1}{|\omega|}\Big[G_1(x,s,\omega,h)e^{-|\omega||x-s|} + G_2(x,s,\omega,h)e^{-|\omega|(2l-x-s)} + G_3(x,s,\omega,h)e^{-|\omega|(2l+x+s)}\Big] \tag{3.3a}$$

$$\omega\left[G(x,s,\omega,h) - G(x,s,\omega,h)\right] = \frac{1}{|\omega|}\Big[G_4(x,s,\omega,h)e^{-|\omega||x-s|} + G_5(x,s,\omega,h)e^{-|\omega|(2l-x-s)} + G_6(x,s,\omega,h)e^{-|\omega|(2l+x+s)}\Big] \tag{3.3b}$$

in which the G_i*'s satisfy*

$$\sup_{\substack{(x,s,\omega,h)\in\mathcal{D}\\ 1\le i\le 6}} |G_i(x,s,\omega,h)| \le M \quad ; \quad \sup_{\substack{(x,s,\omega,h)\in\mathcal{D}\\ 1\le i\le 6}\ ;\ x\neq s} \left|\frac{\partial G_i}{\partial x}(x,s,\omega,h)\right| \le M \tag{3.3c,d}$$

Proof: Utilizing (2.9) it is easy to show that in the limit $\omega \to \infty$, for $x < s$ and positive ω, $G(x,s,\omega,h)$ satisfies (3.3a) with

$$G_1 = |\omega|\frac{1}{2}\frac{(\omega - hs)^{a-1/2}}{(\omega - hx)^{a+1/2}}\exp\left\{\frac{1}{2}h(s^2 - x^2)\right\}\left[1 + \frac{\tilde{G}_1(x,s,\omega,h)}{\omega^2}\right] \tag{3.4a}$$

$$G_2 = |\omega|\frac{1}{2}\frac{(\omega - hl)^{2a}}{(\omega - hx)^{a+1/2}(\omega - hs)^{a+1/2}}\exp\left\{\frac{1}{2}h(2l^2 - s^2 - x^2)\right\}\left[1 + \frac{\tilde{G}_2(x,s,\omega,h)}{\omega^2}\right] \tag{3.4b}$$

$$G_3 = |\omega|\frac{1}{2}\frac{(\omega + hl)^{2a}}{(\omega - hx)^{a+1/2}(\omega - hs)^{a+1/2}}\exp\left\{\frac{1}{2}h(2l^2 - s^2 - x^2)\right\}\left[1 + \frac{\tilde{G}_3(x,s,\omega,h)}{\omega^2}\right], \tag{3.4c}$$

where $\tilde{G}_i$ $(i = 1,2,3)$ and their derivatives with respect to x are bounded in $\mathcal{D}$ for $x < s$ and $\omega \ge 0$. As

$$G(x,s,\omega,h) = G(s,x,\omega,h) = G(-x,-s,-\omega,h)\ , \tag{3.5}$$

(3.3a) is proved. To prove (3.3b) we simply substitute (3.4) into its left-hand-side.

□

We now prove part a) of lemma 1. Let $\chi(L) = \frac{2\pi}{L}$. Upon multiplying (1.6) by $e^{-in\chi y}$ we integrate by parts to obtain, utilizing (3.1a),

$$-\hat{\psi}_L'' + \left[(hx - n\chi)^2 - 1\right]\hat{\psi}_L = -i(hx - n\chi)f_L(x) \tag{3.6a}$$

$$\hat{\psi}_L'(\pm l) = 0 \ . \tag{3.6b}$$

where

$$\hat{\psi}_L(x, n\chi) = \int_0^L e^{-in\chi y}\psi^L(x,y)dy \ . \tag{3.7}$$

Note that (3.6a) is exactly (2.1a) with ω replaced by $n\chi$ and $\psi(x,0)$ by $f_L(x)$. Hence,

$$\hat{\psi}_L(x, n\chi) = -i\int_{-l}^{l}(hs - n\chi)G(x,s,n\chi)f_L(s)ds \ , \tag{3.8}$$

and

$$-\frac{1}{2}[\psi^L(x,0) + \psi^L(x,L)] = \frac{1}{L}\text{P.V.}\sum_{n=-\infty}^{\infty}\hat{\psi}_L(x,n\chi) \ . \tag{3.9}$$

Similarly, multiplying (1.6) by $e^{-i(n+1/2)\chi y}$ and integrating by parts we obtain

$$\frac{1}{2}f_L(x) = \frac{1}{L}\text{P.V.}\sum_{n=-\infty}^{\infty}\tilde{\psi}_L(x,(n+1/2)\chi) \tag{3.10}$$

where

$$\tilde{\psi}_L(x,(n+1/2)\chi) = i\int_{-l}^{l}(hs - (n+1/2)\chi)G(x,s,(n+1/2)\chi)[\psi(s,0) + \psi(s,L)]ds \ . \tag{3.11}$$

We first show that $\|\psi^L(x,0)+\psi^L(x,L)\|_\infty$ is bounded as $L\to\infty$. By (2.3) we have (recall that $\|f_L\|_\infty = 1$)

$$\left|\psi^L(x,0)+\psi^L(x,L)\right| \leq \frac{2}{L}\left\{\sum_{n=-\left[\frac{N}{\chi}\right]}^{\left[\frac{N}{\chi}\right]}\int_{-l}^{l}|G(x,s,n\chi)|ds(hl+N)\right.$$
$$+\sum_{n=\left[\frac{N}{\chi}\right]+1}^{\infty}\int_{-l}^{l}\left[hl|G(x,s,n\chi)+G(x,s,-n\chi)|\right.$$
$$\left.\left.+n\chi|G(x,s,n\chi)-G(x,s,-n\chi)|\right]ds\right\} . \tag{3.12}$$

The first integral on the right-hand-side of (3.12) can be estimated using the uniform boundedness of $G(x,s,\omega,h)$ for $|\omega|\leq N$, i.e.,

$$\int_{-l}^{l}|G(x,s,n\chi)|ds(hl+N)\leq C \tag{3.13}$$

The second integral can be estimated using lemma 2,

$$\int_{-l}^{l}\left[hl|G(x,s,n\chi)+G(x,s,-n\chi)|+n\chi|G(x,s,n\chi)-G(x,s,-n\chi)|\right]ds\leq$$
$$\leq\frac{C}{n\chi}\int_{-l}^{l}\left[e^{-n\chi|x-s|}+e^{-n\chi(2l-x-s)}+e^{-n\chi(2l+x+s)}\right]ds\leq\frac{C}{n^2\chi^2} . \tag{3.14}$$

In the above and in the following C does not necessarily present the same constant. It is, however, always independent of L. Combining (3.12), (3.13), and (3.14) we obtain

$$\left|\psi^L(x,0)+\psi^L(x,L)\right|\leq\frac{C}{L}\left\{\sum_{n=-\left[\frac{N}{\chi}\right]}^{\left[\frac{N}{\chi}\right]}1+\sum_{n=\left[\frac{N}{\chi}\right]+1}^{\infty}\frac{1}{(n\chi)^2}\right\}\leq C \tag{3.15}$$

We now prove equicontinuity of the set $\{f_L\}_{L\geq L_0}$. By (3.10),(3.11), and (3.15) we have

$$
\begin{aligned}
|f_L(x) - f_L(z)| \le \frac{C}{L} \Bigg\{ & \sum_{n=-\left[\frac{N}{\chi}-1\right]}^{\left[\frac{N}{\chi}\right]} \int_{-l}^{l} |G(x,s,(n+1/2)\chi) - G(z,s,(n+1/2)\chi)| ds (hl+N) \\
& + \sum_{n=\left[\frac{N}{\chi}\right]+1}^{\infty} \int_{-l}^{l} \Big[hl \big\{ |G(x,s,(n+1/2)\chi) - G(z,s,(n+1/2)\chi)| \\
& \qquad + |G(x,s,-(n+1/2)\chi) - G(z,s,-(n+1/2)\chi)| \big\} \Big] \\
& \qquad + (n+1/2)\chi \Big| \big[G(x,s,(n+1/2)\chi) - G(x,s,-(n+1/2)\chi) \big] \\
& \qquad - \big[G(z,s,(n+1/2)\chi) - G(z,s,-(n+1/2)\chi) \big] \Big| \Big] ds \Bigg\} .
\end{aligned}
\tag{3.16}
$$

The first sum on the right-hand-side is estimated using the uniform boundedness of $G_x(x,s,\omega)$ for $|\omega| \le N$, i.e.,

$$
\int_{-l}^{l} |G(x,s) - G(z,s)| ds (hl+N) \le C|x-z| \, , \tag{3.17}
$$

(where $G(x,s) \equiv G(x,s,(n+1/2)\chi)$). The second integral is estimated using (3.3). For instance,

$$
\begin{aligned}
\int_{-l}^{l} hl |G(x,s) - G(z,s)| ds \le & \frac{C}{(n+1/2)\chi} \int_{-l}^{l} \Big[|G_1(x,s) - G_1(z,s)| e^{-(n+1/2)\chi|x-s|} \\
& + |G_2(x,s) - G_2(z,s,(n+1/2)\chi)| e^{-(n+1/2)\chi(2l-x-s)} + |G_3(x,s) - G_3(z,s)| e^{-(n+1/2)\chi(2l+x+s)} \Big] ds \\
& + \frac{C}{(n+1/2)\chi} \int_{-l}^{l} \Big[|G_1(z,s)| \left| e^{-(n+1/2)\chi|x-s|} - e^{-(n+1/2)\chi|z-s|} \right| + \\
& |G_2(z,s)| \left| e^{-(n+1/2)\chi(2l-x-s)} - e^{-(n+1/2)\chi(2l-z-s)} \right| \\
& + |G_3(z,s)| \left| e^{-(n+1/2)\chi(2l+x+s)} - e^{-(n+1/2)\chi(2l+z+s)} \right| \Big] ds \, .
\end{aligned}
\tag{3.18}
$$

Utilizing (3.3c,d) we obtain

$$\int_{-l}^{l} hl|G(x,s)-G(z,s)|ds \leq \frac{C}{(n+1/2)\chi}|x-z|\int_{-l}^{l}\Big[e^{-(n+1/2)\chi|x-s|}$$

$$+e^{-(n+1/2)\chi(2l-x-s)}+e^{-(n+1/2)\chi(2l+x+s)}\Big]\,ds$$

$$+\frac{C}{(n+1/2)\chi}\int_{-l}^{l}\Big[\Big|e^{-(n+1/2)\chi|x-s|}-e^{-(n+1/2)\chi|z-s|}\Big|$$

$$+\Big|e^{-(n+1/2)\chi(2l-x-s)}-e^{-(n+1/2)\chi(2l-z-s)}\Big|+\Big|e^{-(n+1/2)\chi(2l+x+s)}-e^{-(n+1/2)\chi(2l+z+s)}\Big|\Big]\,ds \tag{3.19}$$

The integrals on the right-hand-side can be calculated analytically. For instance

$$\int_{-l}^{l}\Big|e^{-\omega|x-s|}-e^{-\omega|z-s|}\Big|\,ds=\frac{1}{\omega}\Big[1-e^{-\omega|x-z|}\Big]\Big[2+e^{-\omega(l-\max(x,z))}+e^{-\omega(l+\min(x,z))}\Big]$$

$$+\frac{2}{\omega}\Big[1-e^{-\omega|x-z|/2}\Big]^2\leq\frac{C}{\omega}\Big[1-e^{-\omega|x-z|}\Big]\;.$$

Hence,

$$\int_{-l}^{l} hl|G(x,s)-G(z,s)|ds \leq \frac{C}{(n+1/2)^2\chi^2}|x-z|+\frac{C\left(1-e^{(n+1/2)\chi|x-z|}\right)}{(n+1/2)^2\chi^2} \tag{3.20}$$

All other integrals on the right-hand-side of (3.16) can be bounded in the same manner to obtain, combining (3.16), (3.17), and (3.20),

$$|f_L(x)-f_L(z)|\leq C|x-z|+\frac{1}{L}\sum_{n=\left[\frac{N}{\chi}\right]+1}^{\infty}\frac{C\left(1-e^{(n+1/2)\chi|x-z|}\right)+C|x-z|}{(n+1/2)^2\chi^2}\;. \tag{3.21}$$

For some $L_0>0$,

$$\sum_{n=\left[\frac{N}{\chi}\right]+1}^{\infty}\frac{1-e^{-(n+1/2)\chi|x-z|}}{(n+1/2)^2\chi^2}\leq C\int_N^{\infty}\frac{1-e^{-\omega|x-z|}}{\omega^2}d\omega$$

$$=C\left[\frac{1-e^{-N|x-z|}}{N}+|x-z|\int_N^{\infty}\frac{e^{-\omega|x-z|}}{\omega}d\omega\right]\leq C|x-z|(1+\log|x-z|)\;, \tag{3.22}$$

for all $L > L_0$. Consequently,

$$|f_L(x) - f_L(z)| \leq C|x-z|(1 + \log|x-z|) \ , \tag{3.23}$$

and hence, the set $\{f_L\}_{L \geq L_0}$ is equicontinuous, and therefore compact.

We now prove part b) of lemma 1. By (3.7) we have for $0 < y < L$

$$\psi^L(x,y) = \frac{1}{L} \sum_{n=-\infty}^{\infty} \hat{\psi}(x, n\chi) e^{in\chi y} \ . \tag{3.24}$$

Applying the Poisson summation formula we obtain

$$\psi^L(x,y) = \sum_{n=-\infty}^{\infty} \phi_L(x, y - nL) \tag{3.25a}$$

in which

$$\phi_L(x,y) = \frac{1}{2\pi} \text{P.V.} \int_{-\infty}^{\infty} e^{i\omega y} \hat{\psi}_L(x,\omega) d\omega \tag{3.25b}$$

Using the same complex plane techniques which were used to derive (2.15) it is easy to show that as $|y| \to \infty$

$$\phi_L(x,y) \sim O(e^{-\gamma_0 \, |y|}) \ . \tag{3.26}$$

We note that in the limit $y \to -\infty$ we have to use a different complex plane path which is the reflection of the path in Fig. 1 with respect to the horizontal axis.

Let $L_k \uparrow \infty$ as $k \to \infty$, and let f_{L_k} be convergent. By (3.2) h_{L_k} is compact too, and hence we shall assume it to be convergent (otherwise we choose the right subsequence). Then, ϕ_{L_k} converges pointwise in the semi-infinite strip $\mathcal{S}$. Denote the limit by ϕ. By (3.25a) and (3.26) $\psi_{L_k} \to \phi$ pointwise. To complete the proof we need yet to show that the various derivatives of ψ_{L_k} converge to the corresponding derivatives of ψ. This can be done by using the Shauder estimates in [18].

As ψ_{L_k} is a Cauchy sequence in $C(\Omega)$, where Ω is any compact subset of $\mathcal{S}$, and since (1.6) together with (1.7) fit into the rather general framework in [18], by theorem 9.3 there ψ_{L_k} is a Cauchy sequence in $C_2(\Omega)$. Hence, $\psi_{L_k} \to \psi$ in $C_2(\Omega)$, which completes the proof of lemma 1.

In addition to the convergence of ψ_{L_k} to a solution of (1.6) together with (1.7) we have also demonstrated convergence of h_{L_k} to one of the critical values of h for which non-trivial solutions to (1.6) together with (1.7) exist. It is not clear, however, if the limit value is indeed the onset field $h_{\mathcal{S}}$. Furthermore, the results of lemma 1 depend all on the boundedness of $\{h_L\}_{L \leq L_0}$ assumed in (3.2). In the next section we prove (3.2) and discuss the possible limit values of h_L.

IV Variational inequalities

The results we have obtained in the previous sections were based on the assumption that the onset fields for a semi-infinite strip $h_{\mathcal{S}}$ and for sufficiently long rectangles h_L are greater than h_{1D} - the onset field for a slab, and are uniformly bounded from above. In the following we prove the existence of these bounds. The proof is based on the estimation of the infimum of the following sesquilinear form

$$\lambda(h,\Omega) = \inf_{\substack{\psi \in H^1(\Omega,\mathbb{C}) \\ \|\psi\|_{L^2(\Omega)}=1}} \int_\Omega \left|(\boldsymbol{\nabla} - ihx\hat{\boldsymbol{j}})\psi\right|^2 dxdy \tag{4.1}$$

where Ω may denote any domain in $\mathbb{R}^2$ whose boundary is in C^2 except perhaps for a finite number of points. Direct methods of the calculus of variation show that for compact domains the $\lambda(h,\Omega)$ is achieved for some value $\psi \in H^1(\Omega,\mathbb{C})$ (cf. for instance

[19]). It is also easy to show (cf. for instance [6]), that when $\lambda(h,R_L)=1$, there exist a solution of (1.6) together with (3.1). By (4.1), $\lambda(h,R_L)$ is the principal eigenvalue of the linear operator associated with the above sesquilinear form, i.e.,

$$-\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)+2ihx\frac{\partial\psi}{\partial y}+h^2x^2\psi\ .$$

Therefore, as the above operator is uniformly elliptic in the bounded rectangle R_L, $\lambda(h,R_L)$ is a branch of a holomorphic function of h, L, and l, and hence must be continuous in all these variables (cf. section VIII-§ 6.5 in [20]).

The foregoing discussion founds the basis for the following proof of the inequality (3.2).

Lemma 3. *$\exists L_0$ s.t. $L\geq L_0\Rightarrow h_{1D}<\tilde{h}\leq h_L<h_1<\infty$.*

proof: To prove the upper bound we first show that $\lambda(h,R_L)\leq\lambda(h,R_{2L})$. Let ψ_{2L} be the minimizer of (4.1) on R_{2L}. Then,

$$\begin{aligned}\lambda(h,R_{2L}) &= \int_{R_{2L}}\left|(\boldsymbol{\nabla}-ihx\hat{\boldsymbol{j}})\psi_{2L}\right|^2dxdy=\\ &=\int_{0\leq y\leq L}\left|(\boldsymbol{\nabla}-ihx\hat{\boldsymbol{j}})\psi_{2L}\right|^2dxdy+\int_{L\leq y\leq 2L}\left|(\boldsymbol{\nabla}-ihx\hat{\boldsymbol{j}})\psi_{2L}\right|^2dxdy\geq\\ &\geq\lambda(h,R_L)\left(\int_{0\leq y\leq L}|\psi_{2L}|^2dxdy+\int_{L\leq y\leq 2L}|\psi_{2L}|^2dxdy\right)=\lambda(h,R_L)\ .\end{aligned}\tag{4.2}$$

We then apply the transformation

$$(x,y)\to\frac{1}{\sqrt{h}}(x,y)$$

to obtain

$$\lambda(h,R_L)=h\lambda\left(1,R_L^{\sqrt{h}}\right)\ ,\tag{4.3}$$

where R_L^α denotes the rectangle $[-\alpha l, \alpha l] \times [0, \alpha L]$. In a similar manner used to show (4.2) we can easily show

$$\lambda\left(1, R_L^2\right) \geq \lambda(1, R_L)$$

and hence,

$$\lambda\left(1, R_L^\alpha\right) \geq \inf_{1\leq\alpha\leq 2} \lambda\left(1, R_L^\alpha\right) \quad \forall\alpha \geq 1 \ . \tag{4.4}$$

Combining (4.2), (4.3), and (4.4), we obtain

$$\lambda(h, R_L) \geq h \inf_{\substack{1\leq\alpha\leq 2\\ L_0\leq L\leq 2L_0}} \lambda\left(1, R_L^\alpha\right) \quad \forall L \geq L_0 \tag{4.5}$$

As $\lambda\left(h, R_L^\alpha\right)$ is a continuous function of both L and α, the infimum on the right-hand-side of (4.5) is positive. Hence, if

$$h > \left[\inf_{\substack{1\leq\alpha\leq 2\\ L_0\leq L\leq 2L_0}} \lambda\left(1, R_L^\alpha\right)\right]^{-1}$$

$\lambda(h, R_L)$ must be greater than 1 and hence, no solution can exist to (1.6) together with (3.1), which proves the upper bound in (4.6).

To prove the lower bound we use (4.5) and the continuous dependence of λ on h. Thus, if we find $\tilde{h}$ such that $\lambda(\tilde{h}, R_L) \leq 1$, then $h_L \geq \tilde{h}$. We then have to prove the existence of $\tilde{h} > h_{1D}$ for which $\lambda(\tilde{h}, R_L) \leq 1$. Let

$$\psi = C(\epsilon, \delta)\left(\psi_\epsilon + \delta\phi\right) \tag{4.6a}$$

where

$$\psi_\epsilon = e^{-\epsilon y} e^{icy} f(x) \quad \text{and} \quad \phi = -ie^{(ic-1)y} f(x)(hx - c) \ . \tag{4.6b,c}$$

In the above f and c satisfy (1.9) for $h = h_{1D}$, $\|f\|_{L^2[-l,l]} = 1$, δ and ϵ are small numbers, and

$$C^2 = 2\epsilon \left[1 + \delta^2 \epsilon A\right]^{-1} \tag{4.6d}$$

where

$$A = \int_{-l}^{l} f^2 (hx - c)^2 dx > 0 \ , \tag{4.6e}$$

so that $\|\psi\|_{L^2[\mathcal{S}]} = 1$.

Let $h = h_{1D}$. Then, integration by parts yields

$$\begin{aligned} I(h_{1D}, \psi, \mathcal{S}) = \int_{\mathcal{S}} \left|(\boldsymbol{\nabla} - ih_{1D}x\hat{\boldsymbol{j}})\psi\right|^2 dxdy = C^2 \int_{\mathcal{S}} \left|(\boldsymbol{\nabla} - ih_{1D}x\hat{\boldsymbol{j}})\psi_\epsilon\right|^2 dxdy + \\ + 2\delta C^2 \Re \left\{ \int_{\mathcal{S}} \phi^* \left[-\boldsymbol{\nabla}^2 \psi_\epsilon + 2ih_{1D}x \frac{\partial \psi_\epsilon}{\partial y} + h_{1D}^2 x^2 \psi_\epsilon \right] dxdy + \right. \\ \left. + \int_{\partial \mathcal{S}} \phi^* (\boldsymbol{\nabla} - ih_{1D}x\hat{\boldsymbol{j}})\psi_\epsilon \cdot \hat{\boldsymbol{n}} ds \right\} + \delta^2 C^2 \int_{\mathcal{S}} \left|(\boldsymbol{\nabla} - ih_{1D}x\hat{\boldsymbol{j}})\phi\right|^2 dxdy = \\ = C^2 \left[\frac{1}{2\epsilon} - A\delta + \left(\frac{1}{2} + \frac{\delta}{1+\epsilon}\right)\epsilon + B\delta^2 \right] \ , \end{aligned} \tag{4.7}$$

wherein B is independent of ϵ and δ. Picking $\delta = \epsilon^{1/2}$ we obtain, since $C^2 < 2\epsilon$

$$I(h_{1D}, \psi, \mathcal{S}) \leq 1 - 2A\epsilon^{3/2} + \tilde{B}\epsilon^2 \ , \tag{4.8}$$

where $\tilde{B}$ is independent of ϵ. Note that the choice (4.6) takes advantage of the fact that the one-dimensional solution $f(x)e^{icy}$ does not satisfy the boundary condition on $y = 0$.

It is easy to show that $I(h_{1D}, \psi, R_L) = I(h_{1D}, \psi, \mathcal{S}) + O(e^{-\epsilon L})$ and that I is continuous in h. Hence for sufficiently large L and for sufficiently small but positive ϵ and $\tilde{h} - h_{1D}$ we have $\lambda(\tilde{h}, R_L) \leq I(\tilde{h}, \psi, R_L) \leq 1$. Hence, h_L (and its corresponding mode

ψ^L) exists and satisfies (3.2).

$\square$

Note that to prove existence of h_L it is enough to use $\lambda(0, R_L) = 0$ together with the result of Giorgi & Philips [2] $\lambda(h, \Omega) \geq C(\Omega)h$ for $h \geq h_0 > 0$ which is valid for any bounded domain in $\mathbb{R}^2$ (as well as domains in $\mathbb{R}^3$). Lemma 3 is necessary in order to prove both the lower bound $\tilde{h} > h_{1D}$ and to prove uniformity of the upper bound for $L \geq L_0$.

In the previous section we proved that any converging sequence of onset fields $\{h_{L_k}\}_{k=1}^{\infty}$ for the rectangles $\{R_{L_k}\}_{k=1}^{\infty}$ tends to a critical field for the semi-infinite strip $\mathcal{S}$, provided that $h_{1D} < \tilde{h} \leq h_L$ for all $L_0 < L$. From lemma 3 together with lemma 1 it is then clear that $h_{\mathcal{S}}$ the onset field for $\mathcal{S}$ exists and is greater than h_{1D}. Theorem 2 is proved.

It is now easy to show that $h_{L_k} \to h_{\mathcal{S}}$. In a similar manner to (4.2) we write

$$\lambda(h, \mathcal{S}) = \sum_{n=0}^{\infty} \int_{nL \leq y \leq (n+1)L} \left|(\boldsymbol{\nabla} - ihx\hat{\boldsymbol{j}})\psi_{\mathcal{S}}\right|^2 dxdy > \lambda(h, R_L)\ . \tag{4.9}$$

Hence, as $h > h_L \Rightarrow \lambda(h, R_L) > 1$, we have $h_{\mathcal{S}} \leq h_L$, which proves that the critical field to which any converging sequence h_{L_k} tends must be the onset field $h_{\mathcal{S}}$. Therefore, $h_L \xrightarrow{L\to\infty} h_{\mathcal{S}}$.

It is not obvious, however, that h_L is strictly greater than $h_{\mathcal{S}}$. To prove the latter we need the following result:

Lemma 4. *Given the conjecture* (2.17), *if* ω_0^+ *is a simple pole of* $\hat{\psi}(x, \omega)$, *and if* $\left|\mathcal{L}_{\omega_0^+} f\right| \neq \left|\mathcal{L}_{\omega_0^+} \tilde{f}\right|$, *then* $\exists L_0$ *s.t.* $L \geq L_0 \Rightarrow h_L > h_{\mathcal{S}}$.

Proof: In a similar fashion to the proof of the previous lemma we show $\lambda(h_{\mathcal{S}}, R_L) < 1$. Let $\psi_{\mathcal{S}}$ denote a solution of (1.6) and (1.7) whose L^2 norm is unity, i.e., $\|\psi_{\mathcal{S}}\|_{L^2(\mathcal{S})} = 1$. Integration by parts yields

$$I(h_{\mathcal{S}}, \psi_{\mathcal{S}}, R_L) = \frac{\int_{R_L} \left|(\boldsymbol{\nabla} - ih_{\mathcal{S}} x\hat{\boldsymbol{j}})\psi\right|^2 dxdy}{\|\psi_{\mathcal{S}}\|_{L^2[R_L]}} = 1 - \frac{\int_{-l}^{l} \frac{\partial|\psi_{\mathcal{S}}|^2}{\partial y}\Big|_{y=L} dx}{\|\psi_{\mathcal{S}}\|_{L^2[R_L]}} . \tag{4.10}$$

Let $L \gg 1$, then picking a and b as in (2.23) we obtain

$$\frac{\partial|\psi_{\mathcal{S}}|^2}{\partial y}\bigg|_{y=L} \sim -\gamma_0 |C|^2 e^{-2\gamma_0 y} < 0 . \tag{4.11}$$

Thus, $\lambda(h_{\mathcal{S}}, R_L) \le I(h_{\mathcal{S}}, \psi_{\mathcal{S}}, R_L) < 1$ and the lemma is proved.

□

Acknowledgements

This research was supported by the Technion V.P.R fund - M. and C. Papo Research Fund and by the David Posnack Memorial Academic Lectureship. The author wishes to thank the Professors Jacob Rubinstein, Yehuda Pinchover and Peter J. Sternberg and also to Ms. Hala Jadallah for many useful comments and discussions.

References

[1] W. Meissner and R. Ochsenfeld, Naturwissenschaffen **21**, 787 (1933).

[2] T. Giorgi and D. Philips, The breakdown of superconductivity due to strong fields for the ginzburg-landau model, SIAM J. Math. Anal. **30**, 341 (1999).

[3] D. Saint-James and P. de Gennes, Onset of superconductivity in decreasing fields, Phys. Let. **7**, 306 (1963).

[4] V. L. Ginzburg and L. D. Landau, On the theory of supercoductivity, Soviet Phys. JETP **20**, 1064 (1950).

[5] C. Bolley and B. Helffer, Rigorous results on ginzburg-landau models in a film submitted to exterior parallel magnetic field i, Nonlinear Studies **3**, 1 (1996).

[6] P. Bauman, D. Philips, and Q. Tang, Stable nucleation for the Ginzburg-Landau system with an applied magnetic field, Arch. Rat. Mech. Anal. **142**, 1 (1998).

[7] K. Lu and X. Pan, Eigenvalue problems of Ginzburg-Landau operator in bounded domains, J. Math. Phys. **50**, 2647 (1999).

[8] M. del Pino, P. L. Felmer, and P. Sternberg, Boundary concentration for eigenvalue problems related to the onset of the superconductivity, preprint.

[9] A. Bernoff and P. Sternberg, Onset of superconductivity in decreasing fields for general domains, J. Math. Phys. **39**, 1272 (1998).

[10] H. Jadallah, J. Rubinstein, and P. Sterenberg, Phase transition curves for mesoscopic superconucting samples, Phys. Rev. Lett. **82**, 2935 (1999).

[11] V. Schweigert and F. Peters, Influence of the confinement geometry on surface superconductivity, Preprint.

[12] Kulik, Vortex structure in superconducting films in a parallel field, Soviet Physics J.E.T.P **25**, 1085 (1967).

[13] T. Boeck and S. J. Chapman, Bifurcation to vortex solution in superconducting films, European J. Appl. Math. **8**, 125 (1997).

[14] S. J. Chapman, Asymptotic analysis of the ginzburg-landau model of superconductivity: reduction to a free boundary model, Quart. Appl. Math. **53**, 601 (1995).

[15] A. A. Abrikosov, On the magnetic properties of superconductors of the second group, Soviet Phys. J.E.T.P. **5**, 1175 (1957).

[16] S. J. Chapman, Nucleation of superconductivity in decreasing fields I, EJAM **5**, 449 (1994).

[17] H. Bateman, *Higher transcendental functions*, McGraw-Hill, 1953.

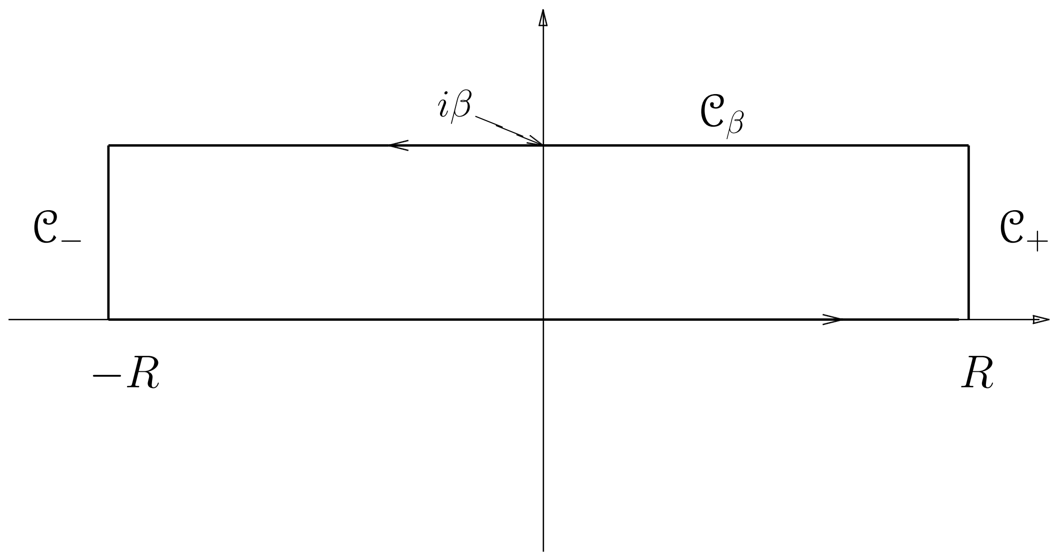
[18] S. Agmon, A. Douglis, and L. Nirenberg, Estimates near the boundary for solutions of partial differential, Comm. Pure Appl. Math. **17**, 35 (1964).

[19] L. C. Evans, *Partial Differential Equations*, AMS, 1st. edition, 1998.

[20] T. Kato, *Perturbation theory for linear operator*, Springer, 3rd. edition, 1980.

Figure captions

Figure 1 The path of integration in the complex plane along which the integral (2.6) is evaluated


$i\beta$
$\mathcal{C}_\beta$
$\mathcal{C}_-$
$\mathcal{C}_+$
$-R$
R