Arbitrary n-vortex self-duality solutions to the Ginzburg-Landau equations satisfying normal state conditions at infinity

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Abstract

It is well known that when the Ginzburg-Landau parameter $\kappa = 1/\sqrt{2}$, the second order Ginzburg-Landau equations may be reduced to the first-order Bogomolnyi equations. It is established in this critical case that, for any given set of vortex locations and orders, these equations possess a unique solution which tends to the purely superconducting state at infinity. In the present contribution we focus on cases in which normal state conditions at infinity are imposed. It is found that, for any given set of vortex locations and orders, an infinite number of solutions satisfying such conditions at infinity exist.

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1 Introduction

We study vortex solutions to the Ginzburg-Landau equations, which serve as the common model for superconductivity. Existence of such solutions was first predicted by Abrikosov [1]. Most studies of these solutions (cf. for instance [8], [9] and [11]) assume the material to be purely superconducting at infinity. When the applied magnetic field is strong enough to insure the presence of regions in which the material is essentially at a normal state, different conditions at infinity need be imposed. Obukhov & Schunck [10] have considered recently single-vortex solutions satisfying normal state conditions at infinity.

In general, when more than a single vortex is present in an infinite domain, the solution is expected to be time-dependent. This conjecture is based on the numerical results of Jacobs & Rebbi [9] which demonstrate that the force between two identical vortices in type II superconductors is repulsive . Multi-vortex solutions have been obtained for the periodic Abrikosov [1] lattice, as well as arbitrary N-vortex solutions [13] for the critical case when the Ginzburg-Landau parameter $\kappa = 1/\sqrt{2}$ (in which case the superconductor exhibits the so-called self-duality [3]). Existence of the latter solutions has been established for purely superconducting state conditions at infinity. Arbitrary N-vortex solutions satisfying normal-state conditions at infinity have never been studied.

The present contribution focuses on the critical case $\kappa = 1/\sqrt{2}$, and considers solutions which satisfy normal-state conditions at infinity. We demonstrate that an infinite number of such solutions exist for a given set of vortex locations and orders.

The steady state Ginzburg-Landau dimensionless equations are [5]

$$\left(\frac{i}{\kappa}\boldsymbol{\nabla}+\boldsymbol{A}\right)^2\psi=\psi\left(1-|\psi|^2\right)\quad , \tag{1.1a}$$

and

$$-\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{A})=\frac{i}{2\kappa}\left(\psi^*\boldsymbol{\nabla}\psi-\psi\boldsymbol{\nabla}\psi^*\right)+|\psi|^2\boldsymbol{A} \tag{1.1b}$$

in which ψ is the (complex) superconducting order parameter, such that $|\psi|$ varies from $|\psi|=0$ (when the material is at a normal state) to $|\psi|=1$ (for the purely superconducting state). The magnetic vector potential is denoted by $\boldsymbol{A}$ (so that the magnetic field is given by $\boldsymbol{H}=\boldsymbol{\nabla}\times\boldsymbol{A}$), and κ is the Ginzburg-Landau parameter which is a material property. Superconductors for which $\kappa<1/\sqrt{2}$ have been termed type I superconductors, and those for which $\kappa>1/\sqrt{2}$ have been termed type II.

We shall consider the situation in the xy plane, in which a uniform magnetic field (directed along the z axis) is applied on an infinite sample of superconductor. The superconductor satisfies in that case, normal-state conditions at infinity, i.e.

$$\lim_{|\boldsymbol{x}|\to\infty}\boldsymbol{\nabla}\times\boldsymbol{A}=h\boldsymbol{k}\quad , \tag{1.1c}$$

and in view of (1.1a)

$$\lim_{|\boldsymbol{x}|\to\infty}\psi=0\quad , \tag{1.1d}$$

wherein $\boldsymbol{k}$ denotes a unit vector in the z direction, $r=|\boldsymbol{x}|$ is a radial coordinate, and h denotes the intensity of the applied magnetic field.

For $\kappa=1/\sqrt{2}$ the Ginzburg-Landau equations exhibit the self-duality property: Some of the solutions of (1.1a) and (1.1b) may be obtained, by solving the system of partial differential equations [4], [13]

$$\sqrt{2}\frac{\partial\psi_1}{\partial x} + A_x\psi_2 - \sqrt{2}\frac{\partial\psi_2}{\partial y} + A_y\psi_1 = 0 \quad , \tag{1.2a}$$

$$\sqrt{2}\frac{\partial\psi_1}{\partial y} + A_y\psi_2 + \sqrt{2}\frac{\partial\psi_2}{\partial x} - A_x\psi_1 = 0 \quad , \tag{1.2b}$$

and

$$H + \frac{1}{\sqrt{2}}(\psi_1^2 + \psi_2^2 - 1) = 0 \quad , \tag{1.2c}$$

wherein ψ_1 and ψ_2 respectively denote the real and imaginary parts of ψ, A_x and A_y are, respectively, the components of $\boldsymbol{A}$ in the x and y directions and H denotes the magnetic field magnitude. It is possible to reduce (1.2) into a scalar second-order non-linear eliptic partial differential equation. Towards this end we set (as in [13])

$$\hat{A} = A_x + iA_y \qquad ; \qquad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$
$$\frac{\partial}{\partial z^*} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \quad . \tag{1.3a,b,c}$$

Equations (1.2a) and (1.2b) may then be written in the form of a single, complex-valued equation

$$2\sqrt{2}\frac{\partial\psi}{\partial z^*} - i\hat{A}\psi = 0 \tag{1.4}$$

which is integrated to obtain

$$\hat{A} = -2\sqrt{2}i\frac{\partial \log\psi}{\partial z^*} \quad , \tag{1.5}$$

Consequently,

$$H = -\sqrt{2}\nabla^2 f \tag{1.6}$$

wherein $f = \log|\psi|$. Combining (1.6) and (1.2c) we obtain

$$\nabla^2 f + \frac{1}{2}(1 - e^{2f}) = 0 \tag{1.7}$$

Taubes [13] examined the case in which $f \to 0$ as $r \to \infty$, i.e., the material is purely superconducting at infinity. For this case he proved the existence and uniqueness of the solution for a given set of vortex locations $(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_N)$ and respective orders $(m_1, \ldots, m_N)$, such that $f \sim m_n \log|\boldsymbol{x} - \boldsymbol{a}_n|$ as $|\boldsymbol{x} - \boldsymbol{a}_n| \to 0$. We shall subsequently demonstrate the existence of a different class of solutions for which $f \to -\infty$ as $r \to \infty$. In this case the superconductor attains a normal state at infinity and the applied magnetic field attains the value $h = 1/\sqrt{2}$ [cf. (1.2c)], or the critical value [5]. In the next section we prove the existence of these solutions and discuss their properties. In the last section we summarize the results obtained in this work and the questions which remain as yet unresolved.

2 Self-duality solutions

Our goal in this section is to prove

Theorem 1 *For a given set of vortex locations* $(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_N)$ *and respective orders* $(m_1, \ldots, m_N)$ *there exist an infinite number of solutions to the equation*

$$\nabla^2 f + \frac{1}{2}(1 - e^{2f}) = 0 \tag{2.1a}$$

satisfying the boundary conditions

$$\lim_{r \to \infty} f = -\infty \quad , \tag{2.1b}$$

and

$$f \sim m_n \log|\boldsymbol{x} - \boldsymbol{a}_n| \quad as \quad |\boldsymbol{x} - \boldsymbol{a}_n| \to 0 \quad n = 1, \ldots, N \quad . \tag{2.1c}$$

We first establish existence of solutions of (2.1a) in a spherical domain containing all the vortices. Explicitly, we seek a solution of the problem

$$\nabla^2 f^R + \frac{1}{2}(1 - e^{2f^R}) = 0 \tag{2.2a}$$

$$f^R\big|_{r=R} = -\frac{1}{8}R^2 + \sum_{n=1}^{N} m_n \log|\boldsymbol{x} - \boldsymbol{a}_n| + C \log R \tag{2.2b}$$

$$f^R \sim m_n \log|\boldsymbol{x} - \boldsymbol{a}_n| \quad \text{as} \quad |\boldsymbol{x} - \boldsymbol{a}_n| \to 0 \quad n = 1, \ldots, N \tag{2.2c}$$

for any $C > 0$. To avoid the singularities at $(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_N)$ it is convenient to introduce the function

$$g^R = f^R - \sum_{n=1}^{N} m_n \log|\boldsymbol{x} - \boldsymbol{a}_n| \tag{2.3}$$

obviously satisfying

$$\nabla^2 g^R + \frac{1}{2}(1 - \prod_{n=1}^{N} |\boldsymbol{x} - \boldsymbol{a}_n|^{2m_n} e^{2g^R}) = 0 \tag{2.4a}$$

$$g^R\big|_{r=R} = -\frac{1}{8}R^2 + C \log R \quad . \tag{2.4b}$$

Existence of solutions of (2.4) is verified by use the following theorem, which is proved in [12] for more general elliptic operators.

Theorem 2 *Let $f(\boldsymbol{x}, u)$ be C^1 with respect to u and Hölder continuous with respect to $\boldsymbol{x}$ for some $0 < \alpha \leq 1$ in a bounded domain $\Omega \in R^n$. Let ϕ be continuous on $\partial\Omega$. Suppose that functions $\bar{u}$ and $\underline{u}$ exist, such that*

$$\nabla^2 \bar{u} + f(\boldsymbol{x}, \bar{u}) \leq 0 \quad in\ \Omega \qquad \bar{u} \geq \phi \quad on\ \partial\Omega$$

$$\nabla^2 \underline{u} + f(\boldsymbol{x}, \underline{u}) \geq 0 \quad in\ \Omega \qquad \underline{u} \leq \phi \quad on\ \partial\Omega$$

Then, a solution to the problem

$$\nabla^2 u + f(\boldsymbol{x}, u) = 0 \quad \textit{in } \Omega \tag{2.5a}$$

$$u = \phi \quad \textit{on } \partial\Omega \tag{2.5b}$$

exists, and satisfies $\underline{u} \leq u \leq \bar{u}$

The functions $\bar{u}$ and $\underline{u}$ are, respectively, a supersolution and a subsolution of (2.5). In order to find such a supersolution and a subsolution for (2.4), consider the function

$$g_0^R = -\frac{1}{8} r^2 + C \log R$$

Since

$$\nabla^2 g_0^R + \frac{1}{2}(1 - \prod_{n=1}^{N} |\boldsymbol{x} - \boldsymbol{a}_n|^{2m_n} e^{2g_0^R}) \leq 0 \,,$$

g_0^R serves as a supersolution for (2.4). Consider now the problem

$$\nabla^2 g_1^R + \frac{1}{2}(1 - \prod_{n=1}^{N} |\boldsymbol{x} - \boldsymbol{a}_n|^{2m_n} e^{2g_0^R}) = 0 \tag{2.6a}$$

$$g_1^R\big|_{r=R} = -\frac{1}{8} R^2 + C \log R \quad . \tag{2.6b}$$

Inasmuch as this problem is governed by a Poisson equation in a bounded domain it is solvable. Since

$$\nabla^2 (g_1^R - g_0^R) = \frac{1}{2} \prod_{n=1}^{N} |\boldsymbol{x} - \boldsymbol{a}_n|^{2m_n} e^{2g_0^R} \leq 0 \quad ,$$

we obtain by the maximum principle

$$g_1^R \leq g_0^R \quad .$$

Hence, by (2.6a)

$$\nabla^2 g_1^R + \frac{1}{2}(1 - \prod_{n=1}^{N} |\boldsymbol{x} - \boldsymbol{a}_n|^{2m_n} e^{2g_1^R}) \geq 0 \quad .$$

Therefore, g_1^R serves as a subsolution for (2.4) and the existence of g^R is proved.

We next turn to prove equicontinuity of the set of functions g^R for $R > R_0$ in the compact domain $B_{R_0} = \{\boldsymbol{x} \mid |\boldsymbol{x}| \leq R_0\}$ containing all the vortices (other than that R_0 is arbitrary). To this end we first need to prove uniform boundedness of the set $\{g^R\}_{R>R_0}$ in B_{R_0}. The above-obtained estimates g_0^R and g_1^R are insufficient for this purpose because for every fixed $\boldsymbol{x} \in \mathbb{R}^2$

$$\lim_{R\to\infty} g_0^R = \infty$$

$$\lim_{R\to\infty} g_1^R = -\infty$$

A useful upper bound may be obtained by subtraction of $\sum_{n=1}^N m_n \log|\boldsymbol{x} - \boldsymbol{a}_n|$ from Taubes' [13] solution. The result, denoted by u, thus satisfies

$$\nabla^2 u + \frac{1}{2}(1 - \prod_{n=1}^N |\boldsymbol{x} - \boldsymbol{a}_n|^{2m_n} e^{2u}) = 0 \quad ,$$

and

$$u \sim -\sum_{n=1}^N m_n \log|\boldsymbol{x} - \boldsymbol{a}_n| \qquad \text{as } |\boldsymbol{x}| \to \infty \quad ,$$

and since for sufficiently large R (say, for $R > R_0$, because we choose arbitrarily large R_0) we have

$$u|_{r=R} > g^R\big|_{r=R} \quad .$$

Thus, u can serve as a supersolution for (2.4), whence

$$u > g^R \quad \forall R > R_0 \quad \forall \boldsymbol{x} \in B_R \quad . \tag{2.6}$$

Thus,

$$M = \sup_{R>R_0} \sup_{\boldsymbol{x}\in B_{R_0}} g^R \leq \sup_{\boldsymbol{x}\in B_{R_0}} u < \infty$$

Next we estimate $\boldsymbol{\nabla} g^R$ in B_{R_0}. Here we need, (as is evident in the following) to derive an upper bound for g^R which tends to $-\infty$ as $|\boldsymbol{x}| \to \infty$ for $R \to \infty$. We thus define the function

$$\bar{g}^R = \begin{cases} M & r \leq R_0 \\ [C + \epsilon(R)] \log \dfrac{r}{R_0} - \dfrac{1}{8}(r^2 - R_0^2) + M & R_0 < r \leq R \end{cases} \tag{2.8a}$$

wherein

$$\epsilon(R) = \frac{C \log R_0 - R_0^2/8 - M}{\log(R/R_0)} \xrightarrow[R\to\infty]{} 0 \quad . \tag{2.8b}$$

By (2.6), $g^R \leq \bar{g}^R$ in B_{R_0}, whereas in the annular domain $R_0 < r < R$ we have

$$\nabla^2 \bar{g}^R + \frac{1}{2}(1 - \prod_{n=1}^{N} |\boldsymbol{x} - \boldsymbol{a}_n|^{2m_n} e^{2\bar{g}^R}) \geq 0$$

$$\bar{g}^R = g^R \qquad \text{on } \partial B_R$$

$$\bar{g}^R \geq g^R \qquad \text{on } \partial B_{R_0}$$

and hence, $\bar{g}^R \geq g^R \ \forall \boldsymbol{x}$.

We now represent (2.4) in the alternative form

$$g^R = -\frac{1}{8} r^2 + C \log R + \frac{1}{2} \int_{|\boldsymbol{\xi}| \leq R} G(\boldsymbol{x}, \boldsymbol{\xi}) \prod_{n=1}^{N} |\boldsymbol{\xi} - \boldsymbol{a}_n|^{2m_n} e^{2g^R} d\xi \quad , \tag{2.9}$$

where

$$G(\boldsymbol{x}, \boldsymbol{\xi}) = \frac{1}{2\pi} \log |\boldsymbol{x} - \boldsymbol{\xi}| - \frac{1}{2\pi} \log \left(\frac{1}{|\boldsymbol{\xi}| R} \big| |\boldsymbol{\xi}|^2 \boldsymbol{x} - R^2 \boldsymbol{\xi} \big| \right)$$

is the Green's function for Laplace equation in B_R. Taking the gradient of (2.9) we obtain

$$\sup_{\boldsymbol{x} \in B_{R_0}} |\boldsymbol{\nabla} g^R| \leq \frac{1}{4} R_0 + \sup_{\boldsymbol{x} \in B_{R_0}} \int_{|\boldsymbol{\xi}| \leq R} \frac{1}{4\pi |\boldsymbol{x} - \boldsymbol{\xi}|} \prod_{n=1}^{N} |\boldsymbol{\xi} - \boldsymbol{a}_n|^{2m_n} e^{2\bar{g}^R} d\xi \, +$$
$$+ \sup_{\boldsymbol{x} \in B_{R_0}} \int_{|\boldsymbol{\xi}| \leq R} \frac{1}{4\pi} \frac{|\boldsymbol{\xi}|^2 \, \big| |\boldsymbol{\xi}|^2 \boldsymbol{x} - 2R^2 \boldsymbol{\xi} \big|}{\big| |\boldsymbol{\xi}|^2 \boldsymbol{x} - R^2 \boldsymbol{\xi} \big|^2} \prod_{n=1}^{N} |\boldsymbol{\xi} - \boldsymbol{a}_n|^{2m_n} e^{2\bar{g}^R} d\xi \tag{2.10}$$

The first two terms on the right-hand-side of (2.10) are bounded as $R \to \infty$, whereas the last term is of $O(1/R^2)$. Hence,

$$\sup_{R>R_0} \sup_{\boldsymbol{x}\in B_{R_0}} |\boldsymbol{\nabla} g^R| \leq L < \infty \quad . \tag{2.11}$$

It remains to prove uniform boundedness from below of g^R for $R > R_0$ in B_{R_0}. Towards this end we prove

Lemma 1 *Let g^R be a solution of (2.4). Then*

$$\int_{|\boldsymbol{x}|\leq R|} \prod_{n=1}^{N} |\boldsymbol{x} - \boldsymbol{a}_n|^{2m_n} e^{2g^R} dx \geq 4\pi(C+\epsilon)$$

wherein ϵ is defined by (2.8b).

Proof: Since $\bar{g}^R - g^R$ is non-negative in B_R and vanishes on ∂B_R, we must have

$$\left.\frac{\partial g^R}{\partial r}\right|_{r=R} \geq \left.\frac{\partial \bar{g}^R}{\partial r}\right|_{r=R} = \frac{C+\epsilon}{R} - \frac{1}{4}R \quad .$$

Hence,

$$\int_{|\boldsymbol{x}|\leq R} \prod_{n=1}^{N} |\boldsymbol{x} - \boldsymbol{a}_n|^{2m_n} e^{2g^R} dx = 2\int_{|\boldsymbol{x}|\leq R} \left(\nabla^2 g^R + \frac{1}{2}\right) dx =$$

$$= 2\int_{|\boldsymbol{x}|=R} \frac{\partial g^R}{\partial r} ds + \pi R^2 \geq 4\pi(C+\epsilon)$$

$\square$

If we suppose that g^R is not uniformly bounded for $R > R_0$ in B_{R_0}, then a sequence $\{R_k\}_{k=1}^{\infty}$ and a point $\boldsymbol{x}_0 \in B_{R_0}$ exist such that

$$\lim_{k\to\infty} g^{R_k}(\boldsymbol{x}_0) = -\infty \quad .$$

In view of (2.11) we must then have

$$\lim_{k\to\infty} g^{R_k}(\boldsymbol{x}) = -\infty \qquad \forall x \in B_{R_0} \quad .$$

Hence,

$$\forall \delta > 0,\ \exists K(\delta) \quad \text{such that} \quad k > K \Rightarrow \exp\{2g^{R_k}\} \leq \delta \quad \forall \boldsymbol{x} \in B_{R_0}$$

In the annular domain $R_0 \leq r \leq R_k$ we can show, similarly to the derivation of (2.8), that

$$g^{R_k} \leq [C + \bar{\epsilon}(R_k)] \log \frac{r}{R_0} - \frac{1}{8}(r^2 - R_0^2) + \frac{1}{2} \log \delta \quad ,$$

wherein

$$\bar{\epsilon}(R_k) = \frac{C \log R_0 - R_0^2/8 - (log \delta)/2}{\log(R_k/R_0)} \quad .$$

Therefore, evidently

$$k > K \Rightarrow \int_{|\boldsymbol{x}| \leq R_k|} \prod_{n=1}^{N} |\boldsymbol{x} - \boldsymbol{a}_n|^{2m_n} \exp\{2g^{R_k}\} dx < A\delta \quad , \tag{2.12}$$

wherein A is independent of both k and δ. The last result contradicts lemma 1, and hence the set of functions $\{g^R\}_{R>R_0}$ must therefore be bounded uniformly from below in B_{R_0}.

From the uniform boundedness of $\{g^R\}_{R>R_0}$ and $|\boldsymbol{\nabla} g^R|_{R>R_0}$ in B_{R_0} and the Arzela-Ascoli theorem follows the existence of an increasing sequence $\{R_k\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} R_k = \infty$ and

$$\sup_{\boldsymbol{x} \in B_{R_0}} |g^{R_k} - g| \xrightarrow[k\to\infty]{} 0$$

where $g \in C(B_{R_0})$. The latter limit is valid for any positive R_0, hence g is well defined and continuous for any $\boldsymbol{x} \in \mathbb{R}^2$. Furthermore, by (2.8)

$$g \leq C \log \left(\frac{r}{\mathfrak{R}_0} \right) - \frac{1}{8}(r^2 - \mathfrak{R}_0^2) + M \qquad \forall r \geq \mathfrak{R}_0 \tag{2.13}$$

in which $\mathfrak{R}_0$ can assume any value of R_0 for which $\bar{g}^R$ serves as an upper bound for g^R.

To complete the proof of theorem 1 we need to show that g satisfies (2.4). We, thus, represent (2.9) in the form

$$g^{R_k} = -\frac{1}{8}r^2 + \left[C - \frac{1}{4\pi}\int_{|\boldsymbol{x}|\leq R_k|}\prod_{n=1}^{N}|\boldsymbol{x}-\boldsymbol{a}_n|^{2m_n}\exp\{2g^{R_k}\}dx\right]\log R_k +$$
$$+\frac{1}{4\pi}\int_{|\boldsymbol{\xi}|\leq R_k}\log|\boldsymbol{x}-\boldsymbol{\xi}|\prod_{n=1}^{N}|\boldsymbol{\xi}-\boldsymbol{a}_n|^{2m_n}\exp\{2g^{R_k}\}d\xi -$$
$$-\frac{1}{8\pi}\int_{|\boldsymbol{\xi}|\leq R_k}\log\left(1+\frac{|\boldsymbol{x}|^2|\boldsymbol{\xi}|^2}{R_k^4}-\frac{2\boldsymbol{x}\cdot\boldsymbol{\xi}}{R_k^2}\right)\prod_{n=1}^{N}|\boldsymbol{\xi}-\boldsymbol{a}_n|^{2m_n}\exp\{2g^{R_k}\}d\xi \quad . \qquad (2.14)$$

In view of (2.13), the last term on the right-hand-side of (2.14) must vanish as $k \to \infty$. For the third term we have

$$\lim_{k\to\infty}\frac{1}{4\pi}\int_{|\boldsymbol{\xi}|\leq R_k}\log|\boldsymbol{x}-\boldsymbol{\xi}|\prod_{n=1}^{N}|\boldsymbol{\xi}-\boldsymbol{a}_n|^{2m_n}\exp\{2g^{R_k}\}d\xi =$$
$$=\frac{1}{4\pi}\int_{\mathbb{R}^2}\log|\boldsymbol{x}-\boldsymbol{\xi}|\prod_{n=1}^{N}|\boldsymbol{\xi}-\boldsymbol{a}_n|^{2m_n}e^{2g}d\xi \quad . \qquad (2.15)$$

In order for g to be single-valued and bounded we must have

$$\lim_{k\to\infty}\left[C - \frac{1}{4\pi}\int_{|\boldsymbol{x}|\leq R_k|}\prod_{n=1}^{N}|\boldsymbol{x}-\boldsymbol{a}_n|^{2m_n}\exp\{2g^{R_k}\}dx\right]\log R_k = b < \infty \quad , \qquad (2.16)$$

hence

$$\frac{1}{4\pi}\int_{\mathbb{R}^2}\prod_{n=1}^{N}|\boldsymbol{x}-\boldsymbol{a}_n|^{2m_n}e^{2g}dx = C \quad . \qquad (2.17)$$

Combination of (2.14), (2.15), and (2.16) yields

$$g = -\frac{1}{8}r^2 + b + \frac{1}{4\pi}\int_{\mathbb{R}^2}\log|\boldsymbol{x}-\boldsymbol{\xi}|\prod_{n=1}^{N}|\boldsymbol{\xi}-\boldsymbol{a}_n|^{2m_n}e^{2g^R}d\xi \quad , \qquad (2.18)$$

consequently,

$$\nabla^2 g + \frac{1}{2}\left(1-\prod_{n=1}^{N}|\boldsymbol{x}-\boldsymbol{a}_n|^{2m_n}e^{2g}\right) = 0 \quad , \qquad (2.19a)$$

and, in view of (2.18),

$$g \sim -\frac{1}{8}r^2 + C\log r \qquad \text{as } r \to \infty \quad . \tag{2.19b}$$

From (2.19) it follows that

$$f = g + \sum_{n=1}^{N} m_n \log |\boldsymbol{x} - \boldsymbol{a}_n|$$

is a solution for (2.1) for any $C > 0$. Theorem 1 is thus proved.

By the very construction of f, at least one solution exists for each $C > 0$. Furthermore, we can arbitrarily choose the center of B_R. As we demonstrate in the following, the latter merely represents a translation of the solution and the locations of the vortices.

The most general solution of (2.19a) can be represented

$$g = -\frac{1}{8}r^2 + \psi + \frac{1}{4\pi}\int_{\mathbb{R}^2} \log|\boldsymbol{x} - \boldsymbol{\xi}| \prod_{n=1}^{N} |\boldsymbol{\xi} - \boldsymbol{a}_n|^{2m_n} e^{2g^R} d\xi \quad , \tag{2.20}$$

where ψ is harmonic in $\mathbb{R}^2$. In order to satisfy (2.1b)

$$\psi = \alpha_1 x + \alpha_2 y + \beta \quad .$$

By relocating the origin we may eliminate the dependence of ψ on x and y. The solution thus obtained corresponds to an appropriately translated set of vortex locations.

In the following we demonstrate that for a fixed origin location and a given value of C the solution must be unique.

Theorem 3 *Let f_1 and f_2 be solutions of (2.1) for a given set of vortex locations $(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_N)$ and respective orders $(m_1, \ldots, m_N)$ and a given origin location. Let*

$$\int_{\mathbb{R}^2} e^{2f_1} dx = 4\pi C_1 \qquad ; \qquad \int_{\mathbb{R}^2} e^{2f_2} dx = 4\pi C_2$$

in which $C_2 > C_1 > 0$. Then, $f_2 > f_1$ $\forall \boldsymbol{x} \in \mathbb{R}^2$. Furthermore, if $C_1 = C_2$ then $f_1 \equiv f_2$

Proof: Set $\phi = f_2 - f_1$. Clearly,

$$\nabla^2 \phi - \frac{1}{2} \frac{e^{2f_1}(e^{2\phi} - 1)}{\phi} \phi = 0$$

and, by (2.20), we have, as $r \to \infty$, $\phi \sim (C_2 - C_1) \log r$. Hence, when $C_2 > C_1$, by the maximum principle, ϕ must be positive everywhere. Suppose now that $C_1 = C_2$, then, $\phi \sim (b_2 - b_1)$ as $r \to \infty$. By the maximum principle we have $\phi \equiv (b_2 - b_1)$. In view of the equation ϕ, it then necessarily vanishes identically.

$\square$

Theorems 1 and 3 demonstrate an analogy between the solutions of the one-dimensional (cf. [7],[5], [2]) and the two-dimensional Bogomolnyi equations. The Taubes' [13] solution is similar to the solution $\psi \equiv 1$, $A \equiv 0$, and the solutions presented in this section are analogous to the symmetric solutions presented by Chapman [5]. Despite this analogy there are major differnces between the one-dimensional and the two dimensional cases, e.g., one-dimensional solutions do not admit vortices.

Finally, we examine the behavior of solutions in the limit $C \to \infty$. Theorem 3 shows that $f(\boldsymbol{x}, C)$ is a monotonically increasing function of C for any $\boldsymbol{x} \in \mathbb{R}^2$ and a given set of vortex locations and orders. Clearly, $f(\boldsymbol{x}, C)$ is bounded by Taubes' solution, and therefore the limit of $f(\boldsymbol{x}, C)$ as $C \to \infty$ must exist. It is possible to show that the limit must be a solution of (2.1) as well . For the radially symmetric case, when at most a single vortex is present, the limit is necessarily Taubes' solution. This is so because if the limit of f as $\boldsymbol{x} \to \infty$ exists, it must be either $-\infty$ or 0. In the former case C must be finite since $\nabla^2 f \sim -1/2$ and hence $f \sim -r^2/8$. Consequently, f must vanish at infinity, and is thus identical with Taubes' solution, which is the unique solution in that case.

When a number (larger than one) of vortices are present the solution is not radially symmetric and $\lim_{r\to\infty} f$ may not exist. It is not clear in that case whether f indeed converges to Taubes' solution as $C \to \infty$.

3 Conclusion

In the following we summarize the main results obtained in this work as well as point out some open questions.

For the critical case $\kappa = h = 1/\sqrt{2}$, we have extended to the real plane the results obtained by Chapman [5] on the real line. Chapman [5] has obtained closed-form solutions satisfying the boundary conditions $\psi \to 0$ and $A' \to 1/\sqrt{2}$ as $|x| \to \infty$, which characterize the normal state. These solutions converge pointwise as $\int \psi^2 dx \to \infty$ to the purely superconducting state $\psi \equiv 1$, $A \equiv 0$. In the two-dimensional case the situation is more complicated. The solutions are characterized in that case by a set of vortex locations and orders. For each set we can find a unique solution which tends to a perfectly superconducting state at infinity. In addition, as is demonstrated in section 2, there exists a whole family of solutions, characterized by the same set of vortex locations and orders which tend to a normal state at infinity. Each solution of this family is characterized by the parameter $C = \int |\psi|^2 dx/4\pi$ which may be interpreted as the total amount of superconductivity in the plane. Furthermore, these solutions increase monotonically with C for any $\boldsymbol{x} \in \mathbb{R}^2$.

The situation in the limit $C \to \infty$, for any arbitrary set of vortex locations and orders, is not yet clear. Naturally, one expects that the limit would be Taubes' [13] solution corresponding to the same set of vortex locations and orders. However, this result was proved here only for radially-symmetric cases, where at most a single vortex is present. In other (non-radially symmetric) cases, it is not clear whether the limit function, which must solve (2.1a), is indeed identical with Taubes' solution.

Radially symmetric solutions, describing a normal state at infinity, have recently been reported by Obukhov & Schunck [10] for values of κ which are not necessarily close to the critical value. Much of the report in [10] is devoted to discussion of results obtained via numerical integration of (1.1) in radially-symmetric cases. The analysis in section 2 proves existence of such solutions for the critical case.

Finally, we comment on the energy of the self-duality solutions whose existence was proved in section 2. Since the energy of the normal state is infinite, in view of (1.1c), all the solutions must have infinite energy. Yet the difference between their energy and the energy of the normal state is finite. In fact this difference identically vanishes for all of these solutions. To demonstrate this fact we first write the difference between the Gibbs free energies of the superconducting solution and the normal state in the form [6]

$$J = \int \left(-|\psi|^2 + \frac{|\psi|^4}{2} + |\boldsymbol{H} - h\hat{\boldsymbol{k}}|^2 + \left| \frac{1}{\kappa} \boldsymbol{\nabla} \psi - i \boldsymbol{A} \psi \right|^2 \right) dx dy$$

Then, rearranging the integrand yields

$$J = \int \left[J_1^2 + J_2^2 + J_3^2 + \nabla \times \left(\sqrt{2}|\psi|^2 \boldsymbol{A} + i(\psi^* \nabla \psi - \psi \nabla \psi^*) \right) \cdot \hat{\boldsymbol{k}} \right] dxdy +$$

$$\int \left[(\sqrt{2} - 2h)H + h^2 - \frac{1}{2} + \left(\frac{1}{\kappa^2} - 2 \right) |\nabla \psi|^2 + i \left(\frac{1}{\kappa} - \sqrt{2} \right) (\psi^* \boldsymbol{A} \cdot \nabla \psi - \psi \boldsymbol{A} \cdot \nabla \psi^*) \right] dxdy$$

wherein J_1, J_2, and J_3 are the respective right hand sides of (1.2a), (1.2b), and (1.2c). Obviously, For the critical case, $h = \kappa = 1/\sqrt{2}$, the minimum is obtained when $J = J_1 = J_2 = J_3 = 0$. Hence, all the self-duality solutions we have obtained, are energetically equivalent to the normal state.

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References

[1] A. A. Abrikosov. On the magnetic properties of superconductors of the second group. *Soviet Phys. J.E.T.P.* **5**, 1175–1204 (1957)

[2] Y. Almog. Asymptotic analysis of the one-dimensional Ginzburg-Landau equations. Quart. Appl. Math. to appear

[3] M. S. Berger. Creation and Breaking of Self-Duality Symmetry - a modern Aspect of Calculus of Variation. *Cont. Math.* **17**, 379–394 (1983)

[4] E. Bogomolnyi. The stability of classical solutions. *Sov. Jour. Nucl. Phy.* **24**, 449–454 (1976)

[5] S. J. Chapman. Nucleation of Superconductivity in Decreasing fields I. *EJAM* **5**, 449–468 (1994)

[6] S. J. Chapman. Asymptotic analysis of the Ginzburg-Landau model of superconductivity: reduction to a free boundary model. *Quart. Appl. Math.* **53**, 601–627 (1995)

[7] S. J. Chapman, S. D. Howison, J. B. Mcleod, and J. R. Ockendon. Normal/Superconducting transition in Landau-Ginzburg Theory. *Proc. Roy. Soc. Edinburgh* **119A**, 117–124 (1991)

[8] S. J. Chapman, S. D. Howison, and J. R. Ockendon. Macroscopic models for superconductivity. *SIAM Review* **34**, 529–560 (1992)

[9] L. Jacobs and C. Rebbi. Interaction energy of superconducting vortices. *Phys. Rev. B* **19**, 4486–4494 (1979)

[10] Y. N. Obukhov and F. E. Schunck. Regular solutions in Abelian gauge model. *Phys. Rev. D* **55**, 2307–2319 (1997)

[11] L. Peres and J. Rubinstein. Vortex dynamics in U(1) Ginzburg-Landau models. *Phys. D* **64**, 299–309 (1993)

[12] J. Smoller. *Shock Waves and Reaction-Diffusion Equations*. Springer (1983)

[13] C. H. Taubes. Arbitrary N-vortex solutions to the first order Ginzburg-Landau equations. *Commn. Math. Phys.* **72**, 277–292 (1980)