On the bifurcation and stability of periodic solutions of the Ginzburg-Landau equations in the plane

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Abstract

The linear bifurcation and stability of periodic solutions to the Ginzburg-Landau Equations in the plane are investigated. In particular, we find new infinite families of solutions, which include the few solutions previously reported in the literature. Then, the vortex structure of these new solutions is examined. In addition, the energy of a large class of solutions is approximated in the limit case for which the fundamental cell is a very thin and long rectangle. In that limit, we

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find that the energy of the solution representing the well-known triangular lattice is the lowest. Finally, we examine the stability of one infinite family of solutions, including both the triangular and square lattices, in an infinite-dimensional space of perturbations (in contrast to a previous work in which stability was examined only in a finite-dimensional space). We find that in addition to the triangular lattice other solutions are stable as well.

1 Introduction

Periodic solutions to the Ginzburg-Landau equations have first been obtained by Abrikosov [1], who analyzed the bifurcation of these solutions from the normal state. Abrikosov [1] focused attention on the square lattice which is the only periodic solution which has one vortex in its fundamental cell. Following the same procedure in [1], Kleiner *et al.* [10] found that the energy of the triangular lattice, which is the solution possessing two vortices in its fundamental cell, is lower than that of the square lattice. The triangular lattice has also been observed in experiments [7], and is therefore believed to be the only stable periodic solution.

In a recent contribution, Chapman [3] presented Abrikosov's anal-

ysis [1] as a formal asymptotic expansion by applying the framework in [11] for the bifurcation of weakly non-linear solutions to the Ginzburg-Landau equations. In addition, Chapman [3] obtained new solutions possessing either three or four vortices within the unit cell. He also examined, in [4] the linear stability of the square lattice and the triangular one, and found that the square lattice is unstable, whereas the triangular lattice was found to be stable for two modes of perturbations.

Evidently, linear bifurcation analysis as well as linear stability analysis are far from being complete. Only very few solutions, out of infinity, have been found. Furthermore, their stability was examined only in a finite dimensional space of perturbations. The present contribution, thus, extends both the linear bifurcation and the linear stability analyzes of periodic solutions. We obtain infinite sets of solutions (including all the solutions that were previously derived), and describe the vortex structures manifested by some of these solutions. We also demonstrate that the energy of the triangular lattice is the lowest, at a certain asymptotic limit, in a class which may contain all possible weakly non-linear periodic solutions. Finally, we examine the linear stability of one of the infinite families of solutions obtained in this

work. In contrast to [4] the space of perturbations in which stability is being examined is infinite-dimensional.

The Ginzburg-Landau energy functional may be represented in the following dimensionless form [5]

$$E = \int \left(-|\psi|^2 + \frac{|\psi|^4}{2} + |\boldsymbol{H}|^2 + \left| \frac{1}{\kappa} \boldsymbol{\nabla} \psi - i \boldsymbol{A} \psi \right|^2 \right) dxdy \qquad (1.1)$$

in which Ψ is the (complex) superconducting order parameter, such that $|\Psi|$ varies from $|\Psi| = 0$ (when the material is at a normal state) to $|\Psi| = 1$ (for the purely superconducting state). The magnetic vector potential is denoted by $\boldsymbol{A}$ (the magnetic field is, then, given by $\boldsymbol{H} = \boldsymbol{\nabla} \times \boldsymbol{A}$), and κ is the Ginzburg-Landau parameter which is a material property. Superconductors for which $\kappa < 1/\sqrt{2}$ are termed type I superconductors, and those for which $\kappa > 1/\sqrt{2}$ have been termed type II. Note that E is invariant to the gauge transformation

$$\Psi \to e^{i\kappa\eta} \psi \qquad ; \qquad \boldsymbol{A} \to \boldsymbol{A} + \boldsymbol{\nabla} \eta \qquad (1.2)$$

We seek periodic local minimizers of E in the xy plane, i.e., we require both $|\Psi|$ and $\boldsymbol{H}$ to be periodic. The Euler-Lagrange equations associated with E (the steady state Ginzburg-Landau equations) are

given by

$$\left(\frac{i}{\kappa}\boldsymbol{\nabla}+\boldsymbol{A}\right)^2\Psi=\Psi\left(1-|\Psi|^2\right) \quad , \tag{1.3a}$$

$$-\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{A})=\frac{i}{2\kappa}\left(\Psi^*\boldsymbol{\nabla}\Psi-\Psi\boldsymbol{\nabla}\Psi^*\right)+|\Psi|^2\boldsymbol{A} \quad , \tag{1.3b}$$

What are the natural boundary conditions of the problem? Let $(\psi,\boldsymbol{A})$ be a solution of (1.3), and let $\psi=\rho(x,y)e^{i\theta(x,y)}$. Both ρ and $\boldsymbol{H}$ have to be periodic on the boundary of any fundamental periodic cell, i.e., any rectangle $R=[x,x+L_x]\times[y,y+L_y]$. Let $\boldsymbol{t}$ denote either $L_x\hat{\boldsymbol{i}}$ or $L_y\hat{\boldsymbol{j}}$, and let $\boldsymbol{x}=(x,y)$. Then,

$$\psi(\boldsymbol{x}+\boldsymbol{t})=e^{i\kappa\zeta(\boldsymbol{x},\boldsymbol{t})}\psi(\boldsymbol{x}) \quad ; \quad \boldsymbol{A}(\boldsymbol{x}+\boldsymbol{t})=\boldsymbol{A}(\boldsymbol{x})+\boldsymbol{\nabla}\Upsilon(\boldsymbol{x},\boldsymbol{t})\ . \tag{1.4}$$

Since the zero set of ψ is discrete [6] we can substitute (1.4) in (1.3) to obtain $\boldsymbol{\nabla}\Upsilon=\boldsymbol{\nabla}\zeta$ [cf. also [12]]. The latter is equivalent to the "boundary condition"

$$\rho\left[\frac{1}{\kappa}\boldsymbol{\nabla}\theta-\boldsymbol{A} \quad --- \quad \text{periodic on } \partial R\right] \tag{1.5}$$

The above condition together with the requirement that ρ and $\boldsymbol{H}$ should both be periodic constitute the natural boundary conditions of the problem. Chapman [3] proposed boundary conditions different than (1.5). Instead he suggested to impose periodicity of $\left(\frac{i}{\kappa}\boldsymbol{\nabla}+\boldsymbol{A}\right)\psi\cdot\hat{\boldsymbol{n}}$. Chapman's condition is, however, too restrictive. In fact, none of the solutions derived in [3] satisfy it.

As the Ginzburg-Landau equations are invariant to (1.2) we may choose the gauge (following [1, 3]) $\boldsymbol{A} = (0, A(x,y), 0)$. Then, $\boldsymbol{H} = (0, 0, H(x,y))$ and $H = \partial A/\partial x$. We investigate the linear bifurcation from the normal state $\Psi \equiv 0$, $A = hx$, with h serving as a bifurcation parameter. Following the same steps taken in [3] we assume first the asymptotic expansion

$$\Psi = \epsilon^{1/2}\psi \tag{1.6a}$$

$$A = hx + \epsilon a \tag{1.6b}$$

$$h = h^{(0)} + \epsilon h^{(1)} + \dots \tag{1.6c}$$

$$a = a^{(0)} + \epsilon a^{(1)} + \dots \tag{1.6d}$$

$$\psi = \psi^{(0)} + \epsilon\psi^{(1)} + \dots \tag{1.6e}$$

The linearized equations possess the form

$$-\frac{1}{\kappa^2}\left(\frac{\partial^2\psi^{(0)}}{\partial x^2} + \frac{\partial^2\psi^{(0)}}{\partial y^2}\right) + \frac{2ih^{(0)}x}{\kappa}\frac{\partial\psi^{(0)}}{\partial y} = \psi^{(0)} - (h^{(0)})^2x^2\psi^{(0)} \quad , \tag{1.7a}$$

$$-\frac{\partial^2 a^{(0)}}{\partial x\partial y} = \frac{i}{2\kappa}\left(\psi^{(0)*}\frac{\partial\psi^{(0)}}{\partial x} - \psi^{(0)}\frac{\partial\psi^{(0)*}}{\partial x}\right), \tag{1.7b}$$

$$\frac{\partial^2 a^{(0)}}{\partial x^2} = h^{(0)})x|\psi^{(0)}|^2 + \frac{i}{2\kappa}\left(\psi^{(0)*}\frac{\partial\psi^{(0)}}{\partial y} - \psi^{(0)}\frac{\partial\psi^{(0)*}}{\partial y}\right). \tag{1.7c}$$

The boundary condition (1.5) for the specific gauge we have chosen

become after linearization

$$\left.\frac{\partial\theta}{\partial x}\right|_{(x,y)}^{(x,y+L_y)} = \left.\frac{\partial\theta}{\partial y}\right|_{(x,y)}^{(x,y+L_y)} = 0 \qquad (1.8a,b)$$

$$\left.\frac{\partial\theta}{\partial x}\right|_{(x,y)}^{(x+L_x,y)} = 0 \quad ; \quad \left.\frac{\partial\theta}{\partial y}\right|_{(x,y)}^{(x+L_x,y)} = \kappa h^{(0)} L_x \qquad (1.8c,d)$$

Integrating (1.8a,b) and (1.8c,d) we obtain

$$\theta|_{(x,y)}^{(x,y+L_y)} = \Lambda \quad ; \quad \theta|_{(x,y)}^{(x+L_x,y)} = \kappa h^{(0)} L_x y + \Theta \,, \qquad (1.9a)$$

where Λ and θ are constants. Hence the overall variation of θ along ∂R is $\kappa h^{(0)} L_x L_y$. Continuity of $\psi^{(0)}$, then, yields

$$\kappa h^{(0)} L_x L_y = 2\pi N \qquad (1.10)$$

where N is the number of vortices within R, or the winding number of $\psi^{(0)}$.

If $\psi(x,y)$ is a solution of (1.7), then so is $e^{-i\Lambda y}\psi(x-\Lambda/\kappa, y)$. Yet, the latter solution is periodic in y in view of (1.9a). We can therefore substitute into (1.7) without any loss of generality the Fourier expansion

$$\psi^{(0)} = \sum_{n=-\infty}^{\infty} e^{ikny} g_n(x) \qquad (1.11)$$

to obtain

$$-\frac{1}{\kappa^2} g_n'' = \left(1 - \frac{k^2 n^2}{\kappa^2} + \frac{2kn}{\kappa} h^{(0)} x - (h^{(0)} x)^2\right) g_n \,. \qquad (1.12)$$

Note that the sequence $\{g_n(x)\}_{n=1}^{\infty}$ must be in ℓ^2 for any fixed value of x. The periods in the x and y directions are given in terms of the new variables k and N, in view of (1.10) and (1.11), as

$$L_y = \frac{2\pi}{k} \quad ; \quad L_x = \frac{kN}{\kappa h^{(0)}} \tag{1.13}$$

Applying the transformation $x \to x - nk/\kappa h^{(0)}$ to (1.12) we obtain

$$\psi^{(0)} = \sum_{n=-\infty}^{\infty} e^{ikny} \left[C_n^1 g^{(1)} \left(x - \frac{nk}{\kappa h^{(0)}} \right) + C_n^2 g^{(2)} \left(x - \frac{nk}{\kappa h^{(0)}} \right) \right] , \tag{1.14}$$

in which $g^{(1)}(x)$ and $g^{(2)}(x)$ are the fundamental solutions of (1.12) for $n = 0$.

In view of (1.13), (1.9) and the periodicity of ρ we have

$$\psi^{(0)}(x + L_x, y) = e^{i(kNy+\Theta)} \psi^{(0)}(x, y) \, . \tag{1.15}$$

Substitution of the above relation into (1.14) yields

$$C_{n+N}^j = e^{i\Theta} C_n^j \quad ; \quad j = 1, 2 \tag{1.16}$$

As the sequence

$$\left\{ C_n^1 g^{(1)} \left(x - \frac{nk}{\kappa h^{(0)}} \right) + C_n^2 g^{(2)} \left(x - \frac{nk}{\kappa h^{(0)}} \right) \right\}_{n=1}^{\infty}$$

must be in ℓ^2 for all x, at least one solution of (1.12) has to be in $L_2(\mathbb{R})$ in view of (1.16). Non-trivial L^2 solutions to (1.12) with $n = 0$

exist when $h^{(0)} = \kappa/(2n+1)$. The largest eigenvalue is $h^{(0)} = \kappa$, with the corresponding eigenfunction

$$g_0 = \exp\left\{-\frac{\kappa^2}{2}\left(x - \frac{k}{\kappa^2}\right)^2\right\} \quad . \tag{1.17}$$

Substituting in (1.14) we obtain the general periodic solution to (1.7a) for $h^{(0)} = \kappa$

$$\psi^{(0)} = \sum_{n=-\infty}^{\infty} C_n \exp\left\{ikny - \frac{\kappa^2}{2}\left(x - \frac{nk}{\kappa^2}\right)^2\right\} \quad . \tag{1.18}$$

where $C_{n+N} = C_n e^{i\Theta}$, N is a natural number, and $\Theta \in [0, 2\pi)$. Since we may remove Θ by applying the transformation $y \to y - \Theta/kN$, we shall require $C_{n+N} = C_n$ in the sequel.

We may now solve for (1.7b) and (1.7c), and then, by applying the expansion (1.6), obtain the next order balance for $\psi^{(1)}$ which is an inhomogeneous version of (1.7a). The orthogonality condition which must be satisfied in order for the inhomogeneous next order balance to be solvable, is

$$\frac{1}{\sqrt{2}}\left(\frac{1}{2\kappa^2} - 1\right) \sum_{r,m=-\infty}^{\infty} C_{n-r+m} C_m^* C_r \exp\left\{-\frac{\kappa^2}{2}\left[(r-m)^2 + (r-n)^2\right]\right\} - \frac{h^{(1)}}{\kappa} C_n = 0 \ ,$$

where $0 \le n \le N-1$. The detailed derivation of the above condition can be found in [3]. It is more convenient to write the above condition

in the form

$$\sum_{r=0}^{N-1}\sum_{m=0}^{N-1} C_{n+r}C^*_{n+r+m}C_{n+m}S_{N,r}S_{N,m} - \bar{h}^{(1)}C_n = 0 \qquad (1.19a)$$

wherein $0 \leq n \leq N-1$,

$$\bar{h}^{(1)} = \frac{2\sqrt{2}\kappa}{1-2\kappa^2}h^{(1)} \ , \qquad (1.19b)$$

and

$$S_{N,r} = \sum_{n=-\infty}^{\infty} q^{(Nn+r)^2} \qquad ; \qquad q = e^{-\frac{k^2}{2\kappa^2}} \qquad (1.19c)$$

As was pointed out in the beginning of this section, a few solutions to (1.19) are already known. The most obvious one is

$$C_n = C \qquad (1.20)$$

which is the only solution for $N = 1$. For $k = \kappa\sqrt{2\pi}$ it is known as Abrikosov's square lattice. For $N = 2$ we have, in addition to (1.20), the solution

$$C_{2n} = C \qquad , \qquad C_{2n+1} = \pm iC \quad , \qquad (1.21)$$

which for $k = \kappa\sqrt{\pi\sqrt{3}}$ is known as the triangular lattice. Chapman [3] obtained some of the solutions for $N = 3$ and $N = 4$.

The following transformations leave (1.19) invariant

$$C_n \to e^{i\theta}C_n \qquad \forall n \in \mathbb{N} \quad ; \quad \theta \in \mathbb{R} \quad , \qquad (1.22a)$$

which changes only the phase of ψ in view of (1.18).

$$C_n \to e^{i2\pi n/N} C_n \quad , \tag{1.22b}$$

which is equivalent to the translation: $y \to y + L_y/N$.

$$C_n \to C_{n+1} \quad , \tag{1.22c}$$

or equivalently, $x \to x + k/\kappa^2$, and a phase change in ψ.

$$C_n \to C_{N-n} \quad , \tag{1.22d}$$

or $x \to -x$, $\psi \to \psi^*$. Another important property of any solution of (1.19) is that $C_n(\bar{h}^{(1)}) = \sqrt{\bar{h}^{(1)}}\, C_n(1)$, as can be verified by direct substitution. This would allow us to consider only $\bar{h}^{(1)} = 1$ in the sequel.

The rest of this contribution is arranged as follows: In the next section we obtain some of the properties of the solutions which are analytic functions of q near $q = 0$. Some of these solutions will be derived in closed form. In § 3 we obtain the lattices, or the vortex structures, embedded in some of the solutions obtained in § 2, and discuss their symmetry properties. In § 4 we demonstrate that in the limit $q \to 0$ (1.21) has the lowest energy of all solutions which are analytic functions of q, and thus support the claim that the triangular

lattice has the lowest energy of all the periodic solutions of (1.3). In § 5 we analyze the local stability of some of the solutions derived in § 2, including the square lattice, the triangular lattice, and the solution which has been obtained for $N = 3$ [3]. In the last section, we summarize the main results of this work and extend the discussion of several key points, insufficiently emphasized within the analysis. Finally, the appendix includes an analysis of the geometry of periodic lattices which are invariant to 180° rotations with respect to each point in the lattice.

2 Solutions as analytic functions of q

In the following we investigate some of the properties of the solutions of (1.19) which are analytic functions of q near $q = 0$, i.e.,

$$C_n = \sum_{k=0}^{\infty} a_{nk} q^k \quad . \tag{2.1}$$

In general, solutions of a polynomial system whose coefficients are analytic functions of a parameter [including (1.19)], need not necessarily be analytic functions of that parameter. For instance the solutions may be meromorphic functions of that parameter.

In this section attention is focused, however, on analytic solutions

only. Though we cannot prove this in general, it appears reasonable to believe that all solutions can essentially be described by (2.1). The exception is, of course, the degrees of freedom the system (1.19) may possess. It easily follows from (1.22a) that at least one degree of freedom always exists: the arbitrary parameter θ. Thus (2.1) can be true only when $\theta(q)$ is analytic near $q = 0$.

Substituting (2.1) into (1.19) we obtain the recurrence relation

$$a_{nk} = \sum_{r^2+s^2 \le k} \sum_{\substack{m,j \ge 0 \\ m+j \le M}} a_{(n+r)j} a^*_{(n+r+s)(M-m-j)} a_{(n+s)m} \quad , \qquad (2.2)$$

where $M(r,s) = k - r^2 - s^2$. (Note that in the above r and s may be negative.) For $k = 0$ it reduces to

$$a_{n0} = |a_{n0}|^2 a_{n0} \quad , \qquad (2.3)$$

and thus, $|a_{n0}| \in \{0, 1\}$. As an immediate corollary of (2.3), it is possible now to show that at least all real solutions of (1.19) must be analytic functions of q near $q = 0$: Any formal series of q derived from (2.2) must converge in some neighborhood of $q = 0$ (cf. [2]). Since the number of real formal series satisfying (2.2) is [by (2.3)] 3^N, which is, by Bezout theorem [9], the maximal number of real solutions to (1.19), every real solution must have the form (2.1) in the vicinity of $q = 0$. This corollary supports our conjecture that any solution of

(1.19) can essentially be described by (2.1).

Substituting (2.3) into (2.2) with $k = 1$ we obtain

$$a_{n1} = [-2 + ib_{n1}]a_{n0} \tag{2.4}$$

where the b_{n1}'s are arbitrary. The apparent non-uniqueness of a_{n1} will be subsequently examined.

Substituting (2.3) and (2.4) in (2.2) with $k = 2$ we obtain the solvability condition

$$\Im\left\{a^*_{(n+2)0}a^2_{(n+1)0}a^*_{n0} + 2a_{(n+1)0}(a^*_{n0})^2a_{(n-1)0} + a^*_{(n-2)0}a^2_{(n-1)0}a^*_{n0}\right\} = 0 \tag{2.5}$$

which must be satisfied whenever $|a_{n0}| = 1$. Setting $a_{n0} = \exp\{i\theta_n\}$ we obtain

$$\sin\chi_{n+1} - 2\sin\chi_n + \sin\chi_{n-1} = 0 \tag{2.6a}$$

where

$$\chi_n = \theta_{n+1} - 2\theta_n + \theta_{n-1} \tag{2.6b}$$

Hence, in the case $|a_{n0}| = 1$ for all n we must have $\sin\chi_n$ =const., or

$$\chi_n \in \{\chi, \pi - \chi\} \quad , \quad \text{where } \chi \in [-\pi/2, \pi/2] \tag{2.7}$$

and hence

$$\cos\chi_n = \alpha_n \cos\chi \tag{2.8}$$

where $\alpha_n \in \{-1, 1\}$. Periodicity then implies,

$$\sum_{n=0}^{N-1} \chi_n = \left(\frac{\pi}{2} - \chi\right) \sum_{n=0}^{N-1} \alpha_n + N\frac{\pi}{2} = 2\pi L \tag{2.9}$$

where $0 \leq L \leq N-1$ is an integer. We may now distinguish between two different cases:

$$\sum_{n=0}^{N-1} \alpha_n \neq 0 \tag{2.10a}$$

and

$$\sum_{n=0}^{N-1} \alpha_n = 0 \quad . \tag{2.10b}$$

In the first case χ can assume at most $N-1$ distinct values. In the second case, solutions may exist if and only if N is divisible by 4, in which case χ may assume any value in the interval $[-\pi/2, \pi/2]$. The foregoing discussion thus explains the non-uniqueness of solutions for $N = 4$ which was discovered in [3], as in this case the solutions depend on an additional arbitrary parameter: χ.

If (2.6) is satisfied, (2.2) with $k = 2$ yields

$$a_{n2} = \frac{1}{2}\left[12 - \cos\chi\left(\alpha_{n+1} + 2\alpha_n + \alpha_{n-1}\right) - b_{n1}^2 + ib_{n2}\right] \tag{2.11}$$

We now focus on the special class of solutions for which $|C_n| = |C|$ for all n. Denote this class by $\mathcal{A}$ ($\boldsymbol{C} \in \mathcal{A}(\subset \mathbb{C}^N) \Rightarrow |C_n| = |C| \quad \forall 0 \leq n \leq$

$N-1$). Clearly, every solution in $\mathcal{A}$ must satisfy $|a_{n0}|=1$. Hence, from (2.11) it follows that

$$\alpha_{n+1}+2\alpha_n+\alpha_{n-1}=\text{const.} \quad . \tag{2.12}$$

Two different solutions for (2.12) exist:

$$\alpha_n=\pm 1 \qquad ; \qquad \alpha_n=\pm(-1)^n \tag{2.13a,b}$$

Since in case (b) $\sum_{n=0}^{N-1}\alpha_n=0$, N must be divisible by 4 in this case in view of (2.9). In case (a) $\chi=2\pi L/N$ for $0\le L\le N-1$, and in case (b) χ is arbitrary.

The foregoing discussion is the basis for the following result, stating the general structure of solutions in class $\mathcal{A}$.

Theorem 1 *Let $\boldsymbol{C}\in\mathcal{A}$ and $C_n=|\boldsymbol{C}|e^{i\bar{\theta}_n}$. Then, either*

$$\bar{\chi}_n=\bar{\theta}_{n+1}-2\bar{\theta}_n+\bar{\theta}_{n-1}=\frac{2\pi L}{N} \qquad ; \qquad 0\le L\le N-1 \tag{2.14a}$$

or

$$\bar{\chi}_n=\frac{\pi}{2}\pm(-1)^n\left(\frac{\pi}{2}-\chi\right) \qquad ; \qquad -\frac{\pi}{2}\le\chi\le\frac{\pi}{2} \tag{2.14b}$$

proof: We first prove that (2.14) indeed represent solutions in class $\mathcal{A}$. To this end, it suffices to show that they satisfy (1.19). Indeed, since

$$(\bar{\theta}_{n+p}-\bar{\theta}_n)-(\bar{\theta}_{n+p+q}-\bar{\theta}_{n+q})=-\sum_{m=0}^{p}\sum_{k=0}^{q}\bar{\chi}_{n+m+k} \tag{2.15}$$

and since $S_{N,p} = S_{N,N-p}$, both (2.14a) and (2.14b) satisfy (1.19).

To prove that *any* $\boldsymbol{C} \in \mathcal{A}$ satisfies (2.14) we first show that for any $\boldsymbol{C} \in \mathcal{A}$ we must have

$$\boldsymbol{C} = \exp\{i\theta(q)\}\tilde{\boldsymbol{C}} \quad , \tag{2.16a}$$

where the coefficients in the power series expansion of $\tilde{\boldsymbol{C}} \in \mathcal{A}$ must satisfy

$$a_{nk} = [f_k + i(-1)^n d_k]\, a_{n0} \qquad \forall k \geq 1 \quad , \tag{2.16b}$$

in which f_k and d_k are real numbers, and $\theta(q)$ is analytic near $q = 0$. We demonstrate in the following that for (2.14a) $d_k = 0$ for all k, as the d_k's reflect possible dependence of χ on q in (2.14b).

We prove (2.16) by invoking inductive arguments. We first show its validity for $n = 1$. To this end we need to examine (2.2) with $k = 3$. In view of (2.4) and (2.11) this relation is solvable *iff*

$$\boldsymbol{A}\boldsymbol{b}_1 = 0 \quad , \tag{2.17}$$

where $\boldsymbol{b}_1 = [b_{11}, \ldots, b_{n1}]^T$ and $\boldsymbol{A}$ is an $N \times N$ symmetric matrix whose

components are given by

$$A_{mj} = \begin{cases} \alpha_{n+1} & (j-m) = 2 \mod N \\ -2\left(\alpha_{n+1} + \alpha_n\right) & (j-m) = 1 \mod N \\ \left(\alpha_{n+1} + 4\alpha_n + (\alpha_{n-1}\right) & j = m \\ -2\left(\alpha_{n-1} + \alpha_n\right) & (m-j) = 1 \mod N \\ \alpha_{n-1} & (m-j) = 2 \mod N \\ 0 & \text{otherwise} \end{cases} . \tag{2.18}$$

For both case (2.14a) and (2.14b) $\boldsymbol{A}$ is a circulant matrix. In case (2.13a) $r(\boldsymbol{A}) = N-1$ and $\ker \boldsymbol{A} = \operatorname{span}[1,\ldots,1]^T$. In case (2.13b) $r(\boldsymbol{A}) = N-2$ and $\ker \boldsymbol{A} = \operatorname{span}\left\{[1,\ldots,1]^T, \left[-1,1,\ldots,(-1)^N\right]^T\right\}$. Hence,

$$a_{n1} = \left[-2 + ib_1 + i(-1)^n d_1\right] a_{n0} \quad ,$$

where $d_1 = 0$ in case (2.13a). Multiplication of $\boldsymbol{C}$ by $\exp\{-ib_1 q\}$ demonstrates the validity of (2.16) for $k = 1$.

Assume by induction the validity of (2.16) for $0 \le k \le K-1$ for any $\boldsymbol{C} \in \mathcal{A}$. In case (2.13a) we assume $d_k = 0$ as well. It is sufficient to consider only $\theta(q) \equiv 0$ in (2.16a). If not, we can obtain the next order term for $e^{-i\theta(q)}\boldsymbol{C}$, which must be in $\mathcal{A}$ as well.

As $|C_n| = |C|$ for all n we have

$$\sum_{m=0}^{k} a_{nm} a^*_{n(k-m)} = F_k \quad . \tag{2.19}$$

Substituting (2.16b) into the above relation with $k = K$ we obtain

$$F_K = \sum_{m=1}^{K} \left(f_m f_{K-m} - d_m d_{K-m} \right) + 2\Re \left(a_{nK} a^*_{n0} \right) \quad ,$$

and hence

$$a_{nK} = \left[f_K + i e_{nK} \right] a_{n0} \quad . \tag{2.20}$$

Substitution of (2.20) into (2.19) with $k = K + 1$ yields

$$\Re \left(a_{n(K+1)} a^*_{n0} \right) = \tilde{f}_{K+1} - e_{nK} d_1 (-1)^n \quad . \tag{2.21}$$

wherein $\tilde{f}_{K+1} \in \mathbb{R}$ We now substitute (2.16b) for $1 \leq k \leq K - 1$ together with (2.20) and (2.21) into (2.2) with $k = K + 2$. A tedious calculation leads to the solvability condition

$$A_{nm} e_{mK} = (-1)^n G_K \quad , \tag{2.22a}$$

where

$$G_K = -2d_1\tilde{f}_{K+1} - 2d_2 f_K + 4d_1 f_K$$
$$+ \sum_{\substack{m,j\geq 1 \\ 3\leq m+j\leq K}} [2f_j f_{K+2-m-j}d_m - f_j d_{K+2-m-j}f_m + d_j d_{K+2-m-j}d_m]$$
$$+ \sum_{\substack{m,j\geq 1 \\ 2\leq m+j\leq K}} [2f_j f_{K+1-m-j}d_m - f_j d_{K+1-m-j}f_m + d_j d_{K+1-m-j}d_m]$$
$$+ \sum_{2\leq r^2+s^2\leq K+2} \sum_{\substack{m,j\geq 1 \\ m+j\leq M}} \left\{f_j f_{M-m-j}d_m\left[(-1)^s + (-1)^r\right] - f_j d_{M-m-j}f_m(-1)^{r+s} + d_j d_{M-m-j}d_m\right\} \tag{2.22b}$$

and $M = K + 2 - r^2 - s^2$. za In case (2.13a) G_K vanishes, and hence $e_{nK} = b_K$. Then, multiplying $\boldsymbol{C}$ by $\exp\{-ib_K q^K\}$ yields a solution in $\mathcal{A}$ which satisfies (2.16). In case (2.13b) if $G_K \neq 0$ no solution to (2.22a) can exist as the right-hand-side is not orthogonal to $\ker \boldsymbol{A}$. Consequently, in such a case $\boldsymbol{C}$ is not a solution of (1.19) and is, therefore, not of any interest. If $G_K = 0$ (2.16) is satisfied with $d_k \neq 0$, which completes the proof of (2.16).

In case (2.13a), the proof of the theorem is complete since by (2.16) any solution must satisfy

$$C_n = e^{i\theta(q)}\left(\sum_{k=0}^{\infty} f_k q^k\right) a_{n0} \quad . \tag{2.23}$$

If the power series on the right-hand-side of (2.23) converges in some neighborhood of $q = 0$ [2], it satisfies (2.14a). In case (2.13b) any

solution must be of the form

$$C_n = e^{i\theta(q)} \left[\sum_{k=0}^{\infty} \left(f_k + i(-1)^n d_k \right) q^k \right] a_{n0} \quad . \tag{2.24}$$

Let

$$\chi(q) = \arctan \left\{ \frac{\sum_{k=0}^{\infty} d_k q^k}{\sum_{k=0}^{\infty} f_k q^k} \right\} \quad ,$$

Which is an analytic function near $q = 0$, as $d_0 = 0$ and $f_0 = 1$. Then,

$$C_n = e^{i\theta(q)} |C(q)| e^{i(-1)^n \chi(q)} a_{n0} \quad . \tag{2.25}$$

It is easy to show that (2.25) satisfies (2.14b).

□

To conclude this sections we present solutions for which $|C_n| \in \{|C|, 0\}$. Two different types of solutions exist in this class, in addition to class $\mathcal{A}$ solutions:

1. Solutions for which

$$\boldsymbol{C} = \left[\underbrace{C_0, 0, \ldots, 0}_{\text{R terms}}, C_1, \ldots, C_{N-1}, \ldots, 0 \right]^T \quad ,$$

where $\bar{\boldsymbol{C}} = [C_0, C_1, \ldots, C_{N-1}]^T \in \mathcal{A}$, and $R \leq 2$ is an integer. In view of (1.18) we obtain, however, $\psi(\boldsymbol{C}, k/R) = \psi(\bar{\boldsymbol{C}}, k)$, and hence, this type of solutions is not of any particular interest.

2. Solutions for which $C_{3n} = 0$, $|C_{3n+1}| = |C_{3n+2}| = |C|$, in which case N must be divisible by 3. As in class $\mathcal{A}$, there are two different cases ($\theta_n = \arg\{C_n\}$):

$$\theta_{3n+4} - 2\theta_{3n+1} + \theta_{3n-2} = \theta_{3n+5} - 2\theta_{3n+2} + \theta_{3n-1} = \frac{2\pi L}{N/3} \tag{2.26a}$$

and

$$\theta_{3n+4} - 2\theta_{3n+1} + \theta_{3n-2} = \theta_{3n+5} - 2\theta_{3n+2} + \theta_{3n-1} = \frac{\pi}{2} \pm (-1)^n \left(\frac{\pi}{2} - \phi\right) \tag{2.26b}$$

where $0 \leq L \leq N/3 - 1$ and $-\pi/2 \leq \phi \leq \pi/2$. In case (2.26b) N must be divisible by 12.

Unlike class $\mathcal{A}$, we do not prove that the above types include all possible solutions for which $|C_n| \in \{|C|, 0\}$. Though it seems reasonable to believe that this is indeed the situation, the proof appears to be technically difficult.

3 Vortex structure

In this section we first present the vortex structures embedded in (2.14a) and (2.26a), and analyze some of their properties. We then

briefly sketch via a simple numerical calculation, the dependence of the vortex location on the parameters k and χ for the solution (2.14a) with $N = 4$.

It is more convenient to present the location of the vortices by the normalized coordinates

$$\xi = x/L_x \quad \text{and} \quad \eta = y/L_y \tag{3.1}$$

rather than by the original coordinates x and y. In terms of these coordinates $\psi^{(0)}$ is expressible in the form

$$\psi^{(0)} = \sum_{n=-\infty}^{\infty} C_n e^{i2\pi n\eta} q^{(N\xi - n)^2} \quad . \tag{3.2}$$

The points at which $\psi^{(0)}$ vanishes, or the vortices, for the class of solutions (2.14a) are given by

$$(\xi_m, \eta_m) = \left(\frac{2m+1}{2N}, \frac{2(M+mL)+1}{2N} \right) \quad , \tag{3.3}$$

where $0 \leq m \leq N-1$, and the integer $0 \leq M \leq N-1$ reflects the possibility of translating the entire lattice by multiples of $1/N$ in η (arbitrary translations are not permitted as $C_{n+N} = C_n$), and L is defined in (2.14a). The validity of (3.3) can be easily verified by substituting it, together with (2.14a) into (3.2). Similarly, when

(2.26b) is satisfied, the vortices are located at

$$(\xi_{3m}, \eta_{3m}) = \left(\frac{3m}{N}, \frac{(2M + 3mL)}{2N} \right) \quad , \tag{3.4a}$$

$$(\xi_{3m+1}, \eta_{3m+1}) = \left(\frac{3m}{N}, \frac{(2M + 3mL)}{2N} + \frac{1}{2} \right) \quad , \tag{3.4b}$$

$$(\xi_{3m+1}, \eta_{3m+1}) = \left(\frac{6m+3}{2N}, \frac{(4M + 6mL + 3)}{4N} \right) \quad . \tag{3.4c}$$

In which $0 \leq M \leq N - 1$ and $0 \leq m \leq N/3 - 1$.

Equation (3.3) represents lattices whose vortices are located along parallel straight lines with equal spacing between them. The separation in the η direction between two adjacent lines is exactly 1 (or L_y in the original coordinates x and y). The lattices which (3.4) represents are characterized by pairs of parallel lines: the spacing between the vortices along one of the lines in the pair is twice larger (or smaller) than the spacing along the other line. The separation in the η direction between two adjacent lines is $1/2$ in that case. It is very easy to show that both (3.3) and (3.4) are invariant to the transformation

$$\xi \to 2\xi_m - \xi \quad ; \quad \eta \to 2\eta_m - \eta \qquad \forall m \quad .$$

Alternatively we may state that all the lattices satisfying either (3.3) or (3.4) are invariant to rotations of 180° with respect to each lattice point (vortex). In appendix A we briefly discuss lattices which are invariant to such rotations. We show that their points must either be

located along parallel straight lines with equal spacing between them [similar to the lattices satisfying (3.3)], or else be located along pairs of parallel straight lines in a similar manner to the lattices described by (3.4).

Figure 1 displays pictures of sample lattices. In part (a) we present two lattices of type (3.3) for the case $N = 6$. The first one, whose vortices are marked by squares corresponds to the case $L = 1$, $M = 0$ (both lattices in figure 1a are translated by $\frac{1}{2N}$ in both the ξ and η directions). All the vortices lie in that case on the principal diagonal of the unit cell, marked by a solid line in the figure. The second lattice corresponds to $L = 5$ and $M = 0$. The vortices are located on the dashed lines which are separated in the ξ direction by $1/5$. It is also possible to present this lattice by substituting $L = -1$ in (3.3) instead of $L = 5$. The vortices, marked by circles, are indeed located on the diagonal connecting $(0, 1)$ and $(1, 0)$ in the unit cell. The transformation $L \to L \pm N$, thus, does not chance the loci of the vortices, but can offer us different explanations of the lattice structure. Figure 1b displays a picture of a lattice of type (3.4) for $N = 9$, $L = 1$, and $M = 0$.

From a physical point of view, the fact that the above mentioned

lattices can exist is not surprising, since the repulsion forces between the vortices must cancel each other in view of the invariance to 180° rotations. As this symmetry exists independently of q, or the ratio between L_x and L_y, the coordinates of the vortices in the $\xi\eta$ plane are independent of q as well [as can be seen directly from (3.3) and (3.4)]. In the following, we prove that if $\boldsymbol{C}$ is an analytic function near $q = 0$ [(2.1) is satisfied], and if $|a_{n0}| = 1$ for all n, then the vortex location, in the $\xi\eta$ plane, must either depend on q, or be of the form (3.3).

We look, then, for lattices for which $d\xi_m/dq = d\eta_m/dq = 0$, for all $0 \leq m \leq N-1$. We look for the first vortex in the strip $0 \leq \xi_0 < 1/N$. We then substitute (2.1) into (3.2), to obtain the leading order balance

$$a_{00}q^{(N\xi_0)^2} + e^{i2\pi\eta_0}a_{10}q^{(N\xi_0-1)^2} \sim O\left(q^{(N\xi_0+1)^2}, q^{(N\xi_0-2)^2}\right) \qquad (3.5)$$

Since $|a_{00}| = |a_{10}|$, a vortex would exist when $\xi_0 \in [0, 1/N)$ only if $\xi_0 = 1/2N$, and

$$a_{00} + e^{i2\pi\eta_0}a_{10} = 0 \quad .$$

Similar considerations would lead to the conclusion that a vortex can exist in the strip $n/N \leq \xi \leq (n+1)/N$ only if

$$\xi_n = \frac{2n+1}{2N} \quad ; \quad a_{n0} + e^{i2\pi\eta_n}a_{(n+1)0} = 0 \quad , \qquad (3.6)$$

It is easy to show by expanding $\psi^{(0)}$ near (ξ_n, η_n) and $q = 0$, that the

vortex which can exist there must be a simple vortex. Hence, (3.6) must be satisfied simultaneously for $n = 0, \ldots, N-1$.

As the $O(q^{5/4})$ balance does not convey any new information on possible lattice geometries we move to present the $O(q^{9/4})$ directly

$$a_{(n-1)0}e^{-i2\pi\eta_n} + a_{n2} + a_{(n+1)2}e^{i2\pi\eta_n} + a_{(n+2)0}e^{i4\pi\eta_n} = 0 \quad . \qquad (3.7)$$

Substitution of (2.11) and (3.6) into (3.7) yields

$$\alpha_{n+2} + 3\alpha_{n+1} - 3\alpha_n - \alpha_{n-1} = 0 \quad ; \quad \forall 0 \leq n \leq N-1 \quad ,$$

admitting the unique solution $\alpha_n = \pm 1$ for all n. Theorem 1 demonstrates that any solution for which $\alpha_n = \pm 1$ is expressible in the form (2.14a), representing the lattice (3.3).

Finally, we consider one of the cases in which equilibrium is not guaranteed by the special symmetry of the lattice. In these cases the lattice depends on $q(k)$. Figure 2 plots the loci of the vortices as a function of $\Delta = L_x/L_y$ in the (ξ, η) plane, for the solution (2.14) with $N = 4$ and $\chi = 0$. In general, four distinct vortices exist, two of them are located along the full curves, and the other two along the dashed curves. Note that when $L_x = L_y$, two double vortices exist, a situation which must be highly unstable.

4 Energies

Abrikosov [1] demonstrates that the energy functional is proportional to

$$\frac{1}{2} + \bar{H}^2 - \frac{\kappa - \bar{H}}{1 + (2\kappa^2 - 1)\,\beta}$$

where $\bar{H}$ is the average magnetic field, and

$$\beta = \frac{\overline{|\psi^{(0)}|^4}}{\left(\overline{|\psi^{(0)}|^2}\right)^2} \quad .$$

Thus, for $\kappa > 1/\sqrt{2}$ and fixed $\bar{H}$, the free energy is minimized by minimizing β. At a critical point, when $\boldsymbol{C}$ is a solution of (1.19), we have

$$\beta = N\frac{k}{\kappa\sqrt{2\pi}}\|\boldsymbol{C}\|^{-2} \quad . \tag{4.1}$$

Hence, when $\|\boldsymbol{C}\|^2$ is maximal β is minimal and vice versa.

We now present an asymptotic calculation of $\|\boldsymbol{C}\|^2$ in the limit $q \to 0$, for the set of all the solution of (1.19) which obey (2.1). Since

$$\|\boldsymbol{C}\|^2 = \sum_{n=0}^{N-1}\sum_{k=0}^{\infty}\sum_{m=0}^{k} a_{nm}a^*_{n(k-m)}q^k \tag{4.2}$$

we have

$$\|\boldsymbol{C}\|^2 = \sum_{n=0}^{N-1}|a_{n0}|^2 + O(q) \quad .$$

The above relation shows that as $q \to 0$ solutions for which $|a_{n0}| = 1$ for all n have lower energy than those for which for at least one value of n, we have $|a_{n0}| = 0$.

Suppose now that $|a_{n0}| = 1$ for all n. Then, in view of (4.2), (2.4), and (2.11)

$$\|\boldsymbol{C}\|^2 = N\left[1 - 4q + \left(16 - \frac{4}{N}\cos\chi\sum_{n=0}^{N-1}\alpha_n\right)q^2 + O(q^3)\right] \qquad (4.3a)$$

or,

$$\beta = \sqrt{\frac{\log(1/q)}{\pi}}\left[1 + 4q + \frac{4}{N}\cos\chi\sum_{n=0}^{N-1}\alpha_n q^2 + O(q^3)\right] \quad . \qquad (4.3b)$$

Hence, β is minimized when $\chi = \pi$ and $\alpha_n = 1$ for all n, or alternatively, in view of (2.6), when $\chi_n = \pi$ for all n. It is easy to show, from (2.6) that the only solution satisfying this requirement is (1.21).

The above calculation demonstrates that (1.21), which represents the triangular lattice, has the lowest possible energy among all the solutions of (1.19) which are analytic near $q = 0$, in the limit $q \to 0$. The preferred value of q is found in the literature [1, 10, 3] by minimizing $\beta(\boldsymbol{C}, q)$ for fixed values of $\boldsymbol{C}$. The exact expression for β, for solutions of the type (2.14a) is given by

$$\beta(\chi, q) = \sqrt{\frac{\log(1/q)}{\pi}}\sum_{m,r=-\infty}^{\infty} q^{m^2+r^2}e^{-i\chi mr} \quad . \qquad (4.4)$$

For the triangular lattice $\chi = \pi$ and β is minimized at $q = e^{-\pi\sqrt{3}/2}$ [10, 3]. Substituting these values into both (4.3b) and (4.4) we obtain, by comparison, a relative error of $O(10^{-5})$. The error for the square lattice is of the same order of magnitude.

We conclude this section by comparing the energies of the different solutions of the form (2.14a). Figure 3 plots the dependence of the value of β, after minimization with respect to q, as a function of χ/π. Applying the transformation $m \to -m$ in (4.4) we obtain $\beta(\chi, q) = \beta(-\chi, q)$, and hence, we plot β for $0 \leq \chi \leq \pi$ only. We note that the square lattice ($\chi = 0$) has the highest energy, whereas the triangular lattice has the lowest.

5 Linear stability

The time-dependent Ginzburg-Landau equations are [8]

$$\frac{\alpha}{\kappa^2}\frac{\partial \psi}{\partial t} + \frac{\alpha i \phi}{\kappa} + \left(\frac{i}{\kappa}\boldsymbol{\nabla} + \boldsymbol{A}\right)^2 \Psi = \Psi\left(1 - |\Psi|^2\right) \quad , \tag{5.1a}$$

$$-\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{A})\frac{\partial \psi}{\partial t} + \boldsymbol{\nabla}\phi = \frac{i}{2\kappa}\left(\Psi^* \boldsymbol{\nabla}\Psi - \Psi \boldsymbol{\nabla}\Psi^*\right) + |\Psi|^2 \boldsymbol{A} \quad , \tag{5.1b}$$

where α is a time scale and ϕ is the magnetic potential. Following [4] we consider a small perturbation of a steady solution of (1.3). Thus,

$$\Psi = \Psi_0 + \delta e^{\sigma t}\Psi_1 \tag{5.2a}$$

$$\boldsymbol{A} = \boldsymbol{A}_0 + \delta e^{\sigma t}\boldsymbol{A}_1 \tag{5.2b}$$

$$\Phi = \delta e^{\sigma t}\Phi_1 \quad , \tag{5.2c}$$

wherein $0 < \delta \ll \epsilon \ll 1$, Ψ_0 and $\boldsymbol{A}_0$ are given by the expansion (1.6), and Ψ_1, $\boldsymbol{A}_1 = (0, A_1(x,y), 0)$, Φ_1, and σ may be similarly expanded, i.e.,

$$\Psi_1 = \epsilon^{1/2}\psi_1 \tag{5.3a}$$

$$A_1 = \epsilon a_1 \tag{5.3b}$$

$$\Phi_1 = \epsilon\phi_1 \tag{5.3c}$$

$$\psi_1 = \psi_1^{(0)} + \epsilon\psi_1^{(1)} + \dots \tag{5.3d}$$

$$a_1 = a_1^{(0)} + \epsilon a_1^{(1)} + \dots \tag{5.3e}$$

$$\phi_1 = \phi_1^{(0)} + \epsilon\phi_1^{(1)} + \dots \tag{5.3f}$$

$$\sigma = \sigma^{(0)} + \epsilon\sigma^{(1)} + \dots \quad . \tag{5.3g}$$

Substituting (5.2) and (5.3) into (5.1) we obtain the leading order balance

$$\frac{\alpha\sigma^{(0)}}{\kappa^2}\psi_1^{(0)} - \frac{1}{\kappa^2}\left(\frac{\partial^2\psi_1^{(0)}}{\partial x^2} + \frac{\partial^2\psi_1^{(0)}}{\partial y^2}\right) + \frac{2ih^{(0)}x}{\kappa}\frac{\partial\psi_1^{(0)}}{\partial y} = \psi_1^{(0)} - (h^{(0)})^2x^2\psi_1^{(0)} \quad , \tag{5.4a}$$

$$-\frac{\partial^2 a_1^{(0)}}{\partial x\partial y} - \frac{\partial\phi_1^{(0)}}{\partial x} = \frac{i}{2\kappa}\left(\psi_0^{(0)*}\frac{\partial\psi_1^{(0)}}{\partial x} + \psi_1^{(0)*}\frac{\partial\psi_0^{(0)}}{\partial x} - \psi_0^{(0)}\frac{\partial\psi_1^{(0)*}}{\partial x} - \psi_0^{(0)}\frac{\partial\psi_1^{(0)*}}{\partial x}\right) \quad , \tag{5.4b}$$

$$\frac{\partial^2 a_1^{(0)}}{\partial x^2} - \frac{\partial\phi_1^{(0)}}{\partial y} - \sigma^{(0)}a_1^{(0)} = \frac{i}{2\kappa}\left(\psi_0^{(0)*}\frac{\partial\psi_1^{(0)}}{\partial y} + \psi_1^{(0)*}\frac{\partial\psi_0^{(0)}}{\partial y} - \psi_0^{(0)}\frac{\partial\psi_1^{(0)*}}{\partial y} - \psi_0^{(0)}\frac{\partial\psi_1^{(0)*}}{\partial y}\right)$$
$$+h^{(0)}x\left(\psi_1^{(0)}\psi_0^{(0)*} + \psi_1^{(0)*}\psi_0^{(0)}\right) \quad . \tag{5.4c}$$

Equation (5.4a) possesses nontrivial solutions whenever

$$\sigma^{(0)} = \frac{\kappa h^{(0)}}{\alpha}\left(\frac{\kappa}{h^{(0)}} - 2n - 1\right) \quad . \tag{5.5}$$

When $h^{(0)} = \kappa$ all the modes are stable except for $n = 0$ for which $\sigma^{(0)} = 0$. The periodic modes in this case are given by

$$\psi_1^{(0)} = \sum_{n=-\infty}^{\infty} B_n \exp\left\{ikny - \frac{\kappa^2}{2}\left(x - \frac{nk}{\kappa^2}\right)^2\right\} \quad . \tag{5.6}$$

and $B_n = B_{n+\bar{N}}$ for some natural $\bar{N}$. More generally we should consider $B_n = B_{n+\bar{N}}e^{i\Theta}$, as translated mode of perturbations are also of interest. We shall, however, examine this case only later.

Proceeding with the next order balance (in powers of ϵ) we obtain the solvability condition [4]

$$\frac{2\sqrt{2}\alpha\sigma^{(1)}}{1-2\kappa^2}B_n = \sum_{r,m}\left[2B_{n-r+m}C_m^*C_r + C_{n-r+m}B_m^*C_r - \frac{C_rC_m^*C_{n-r+m}}{C_n}B_n\right]\cdot$$
$$\cdot \exp\left\{-\frac{\kappa^2}{2}\left[(r-m)^2 + (r-n)^2\right] \quad .\right\}$$

With the aid of (1.19) we obtain (for $\bar{h}^{(1)} = 1$)

$$\sum_{r=0}^{N-1}\sum_{m=0}^{N-1}\left[2C_{n+r}C_{n+r+m}^*B_{n+m} + C_{n+r}B_{n+r+m}^*C_{n+m}\right]S_{N,r}S_{N,m} - B_n = \bar{\sigma}^{(1)}B_n \tag{5.7a}$$

wherein

$$\bar{\sigma}^{(1)} = \frac{2\sqrt{2}\alpha}{1-2\kappa^2}\sigma^{(1)} \tag{5.7b}$$

The left hand side of (5.7) is a linear transformation of the vector $\boldsymbol{B} = [B_0, \ldots, B_{N-1}]^T \in \mathbb{C}^N$. Denote this transformation by $T : \mathbb{C}^N \to \mathbb{C}^N$. The field above which we define $\mathbb{C}^N$ must be $\mathbb{R}$ rather than $\mathbb{C}$, otherwise $T(\boldsymbol{B}, \boldsymbol{B}^*)$ would not be a linear transformation. Our goal is than to find the eigenvalues of T. Since the field of scalars is $\mathbb{R}$ we have $2N$ eigenvalues. Naturally, the eigenvalues depend on the steady solution $\boldsymbol{C}$, whose stability is examined. There is, however, an infinite number of presentations for every solution. Consider for instance (1.21): we may present it as a vector in $\mathbb{C}^2$

$$\boldsymbol{C} = [C, iC]^T \quad , \tag{5.8a}$$

or as a vector in $\mathbb{C}^{2P}$

$$\boldsymbol{C} = [C, iC, C, iC, \ldots, C, iC]^T \quad , \tag{5.8b}$$

or as a vector in $\mathbb{C}^{2PR}$

$$\boldsymbol{C} = [\underbrace{C, 0, \ldots, 0}_{\text{R terms}}, iC, 0, \ldots, 0, C, \ldots, iC]^T \quad . \tag{5.8c}$$

In the latter case, in order to maintain L_y invariant, we need to substitute k/R instead of k in (1.18).

The difference between the various presentations is not only semantic. If we adopt the presentation (5.8a) we will be forced to present

the eigenvalue problem (5.7) in $\mathbb{C}^2$. Such an analysis has been performed by Chapman [4]. As $\boldsymbol{B} \in \mathbb{C}^2$ the analysis is confined to a four-dimensional perturbation space.

Adopting the presentation (5.8a), we may consider $\boldsymbol{B} \in \mathbb{C}^{2P}$. Such a presentation allows us to consider perturbations whose period in the y direction is still L_y, but their period in the x direction would be PL_x where P can be set to be arbitrarily large. Furthermore, by applying the transformation (1.22b) we may consider translations by αL_y in the y direction of the perturbation (5.6), where α is a rational number (irrational values of α can be allowed only if we replace the requirement $B_n = B_{n+\bar{N}}$ by $B_n = B_{n+\bar{N}} e^{i\Theta}$). The most general class of periodic perturbations, which can be examined within the present framework, is obtained by adopting the presentation (5.8c). Such a presentation allows for perturbations whose respective periods in the x and y directions are PL_x and RL_y, where both P and R can be set to be arbitrarily large.

It is convenient to rewrite (5.7) in the form

$$\bar{\sigma}\boldsymbol{B} = T_1\boldsymbol{B} + T_2\boldsymbol{B}^* \quad , \tag{5.9a}$$

where

$$[T_1\boldsymbol{B}]_n = \sum_{r=0}^{\bar{N}-1}\sum_{m=0}^{\bar{N}-1} 2C_{n+r}C^*_{n+r+m}B_{n+m}S_{N,r}S_{N,m} - B_n \quad , \qquad (5.9b)$$

and

$$[T_2\boldsymbol{B}^*]_n = \sum_{r=0}^{\bar{N}-1}\sum_{m=0}^{\bar{N}-1} C_{n+r}B^*_{n+r+m}C_{n+m}S_{N,r}S_{N,m} \quad . \qquad (5.9c)$$

We examine the stability of steady solutions of the form (2.14a). The transformation T_1 can be represented in this case by an Hermitian matrix whose eigenvectors are given by

$$B_n^{(r,s)} = \frac{1}{|C|(\bar{N}/R)^{1/2}} \begin{cases} C_l\epsilon_r^l & n = lR+s \\ 0 & \text{otherwise} \end{cases} \qquad (5.10)$$

where $0 \le l, r \le \bar{N}/R - 1$, $0 \le s \le R-1$, and

$$\epsilon_r = e^{i\frac{2\pi}{\bar{N}/R}r} \quad .$$

The eigenvalues of T_1 are given by

$$\lambda^{(r,s)} = 2|C|^2 \sum_{l,m=-\infty}^{\infty} q^{l^2+(m-\nu)^2} e^{i(-lm\chi+2l\mu)} - 1 \qquad (5.11a)$$

in which

$$\mu = \frac{\pi r}{\bar{N}/R} \quad , \qquad (5.11b)$$

and

$$|C|^{-2} = \sum_{l,m=-\infty}^{\infty} q^{l^2+m^2} e^{-ilm\chi} \quad . \tag{5.11c}$$

In the above χ is given by (2.14a) with N replaced by $\bar{N}/R$. We note that $\lambda^{(r,s)}$ is real for all r and s.

Since the eigenvectors of T_1 constitute an orthonormal basis to $\mathbb{C}^{\bar{N}}$ above $\mathbb{C}$, we have

$$\boldsymbol{B}^{(m,j)^*} \cdot T_1 \boldsymbol{B}^{(r,s)} = \lambda^{(r,s)} \delta_{mr} \delta_{js} \quad , \tag{5.12}$$

wherein the symbol $\cdot$ represents the ordinary scalar product in $\mathbb{R}^{\bar{N}}$. Furthermore, it can be easily shown that

$$\boldsymbol{B}^{(m,j)^*} \cdot T_2 \boldsymbol{B}^{(r,s)^*} = \gamma^{(r,s)} \delta_{m(\bar{N}/R-r+L)} \delta_{j(R-s)} \quad , \tag{5.13}$$

in which

$$\gamma^{(r,s)} = (-1)^L \exp\left\{-i\frac{2\pi}{\bar{N}/R} J\right\} |C|^2 e^{i\left(\frac{\chi}{2}-2\mu\right)} \sum_{l,m=-\infty}^{\infty} q^{(l+\nu)^2+(m-\nu)^2} e^{i[-lm\chi+2(m+l)\mu]} \quad , \tag{5.14a}$$

where

$$\nu = \frac{s}{R} \quad , \tag{5.14b}$$

and J is an integer which can be set equal to 0 by appropriately applying (1.19) to $\boldsymbol{C}$. Note that

$$\lambda^{(r,s)} = \lambda^{(\bar{N}/R-r+L,R-S)} \qquad ; \qquad \gamma^{(r,s)} = \gamma^{(\bar{N}/R-r+L,R-S)} \quad . \tag{5.15}$$

In view of (5.12) and (5.13) we look for eigenvectors of T in the form

$$(a_1 + ia_2)\,\boldsymbol{B}^{(r,s)} + (b_1 + ib_2)\,\boldsymbol{B}^{(\bar{N}/R-r+L,R-S)}$$

We find that the linear transformation T possesses the eigenvalues

$$\bar{\sigma}^{(r,s)} = \lambda^{(r,s)} \pm \left|\gamma^{(r,s)}\right| \quad . \tag{5.16}$$

For $\bar{N}$ even and $L \neq 0$, there are $\bar{N}$ distinct eigenvalues, in view of (5.15), and each eigenvalue corresponds to two independent eigenvectors. Hence, in view of (5.16) and (5.7b), the steady solution described by (1.18) and (2.14a) is linearly stable only if

$$\lambda^{(r,s)} - \left|\gamma^{(r,s)}\right| \geq 0 \quad . \tag{5.17}$$

We allow for a weak inequality in the above since $\lambda^{(0,0)} = |\gamma^{(0,0)}| = 1$ independently of χ. It can be easily verified from (5.7) that the mode which is always marginally stable is $\boldsymbol{B} = i\boldsymbol{C}$ (or $\Psi_1 = i\Psi_0$). This mode corresponds to an infinitesimal gauge transformation [13], i.e.,

$$\Psi = \Psi_0(1 + i\delta) + O\left(\delta^2\right) = e^{i\delta}\Psi_0 + O\left(\delta^2\right)$$

Naturally, such a transformation must be marginally stable as the Ginzburg-Landau equations are invariant to the transformation (1.2). Other than this special mode, we require that all modes would be strictly stable.

As $\bar{N}/R$ and R may be set arbitrarily large, it is only natural to consider continuous μ, ν, and χ. The condition (5.17) may then be written in the form

$$\lambda(\chi,\mu,\nu) - |\gamma(\chi,\mu,\nu)| > 0 \qquad ; \qquad 0 < \mu < \pi \quad 0 < \nu < 1 \quad (5.18)$$

In the above χ represents the steady solution whose stability is examined , and μ and ν represent the mode of the perturbation. The above inequality can be solved numerically. Special care should be given to the vicinity of $\mu = \nu = 0$ where the solution is marginally stable. The numerical technique we utilized included thus an asymptotic expansion of $\lambda - |\gamma|$ near $\mu = \nu = 0$, together with a numerical calculation on a 1000×1000 grid. We have examined only the interval $0 \leq q \leq 0.12$. It is reasonable to expect that stable solutions can be found in another interval of q values would exist, as the minimal energy of (1.21) is obtained for two different values of $k(q)$: (a) $k = \kappa\sqrt{\pi\sqrt{3}}$, (b) $k = \kappa\sqrt{\pi/\sqrt{3}}$.

Figure 4 displays the dependence of the interval of k values for which (5.18) is satisfied on the value of χ. We plot it for $0 \leq \chi \leq \pi$ only as

$$\lambda(\chi,\mu,\nu) = \lambda(-\chi,\pi-\mu,\nu)$$

$$\gamma(\chi,\mu,\nu) = \gamma^*(-\chi,\pi-\mu,\nu)\ .$$

The part of the curve which lies left to the minimum point, denotes the minimal value of $(\log(1/q)/\pi)^2$ for which (5.18) is satisfied as a function of χ/π, whereas the part of the curve right to it denotes the maximal one. It can be seen that a non-empty interval of k values where the solution is stable exists whenever $3\pi/5 \sim\leq \chi \leq \pi$. The largest interval is obtained for $\chi = \pi$. The latter fact my be alternatively described by stating that (1.21) representing the triangular lattice is the "most stable" among the solutions (2.14a). It is important to emphasize here that solutions have been found stable only to perturbations for which the ratio between the periods in the x and y directions and the respective periods of the steady solutions are rational numbers. The stability to other modes of perturbations is left to future research.

6 Conclusion

Previous works [1, 3] show that periodic solutions to the Ginzburg-Landau equations (1.3) near their bifurcation from the normal state can be presented by a vector $\boldsymbol{C} \in \mathbb{C}^N$, where N is the number of vortices inside the unit cell. The components of this vector are found by simultaneously solving N polynomial equations of the third order [Eqs. (1.19)]. The solutions depend on the parameter q which in turn depends on the ratio between the sides of the unit cell.

In § 2 we discuss solutions which are analytic functions of q near $q = 0$. We show that all real solutions of (1.19) must be analytic functions of q near $q = 0$. For complex solutions the situation is more complicated. If we allow for non-real $\boldsymbol{C}$, the solutions can be determined, in view of (1.22a), only up to an arbitrary parameter θ which need not be an analytic function of q. Another arbitrary parameter χ arises, when N is divisible by 4, in the case (2.10b). For fixed θ and χ it seems reasonable to conjecture that any $\boldsymbol{C}$ satisfying $|C_n| = 1$ at $q = 0$ for all n, must be analytic near $q = 0$.

In the remainder of § 2 we focus on analytic solutions for which $|C_n| = |C|$ for all n (class $\mathcal{A}$). We derive the closed forms (2.14) and prove, in theorem 1, that any solution in this class is representable

by one of these forms. We also discuss briefly solutions for which $|C_n| \in \{|C|, 0\}$ for all n. We derive the closed forms (2.26a,b) which are respectively analogous to (2.14a,b).

In § 3 we analyze the geometrical structure of the vortex lattices which the closed forms (2.14) and (2.26) represent. We find that for (2.14a) the vortices are arranged along parallel lines with equal spacing between them. For (2.26a) all lattices are characterized by pairs of parallel lines: the spacing between the vortices along one of the lines in the pair is twice larger (or smaller) than the spacing along the other line. The separation in the η direction between two adjacent lines is $1/2$ in that case.

Both the aforementioned lattice geometries are invariant to 180° rotations with respect to each point in the lattice. In appendix A we show that any lattice which has this property must be arranged according to one of the above-mentioned geometries. Physically it means that the repulsion forces between the vortices must balance each other. Such lattice geometries are therefore possible independently of the ratio between the sides of the unit cell (or q). We show that any solution for which $|C_n| = 1$ at $q = 0$ for all n and the normalized coordinates ξ and η defined in (3.1) are independent of q must represent the lattice

geometry (3.2).

In § 4 we calculate the energies of all the solutions of (1.19) which are analytic near $q = 0$ in the limit $q \to 0$ up to $O(q^2)$ terms. We find that (1.21) has the lowest energy in that limit. We then compare between the minimal values, in $q \in [0, 1)$, of the energies of the solutions (2.14a) and find again that the triangular lattice has the lowest energy. We note that these minimal values can be obtained up to the fourth decimal point by using the expansion (4.3) in the limit $q \to 0$ instead of the exact expression (4.4).

In § 5 we present a linear stability analysis of the solutions (2.14a). In contrast to [4] in which the local stability is examined in a finite-dimensional subspace corresponding to the largest eigenvalue of (5.4a), we use an infinite-dimensional perturbation subspace including all periodic perturbations whose periods in the normalized coordinates (3.1) are rational numbers. We find that (1.21) is not the only solution in the family (2.14a) which is linearly stable. It is stable, however, for the largest interval of q values compared to other solutions in this family.

In section 1 we have derived the natural boundary conditions of the problem by showing that any periodic solution of (1.3) must satisfy

(1.5) (cf. also [12]). Another intersting question is in which space can $(\psi, \boldsymbol{A})$ serve as a minimizer of E? Let $\psi = \rho(x,y)e^{i\theta(x,y)}$. Consider the real valued function $e(\tau) = E(\psi + \tau\phi(\tau), \boldsymbol{A})$ where $\phi(\tau)$ must be chosen so that $\psi + \tau\phi$ belongs to the space in which we minimize E. We shall assume here that this space is a subset of the space of $W^{1,2}(R)$ functions whose absolute value is periodic on the boundaries of the fundumental cell ∂R. Hence,

$$\psi + \tau\phi = (\rho + \tau r)e^{i(\theta+\tau\zeta)} \,, \tag{6.1}$$

where r must be periodic and ζ can be any C^1 function. Clearly, $(\psi, \boldsymbol{A})$ can serve as a local minimizer of E only if $e'(0) = 0$ for all appropriate ϕ. Since ϕ is bounded as $\tau \to 0$ we have

$$\begin{aligned} e'(0) &= \lim_{\tau\to 0} \frac{2}{\kappa} \int_{\partial R} |\phi| \left[\frac{1}{\kappa} \boldsymbol{\nabla}\rho \cos(\theta - \arg\phi) + \rho \left(\frac{1}{\kappa}\boldsymbol{\nabla}\theta - \boldsymbol{A} \right) \sin(\theta - \arg\phi) \right] \cdot \hat{\boldsymbol{n}} dl \\ &= \int_{\partial R} r \left[\frac{1}{\kappa}\boldsymbol{\nabla}\rho - \rho\zeta \left(\frac{1}{\kappa}\boldsymbol{\nabla}\theta - \boldsymbol{A} \right) \right] \cdot \hat{\boldsymbol{n}} dl \end{aligned} \tag{6.2}$$

by (1.5) $e'(0)$ would vanish for any periodic r *iff* ζ is periodic on ∂R. Hence, $\psi + \tau\phi$ must satisfy (1.5) for all τ. Thus, Abrikosov's solutions can serve as minimizers for E in a rather limited sense, if we focus on subspaecs of periodic functions with given periods in the x and y directions.

The stability results we have obtained indicate that the triangular

lattice and other periodic solutions are minimizers in the above space of $W^{1,2}(R)$ periodic functions satisfying (1.5). The periods can be any product by integers of the minimal periods of the cell. Further research is necessary in order to determine the structure of the space, when we let the above integers tend to infinity.

The existence of stable periodic solutions is definitely a surprising result. There might be several explanation to the fact that stable periodic solutions to (1.3) exist in addition to the triangular lattice, despite the fact that none of them has been observed in experiments. For instance: the initial conditions in their domain of attraction cannot be set in real situations, they may become unstable shortly after the bifurcation when the magnetic field is further decreased, the interaction with the boundaries may play an important role, etc.

Acknowledgements

The author wishes to thank the professor Itai Shafrir and Jacob Rubinstein for their suggestions and comments. The author was supported by the David Posnack Memorial lectureship fund.

A Lattices invariant to 180° rotation

Consider the set $(\eta_k, \xi_k) \in \mathbb{R}^2/\mathbb{Z}^2$, where $0 \leq k \leq N-1$. We look for the geometrical structure of the lattices which are invariant to 180° rotation, or,

$$\forall k, j \; \exists n \quad \text{s.t.} \quad \xi_j - \xi_k = -(\xi_n - \xi_k) \quad ; \quad \eta_j - \eta_k = -(\eta_n - \eta_k) \quad .$$

Let $\xi_0 = \eta_0 = 0$. Then, invariance to rotation implies that the ξ components are equally spaced on [0,1], i.e.,

$$\exists \, 0 \leq M \leq N-1 \quad \text{s.t.} \quad \forall \, 0 \leq m \leq M-1 \quad \exists k \quad \text{s.t.} \quad \xi_k = \frac{m}{M} \quad ,$$

where $1 \leq M \leq N$. Denote by l_m the number of lattice points for which $\xi = m/M$. The lattice's symmetry implies that the l_m's must be arranged in pairs, i.e., $l_{2j} = l_0$ and $l_{2j+1} = l_1$ $(0 \leq j \leq M/2-1)$. If M is odd we must have $l_1 = l_0$. Otherwise, assume without loss of generality $l_0 \leq l_1$. Suppose that $\xi_1 = 1/M$. Symmetry implies the existence of a lattice point at $(2\xi_1, 2\eta_1)$. Another lattice point must exist at $(\xi_1, \eta_1 + 1/l_1)$. Rotating the the lattice with respect to the latter point by 180°, we find that $(2\xi_1, 2\eta_1)$ becomes $(0, 2/l_1)$ at which another lattice point must reside. Hence,

$$\frac{2}{l_1} = \frac{p}{l_0} \quad ,$$

and hence, as $l_0 \leq l_1$, we must have $p \leq 2$. If $p = 1$, then $l_1 = 2l_0$, otherwise $l_1 = l_0$ for $p = 2$.

We now rescale η by l_0, i.e., $\bar{\eta} = \eta/l_0$. The lattice is still periodic in $\bar{\eta}$, with period $L_\eta = 1$. In the rescaled unit cell we have $l_0 = 1$ and $l_1 \in \{1, 2\}$. In the case $l_1 = 1$ the lattice points are located along parallel straight lines with equal spaces between them. The separation in the $\bar{\eta}$ direction between two adjacent lines is exactly 1. In the case $l_1 = 2$, the lattice is characterized by pairs of parallel lines: the spacing between the lattice points along one of the lines in the pair is twice larger (or smaller) than the spacing along the other line. The separation in the $\bar{\eta}$ direction between two adjacent lines is $1/2$ in that case.

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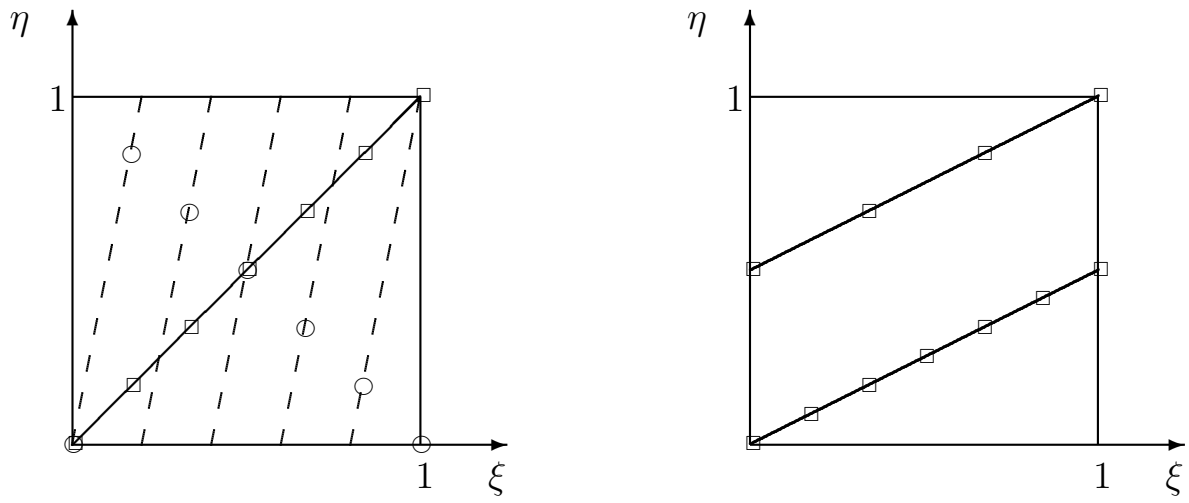

(a) (b)

Figure 1: Sample lattices of type (3.3) (a) and (3.4) (b)

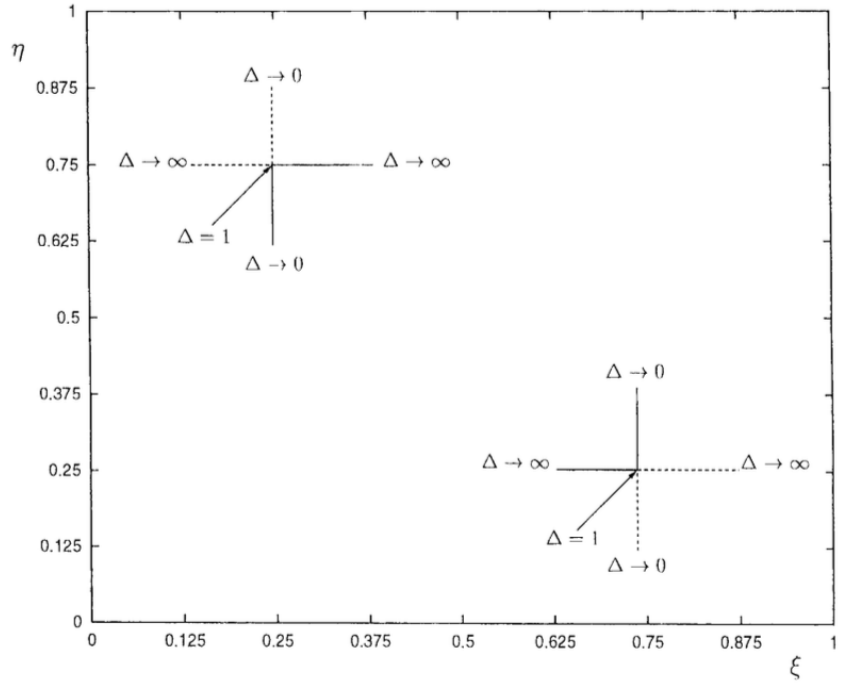

Figure 2: The loci of vortices as a function of $\Delta = L_x/L_y$ in the normalized unit cell for the case (2.14b) with $N = 4$ and $\chi = 0$.

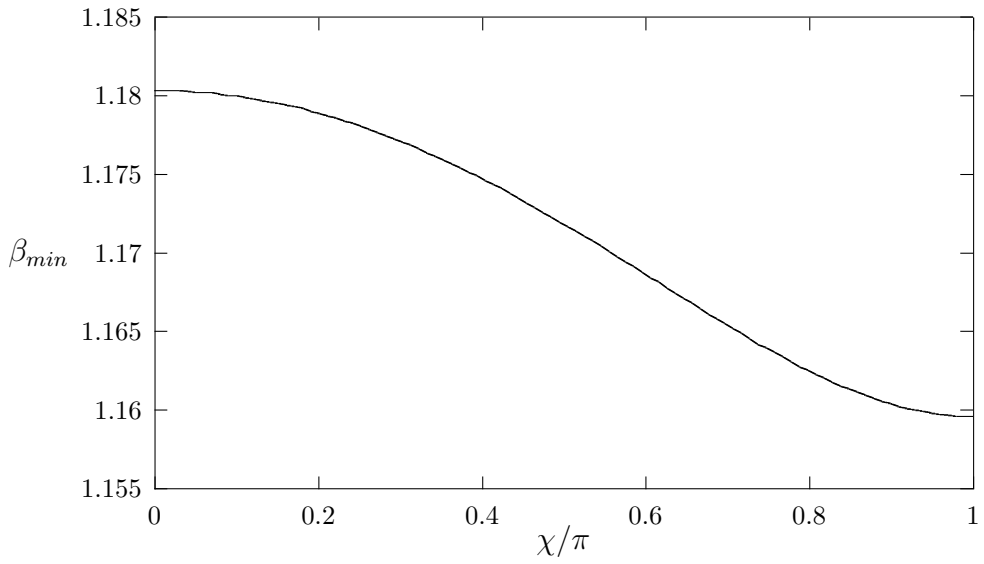

Figure 3: The dependence of the minimal value of β, with respect to q, for the solutions (2.14a), on χ

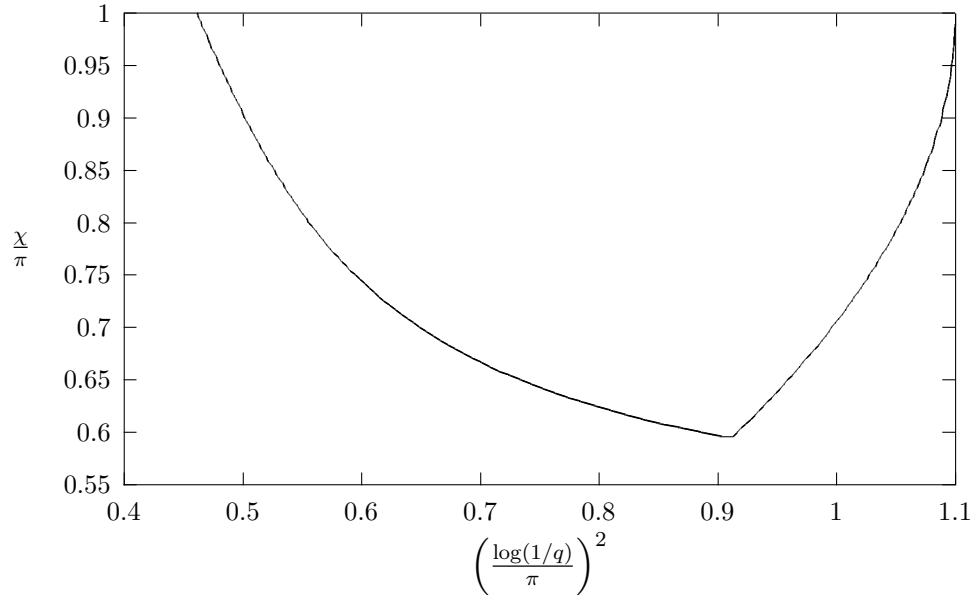

Figure 4: The domain of stability of the solutions (2.14a). The area above the curve denotes the region of stability in the $\left(\frac{\chi}{\pi}, \left(\frac{\log(1/q)}{\pi}\right)^2\right)$ plane.