The distribution of surface superconductivity along the boundary : on a conjecture of X. B. Pan

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Abstract

We consider the Ginzburg-Landau model of superconductivity in two dimensions in the large κ limit. For applied magnetic fields weaker than the onset field H_{C_3} but greater than H_{C_2} it is well known that the superconductivity order parameter decays exponentially fast away from the boundary. It has been conjectured by X. B. Pan that this surface superconductivity solution converges pointwise to a constant along the boundary. For applied fields that are in some sense between H_{C_2} and H_{C_3}, we prove that the solution indeed converges to a constant but in a much weaker sense.

1 Introduction

The Ginzburg-Landau energy functional of superconductivity is given in the form

$$\mathcal{J}(\Psi, A) = \int_\Omega \left(-|\Psi|^2 + \frac{|\Psi|^4}{2} + |h - h_{ex}|^2 + \left| \left(\frac{i}{\kappa}\nabla + A \right) \Psi \right|^2 \right) dx_1 dx_2 \,, \tag{1.1}$$

in which $\Omega \subset\subset \mathbb{R}^2$ is smooth, Ψ is the (complex) superconducting order parameter, such that $|\Psi|$ varies from $|\Psi| = 0$ (when the material is at a normal state) to $|\Psi| = 1$ (for the purely superconducting state). The magnetic vector potential is denoted by A (the magnetic field is, then, given by $h = \nabla \times A$), h_{ex} is the constant applied magnetic field, and κ is the Ginzburg-Landau parameter which is a property of the material. The functional $\mathcal{J}$ is invariant under the gauge transformation

$$\Psi \to e^{i\kappa\eta}\Psi \ , \ A \to A + \nabla\eta \,, \tag{1.2}$$

where η is a smooth function. We focus here on the properties, for a given h_{ex}, of the global minimizers[1] (Ψ_κ, A_κ) of $\mathcal{J}$ in $H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^2)$ for type II superconductors (for which $\kappa > 1/\sqrt{2}$). Note that every global minimizer represents actually an orbit of minimizers associated to the group of transformations (1.2).

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[1] We could have actually written more precisely $(\Psi_{\kappa,h_{ex}}, A_{\kappa,h_{ex}})$ but will omit the reference to h_{ex} in order to avoid cumbersome notations.

1.1 The onset of superconductivity

It is known both from experiments [18] and rigorous analysis [13] that for a sufficiently strong magnetic field the normal state ($\Psi \equiv 0$, $h = h_{ex}$) would prevail. If the field is then decreased, there is a critical field, depending on the sample's geometry, where the material would enter the superconducting state. For samples with boundaries, this field is known as the onset critical field (or nucleation field) and is called H_{C_3}. This leads to the definition (cf. [19, 15, 10] for instance)

$$H_{C_3}(\kappa) = \inf\{h_{ex} > 0 \,:\, (0, \hat{A}) \text{ is the unique global minimizer of } \mathcal{J}\}, \tag{1.3}$$

where $\hat{A} : \Omega \to \mathbb{R}^2$ satisfies $\nabla \times \hat{A} = h_{ex}$. The minimizer $(0, \hat{A})$ is unique in the sense that any other minimizer is gauge equivalent to it, i.e. it should be in the form $(0, \hat{A} + \nabla\eta)$. We note that in that for our choice of scaling in (1.1) we have $H_{C_3} \sim \frac{\kappa}{\Theta_0}$ as $\kappa \to \infty$ for smooth Ω [17], where Θ_0 is approximately 0.59.

The simplest case in which the bifurcation from the normal state $(0, \hat{A})$ to the superconducting one was described is the case of a half-plane [20]. The analysis in this case is one dimensional : the linearized Ginzburg-Landau equations were solved on $\mathbb{R}_+$. A similar situation occurs in two dimensions : it was proved in [17] and [7] that the bifurcating mode in $\mathbb{R}^2_+$ is one-dimensional and that the value of H_{C_3} is exactly the same as in the one-dimensional case.

In addition, Saint-James and de Gennes [20] found that superconductivity appears first near the boundary for a half-plane, i.e. the order parameter Ψ_κ decays exponentially fast away from the boundary. This phenomenon, which appears only in the presence of boundaries, is therefore called surface superconductivity. It was later proved for general two-dimensional domains with smooth boundaries [17, 7], that as the domain's scale tends to infinity the onset field tends to de Gennes' value, and that Ψ_κ decays exponentially fast away from the boundary.

Another related problem that has been considered in the literature is the distribution of $|\Psi_\kappa|$ along the boundary near the critical field. In [4], this distribution was formally obtained. This led to the conjecture that $|\Psi_\kappa|$ should be maximal at the point of maximal curvature along the boundary. This was indeed proved a few years later [15, 16, 11, 12]. Furthermore, it was shown that Ψ_κ decays exponentially fast away from the points of maximal curvature along the boundary.

1.2 Weakly non-linear analysis

Suppose now that h_{ex} is further decreased below H_{C_3}. While the minimizer Ψ_κ still decays exponentially fast away from the boundary much after the nucleation in the highly non-linear regime when $\kappa < h_{ex} < H_{C_3}$ [1, 19, 2], the exponential decay along the boundary disappears quite rapidly as h_{ex} decreases. More precisely, if we introduce the distance to the nucleation field ρ by

$$\rho(\kappa) = H_{C_3}(\kappa) - h_{ex}\,,$$

then, exponential rate of decay along the boundary (far from the points of maximal curvature) is guaranteed only when

$$\rho(\kappa) \xrightarrow[\kappa\to\infty]{} 0\,.$$

Furthermore, it was proved in [10] that if ρ satisfies

$$\lim_{\kappa\to\infty}\rho(\kappa)=\infty \quad ; \quad \lim_{\kappa\to\infty}\frac{\rho(\kappa)}{\kappa^{1/2}}=0\,,$$

then there exists $u\in\mathcal{S}(\overline{\mathbb{R}^+})$ such that

$$\int_\Omega \left||\Psi_\kappa(x)|^2-\frac{\rho}{\kappa}u\left(\frac{\kappa}{\sqrt{\lambda}}\,t(x)\right)\right|^2\,dx=o(\rho\kappa^{-3})\,, \tag{1.4}$$

where $t=d(x,\partial\Omega)$, $\lambda=\kappa/h_{ex}$.
This leaves open the situation when $\rho(\kappa)/\kappa^{\frac{1}{2}}$ does not tend to 0 as $\kappa\to\infty$ and in particular becomes of the order of $\kappa^{\frac{1}{2}}$. This is this last case, which will be considered in this article.

1.3 Highly non-linear Analysis : Pan's conjecture

Given some $\lambda\in]\beta_0,1[$, let $(\kappa_n,h_{ex}^n)_{n\in\mathbb{N}}$, denote a pair of sequences satisfying

$$\lim_{n\to+\infty}\kappa_n=\infty \quad ; \quad \lim_{n\to+\infty}\frac{\kappa_n}{h_{ex}^n}=\lambda\,.$$

In the above $\beta_0=\lim_{\kappa\to\infty}\kappa/H_{C_3}(\kappa)$ (we provide a better definition of β_0 in the next section). In [19] (Conjecture 1) X.B. Pan conjectures the existence of a function $]\beta_0,1[\ni\lambda\mapsto C(\lambda)\in\mathbb{R}^+$ such that, for any sequence as above,

$$|\Psi_{\kappa_n}(x)|\to C(\lambda)\;,\;\forall x\in\partial\Omega\,. \tag{1.5}$$

While the conjecture appears to be correct in its essence – any minimizer, as the results in [10] and in the present contribution suggest, does tend in some weak sense to a constant along the boundary – we believe that either the convergence assumed in (1.5) cannot be uniform, or else that the global minimizer must be discontinuous in h_{ex} and κ. Let us sketch the heuristic arguments supporting this belief. We first write the Euler-Lagrange equations associated with (1.1) (or the Ginzburg-Landau equations) :

$$\left(\frac{i}{\kappa}\nabla+A_\kappa\right)^2\Psi_\kappa=\Psi_\kappa\left(1-|\Psi_\kappa|^2\right) \tag{1.6a}$$

$$-\nabla\times\nabla\times A_\kappa=\frac{i}{2\kappa}\left(\Psi_\kappa^*\nabla\Psi_\kappa-\Psi_\kappa\nabla\Psi_\kappa^*\right)+|\Psi_\kappa|^2A_\kappa\,. \tag{1.6b}$$

If $|\Psi_\kappa|>0$ for all $x\in\partial\Omega$ (and this is indeed the case if we assume uniform convergence in (1.5)), we can divide (1.6b) by $|\Psi_\kappa|^2$ and integrate over $\partial\Omega$ (the measure on $\partial\Omega$ being denoted by ds), to obtain the existence of an integer $N(\Psi_\kappa)$ such that

$$\int_{\partial\Omega}\frac{\nabla\times(h_\kappa-h_{ex})}{|\Psi|^2}ds+\int_\Omega h_\kappa\,dx=\frac{2\pi}{\kappa}N(\Psi_\kappa)\,,$$

where $h_\kappa = \nabla \times A_\kappa$ is the induced magnetic field.
The integer $N(\Psi_\kappa) \in \mathbb{Z}$ is the winding number (or the degree) of Ψ_κ around $\partial\Omega$, (which is invariant under the transformation (1.2) since η must be smooth).
In [9] it is proved that Ψ_κ vanishes at isolated points or curves which should end on $\partial\Omega$. If $|\Psi_\kappa|$ does not vanish on the boundary, as implied by (1.5), it is clear that Ψ_κ can only vanish at isolated points. Thus, we can conclude that $N(\Psi_\kappa)$ is the number of vortices of Ψ_κ, including multiplicities, in Ω.

In [16], it is proved (see [19] for an extension to the case which is considered here) that, for any $\epsilon_0 > 0$, there exist $C > 0$ and κ_0, such that, if $\kappa/h_{ex} \in [\beta_0 + \epsilon_0, 1 - \epsilon_0]$ and $\kappa \geq \kappa_0$, then

$$||h - h_{ex}||_{L^\infty(\Omega)} + \|\nabla(h_\kappa - h_{ext})\|_{L^\infty(\Omega)} \leq C\,.$$

Hence there exists a constant C such that :

$$\left|N(\Psi_\kappa) - \kappa h_{ex}\frac{|\Omega|}{2\pi}\right| \leq C\kappa\,. \tag{1.7}$$

Suppose now that the minimizer is unique when varying h_{ex} and κ as above. It is in this case reasonable to think that (Ψ_κ, A_κ) varies continuously. If there exist κ_0, $\epsilon_1 > 0$ and $C > 0$ such that

$$|\Psi_\kappa| \geq \frac{1}{C} \text{ on } \partial\Omega\ ,\ \forall \kappa \geq \kappa_0\ s.t.\ \frac{\kappa}{h_{ex}} \in [\lambda - \epsilon_1, \lambda + \epsilon_1]\ ,$$

(which would be a consequence of a uniform version of (1.5)), then $N(\Psi_\kappa)$ must be fixed, by continuity, for all $\kappa \geq \kappa_0$ such that $\kappa/h_{ex} \in [\lambda - \epsilon_1, \lambda + \epsilon_1]$, and this is in contradiction with (1.7).

The above argument works not only for (ψ_κ, A_κ) but for any solution of (1.6). If indeed critical points of (1.1) are continuous functions of κ and h_{ex} in this regime, then (1.7) would contradict another conjecture of Pan (conjecture 2 in [19]), implying that any solution of (1.6) converges to a constant along the boundary when $\kappa \to \infty$ and $\kappa/h_{ex} \in (\beta_0, 1)$. While the existence of continuous branches of critical points appears to be reasonable, two counter-examples come to our mind while discussing the continuity of the global minimizer:

1. Serfaty, [21] proves, for much lower external fields, that a large number of local minimizers of (1.1) in a disc, characterized by different winding numbers, exist for sufficiently large κ whenever $\frac{1}{C}\kappa^{-1} \leq h_{ex} \leq C\kappa^{\alpha-1}$ for some $0 < \alpha < 1$. In this regime of applied magnetic field values the magnetic field in non-uniform inside the domain, and hence the vortices are kept near the disc's center, which minimizes the magnetic field term in (1.1).

   While in the present case h_{ex} and κ have same order, if we allow for an $O(1)$ change in the applied magnetic field we might still encounter a global minimizer which turns into a local minimizer (or a critical point) and vice versa. Thus, this result suggests that the contradiction between (1.7) and the convergence to a uniform constant along the boundary might be explained by arguing that the global minimizer is discontinuous. However, unlike the case discussed in [21], no equivalent mechanism which keeps the vortices away from the boundary is presently known as the magnetic field uniformly converges to h_{ex} in Ω [2].

2. Bauman *et al.* [3] found radially symmetric solutions of the linearized version of (1.6). These solutions are characterized by a "fat" vortex at the disc's center. The degree of the vortex is determined, to leading order in the large κ limit, by the magnetic flux through the disc. Thus, there is a sequence of critical flux values where the bifurcating mode changes its winding number. It is shown in [3] that the bifurcating mode is locally stable near the bifurcation for κ large enough.

   Based on the results in [3] one can argue that the minimizer undergoes an abrupt change when the flux varies around one of the above critical values (and when κ is appropriately tuned to guarantee that weakly non-linear analysis still holds). However, this result seems to follow from the special geometry, and in general, for different geometries or away from the linear regime, nothing would hold the vortices in the center.

1.4 Statement of the main result

In the present contribution we focus on the case

$$\lim_{\kappa\to\infty} \frac{h_{ex}(\kappa)}{\kappa} = \frac{1}{\lambda}\,,$$

with λ close to β_0. We prove the following theorem :

Theorem 1.1
Let $\delta > 0$ be sufficiently small, so that $t = d(x, \partial\Omega)$ is a smooth function of x for $0 \le t \le \delta$, and let

$$\Omega_\delta = \left\{ x \in \bar{\Omega} \,:\, d(x,\partial\Omega) \le \delta \right\}\,.$$

Then there exist $\epsilon > 0$, a function

$$[0,+\infty[\times]\beta_0, \beta_0+\epsilon[\ni (\tau,\lambda) \mapsto U(\tau,\lambda) \in \mathbb{R}^+\,,$$

a constant $C > 0$ and κ_0, such that, for $\kappa \ge \kappa_0$ and $h_{ex} = \frac{\kappa}{\lambda}$ with $\lambda \in]\beta_0, \beta_0+\epsilon[$,

$$\int_{\Omega_\delta} \left[|\Psi_\kappa(x)|^2 - U\Big(\frac{\kappa}{\sqrt{\lambda}} t(x), \lambda\Big) \right]^2 dx_1 dx_2 \le \frac{C}{\kappa^2}\,, \tag{1.8a}$$

$$\int_{\partial\Omega} \left[\left| \Psi_\kappa \big/_{\partial\Omega} \right|^2 - U(0,\lambda) \right]^2 ds \le \frac{C}{\kappa^{1/2}}\,. \tag{1.8b}$$

The function $U(\tau,\lambda)$ is defined for $\tau \in \overline{\mathbb{R}^+}$ by

$$U(\tau,\lambda) = |f_{\zeta(\lambda)}(\tau;\lambda)|^2\,,$$

where $f_z(\tau;\lambda)$ and $\zeta(\lambda)$ are associated to minimizers of a family of one-dimensional problems, which will be analyzed in Section 2. The second statement in the theorem gives the $L^2(\partial\Omega)$ convergence of $|\Psi_\kappa|^2$ to a constant and is consequently a weak form of Pan's conjecture.

The rest of the contribution is arranged as follows :
In Section 2 we consider a one-dimensional differential operator and prove that it is positive for $\beta_0 < \lambda < \beta_0 + \epsilon$. In Section 3 we use the results of Section 2 to analyze a simplified two-dimensional minimization problem, which was proved in [19] to be a good approximation of the full Ginzburg-Landau model for $\beta_0 < \lambda < 1$. The last section gives the proof of Theorem 1.1.

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2 A one-dimensional problem

Let

$$\beta(z) = \inf_{\phi\in H^1_{mag}(]0,\infty[)\setminus\{0\}} \frac{\int_0^\infty |\phi'(\tau)|^2 + (\tau+z)^2|\phi(\tau)|^2\,d\tau}{\int_0^\infty |\phi(\tau)|^2\,d\tau}\,. \tag{2.1}$$

Here

$$H^1_{mag}\left(]0,+\infty[\right) = \left\{u \in L^2\left(]0,+\infty[\right)\ ,\ u' \in L^2\left(]0,+\infty[\right) \text{ and } \tau u \in L^2\left(]0,+\infty[\right)\right\}.$$

It is well known (see [5]) that $\beta(z)$ has a unique local minimum at $z_0 < 0$, where

$$\beta(z_0) = \beta_0 = z_0^2\,.$$

Furthermore, $\beta(z) \xrightarrow[z\to\infty]{} \infty$, and $\beta(z) \xrightarrow[z\to-\infty]{} 1$. Clearly, for $\beta_0 < \lambda < 1$ there exist $z_1(\lambda) < z_0 < z_2(\lambda)$, such that

$$]z_1(\lambda), z_2(\lambda)[= \beta^{-1}([\beta_0,\lambda[)\,.$$

It is also easy to show [6], that

$$\beta''(z_0) = -2z_0\phi^2(0) > 0\,, \tag{2.2}$$

where ϕ is the minimizer of (2.1) whose $L^2(\mathbb{R}_+)$ norm is unity.

Let $f_z(\tau;\lambda)$ denote the minimizer of

$$\mathcal{E}_{z,\lambda}(\phi) = \int_0^\infty |\phi'(\tau)|^2 + (\tau+z)^2|\phi(\tau)|^2 + \frac{\lambda}{2}|\phi(\tau)|^4 - \lambda|\phi(\tau)|^2\,d\tau \tag{2.3}$$

in $H^1_{mag}(]0,\infty[)$.
The Euler-Lagrange equation associated with (2.3) is

$$-f_z''(\tau;\lambda) + (\tau+z)^2 f_z(\tau;\lambda) = \lambda f_z(\tau;\lambda)(1-f_z(\tau;\lambda)^2)\,. \tag{2.4}$$

It has been proved in [19] (Theorems 3.1 and 3.3) that, whenever $z_1(\lambda) < z < z_2(\lambda)$, there exists a unique positive global minimizer to (2.3). Furthermore, let

$$b(z,\lambda) = \inf_{\varphi\in H^1_{mag}(]0,\infty[)} \mathcal{E}_{z,\lambda}(\varphi)\,. \tag{2.5}$$

Then, there exists $\zeta(\lambda) \in]z_1(\lambda), z_2(\lambda)[$, where $z \mapsto b(z,\lambda)$ attains its minimum over $\mathbb{R}$:

$$b(\zeta(\lambda),\lambda) = \inf_z b(z,\lambda)\,.$$

Moreover,

$$\int_0^\infty (\tau+\zeta(\lambda))|f_{\zeta(\lambda)}(\tau;\lambda)|^2\,d\tau = 0\,. \tag{2.6}$$

Remark 2.1
Note that when $z \notin]z_1(\lambda), z_2(\lambda)[$, then $b(z,\lambda)=0$, and the minimizer of $\mathcal{E}_{z,\lambda}$ is the 0-function. In particular,

$$b(\zeta(\lambda),\lambda) < 0\,,\quad \textit{if } \lambda > \beta_0\,.$$

The following lemma will play a crucial role in the analysis of the two-dimensional problem in Section 3.

Lemma 2.2
Let

$$\gamma(\alpha,\lambda) = \inf_{\phi\in H^1_{mag}(]0,\infty[)} \frac{\int_0^\infty |\phi'(\tau)|^2 + (\tau+\zeta+\alpha)^2|\phi(\tau)|^2 - \lambda(1-f_\zeta(\tau;\lambda)^2)|\phi(\tau)|^2\,d\tau}{\int_0^\infty |\phi(\tau)|^2\,d\tau}\,, \tag{2.7}$$

with $\zeta = \zeta(\lambda)$.
Then, there exists $\epsilon > 0$ such that, for $\lambda \in [\beta_0, \beta_0+\epsilon[$,

$$\min_{\alpha\in\mathbb{R}} \gamma(\alpha,\lambda) = 0\,. \tag{2.8}$$

Proof :
We divide the proof into three steps.
Step 1 : $\gamma(0,\lambda) = \gamma_\alpha(0,\lambda) = 0$**.**
Let $\mathbb{R}_+ \ni \tau \mapsto u(\tau;\alpha,\lambda)$ denote the positive minimizer of (2.7), whose $L^2(\mathbb{R}_+)$ norm is one. Then, u satisfies

$$-u''(\tau;\alpha,\lambda) + (\tau+\alpha+\zeta)^2 u(\tau;\alpha,\lambda) - \lambda(1-f_\zeta(\tau;\lambda)^2)u(\tau;\alpha,\lambda) = \gamma(\alpha,\lambda)u(\tau;\alpha,\lambda)\,, \tag{2.9a}$$

$$u'(0;\alpha,\lambda) = 0\,. \tag{2.9b}$$

For $\alpha = 0$, we multiply (2.9a) by f_ζ and integrate over $\mathbb{R}_+$ to obtain

$$\gamma(0,\lambda)\int_0^\infty f_\zeta(\tau;\lambda)u(\tau;\alpha,\lambda)\,d\tau = 0\,.$$

Since both u and f_ζ are positive, we have using (2.4),

$$\gamma(0,\lambda) = 0\,,\quad u(\tau,0,\lambda) = \frac{f_\zeta(\tau;\lambda)}{\|f_\zeta\|_2}\,, \tag{2.10}$$

where, for $p \in [1,+\infty]$, $\|\cdot\|_p = \|\cdot\|_{L^p(\mathbb{R}_+)}$.
Next, we differentiate (2.9) with respect to α to obtain, having in mind (2.4),

$$-u_\alpha'' + (\tau+\alpha+\zeta)^2 u_\alpha - \lambda(1-f_\zeta^2)u_\alpha = \gamma u_\alpha + \gamma_\alpha u - 2(\tau+\alpha+\zeta)u\,, \tag{2.11a}$$

$$u_\alpha'(0) = 0\,, \tag{2.11b}$$

where $u_\alpha(\tau;\alpha,\lambda) = (\frac{\partial u}{\partial\alpha})(\tau;\alpha,\lambda)$ and $\gamma_\alpha(\alpha,\lambda) = \frac{\partial\gamma}{\partial\alpha}(\alpha,\lambda)$.
Multiplying (2.11a) by u and integrating by parts, we obtain

$$\gamma_\alpha(\alpha,\lambda) = 2\int_0^\infty (\tau+\alpha+\zeta(\lambda))|u(\tau;\alpha,\lambda)|^2\,d\tau\,. \tag{2.12}$$

In view of (2.6) and (2.10), we thus have

$$\gamma_\alpha(0,\lambda) = 0\,. \tag{2.13}$$

Step 2 : $\exists\epsilon_1 > 0\ :\ \lambda < \beta_0 + \epsilon_1 \Rightarrow \gamma_{\alpha\alpha}(0,\lambda) > \frac{1}{2}\beta''(z_0) > 0\,.$ (2.14)

To prove the above statement we notice that $z_1(\lambda) \uparrow z_0$ and $z_2(\lambda) \downarrow z_0$ as $\lambda \to \beta_0$. Hence, since $z_1(\lambda) < \zeta(\lambda) < z_2(\lambda)$, we have

$$\zeta(\lambda) \xrightarrow[\lambda\to\beta_0]{} z_0\,. \tag{2.15}$$

Moreover, one gets from the fact that f_z is a minimizer the property that

$$\mathcal{E}_{z,\lambda}(f_z) \le 0\,.$$

From this inequality, and (2.1), we easily obtain :

$$\frac{1}{2}||f_z||_4^4 \le \frac{(\lambda-\beta_0)}{\lambda}||f_z||_2^2\,,$$

$$||(\tau+z)f_z||_2^2 \le \lambda||f_z||_2^2\,, \tag{2.16}$$

and

$$||f_z||_{H^1}^2 \leq (\lambda+1)||f_z||_2^2\,.$$

Let $z = \zeta(\lambda)$. Since $|\zeta(\lambda)|$ is bounded in some right semi-neighborhood of β_0, it follows immediately from (2.16) that for R large enough we get

$$||f_\zeta||_2^2 \leq 2||f_\zeta||_{L^2(]0,R[)}^2\,.$$

We now observe that :

$$||f_\zeta||_4^4 \leq C(\lambda-\beta_0)||f_\zeta||_{L^2(]0,R[)}^2 \leq C(\lambda-\beta_0)R^{\frac{1}{2}}||f_\zeta||_4^2\,.$$

This gives first that :

$$||f_\zeta||_4 \leq \tilde{C}(\lambda-\beta_0)^{\frac{1}{2}}\,,$$

and hence that

$$||f_\zeta||_2 \leq \hat{C}(\lambda-\beta_0)^{\frac{1}{2}}\,.$$

By interpolation, we obtain

$$||f_\zeta||_\infty \leq C||f_\zeta||_2^{\frac{1}{2}}\,||f_\zeta'||_2^{\frac{1}{2}} \leq C'(\lambda-\beta_0)^{\frac{1}{4}}\,, \tag{2.17}$$

which implies that

$$\lim_{\lambda\to\beta_0}\|f_{\zeta(\lambda)}(\,\cdot\,;\lambda)\|_\infty = 0\,.$$

Substituting the above and (2.15) into (2.7) yields

$$\gamma(\alpha,\lambda)\xrightarrow[\lambda\to\beta_0]{}\beta(\alpha+z_0)\,,$$

where the convergence is uniform on every compact set in $\mathbb{R}$. Since γ is holomorphic in α, its derivatives must uniformly converge as well, and hence

$$\gamma_{\alpha\alpha}(\alpha,\lambda)\xrightarrow[\lambda\to\beta_0]{}\beta''(\alpha+z_0)\,, \tag{2.18}$$

from which (2.14) easily follows.
We note that a tedious calculation shows that

$$\gamma_{\alpha\alpha}(0,\lambda) = -2\zeta\frac{f_\zeta^2(0;\lambda)}{\|f_\zeta\|_2^2} + \frac{6\lambda^2}{\|f_\zeta\|_2^2}\int_0^\infty f_\zeta^6(\tau;\lambda)\,d\tau\; - \frac{2\lambda}{3}\int_0^\infty f_\zeta^4(\tau;\lambda)[\lambda-(\tau+\zeta)^2]d\tau\,,$$

with $\zeta = \zeta(\lambda)$, from which one can easily prove (2.14) as well.

From (2.14) we obtain that

$$\exists\alpha_0 > 0:\; \lambda < \beta_0+\epsilon_1 \Rightarrow \gamma(\alpha,\lambda) \geq 0\;,\; \forall|\alpha| \leq \alpha_0\,.$$

The last step would thus be to prove the above statement for $|\alpha| > \alpha_0$.

Step 3 : **Proof of** (2.8)**.**

From the definition of γ (2.7), it follows that

$$\gamma(\alpha,\lambda) \geq \beta(\zeta+\alpha) - \lambda\,.$$

Clearly, for any $\alpha_1 > 0$, there exists $\epsilon_2 > 0$, such that, if $\lambda \leq \beta_0 + \epsilon_2$, then $[z_1(\lambda), z_2(\lambda)] \subset [z_0 - \alpha_1, z_0 + \alpha_1]$. We now take $\alpha_1 = \alpha_0$. This gives that $\beta(\zeta+\alpha) \geq \lambda$ for all $|\alpha| \geq \alpha_0$, and (2.8) follows.

□

3 On two-dimensional models on half cylinders

We can now prove the following theorem.

Theorem 3.1
For $\omega \in]0,+\infty[$ and $\lambda \in [\beta_0,+\infty[$, let us consider the functional

$$\mathcal{H}_\omega \ni \psi \mapsto E_\omega(\psi,\lambda) = \int_{-\pi/\omega}^{\pi/\omega}\int_0^\infty \left[|(i\nabla + \xi_1 \hat{i}_2)\psi|^2 + \frac{1}{2}\lambda|\psi|^4 - \lambda|\psi|^2\right] d\xi_1 d\xi_2\,, \tag{3.1}$$

where

$$|(i\nabla + \xi_1\hat{i}_2)\psi|^2 = |i\partial_{\xi_1}\psi|^2 + |(i\partial_{\xi_2} + \xi_1)\psi|^2\,,$$

and

$$\mathcal{H}_\omega = \Big\{\psi \in H^1_{mag}\big(\mathbb{R}_+ \times]-L,L[,\mathbb{C}\big)\,,\ \forall L>0 \,\Big|\, \exists z \in \mathbb{R}:\ \psi(\xi_1,\xi_2+2\pi/\omega) = e^{-iz\frac{2\pi}{\omega}}\psi(\xi_1,\xi_2)\Big\}\,.$$

Let ψ_λ the function

$$(\mathbb{R}_+\times\mathbb{R}) \ni (\xi_1,\xi_2) \mapsto \psi_\lambda(\xi_1,\xi_2) := e^{-i\zeta(\lambda)\xi_2} f_{\zeta(\lambda)}(\xi_1;\lambda)\,. \tag{3.2}$$

Then, there exists $\epsilon > 0$ such that :

$$E_\omega(\psi,\lambda) \geq E_\omega(\psi_\lambda,\lambda)\,,\ \forall\lambda \in]\beta_0,\beta_0+\epsilon[, \forall\omega>0 \text{ and } \forall\psi\in\mathcal{H}_\omega\,. \tag{3.3}$$

Remark 3.2
Clearly ψ_λ is in $\mathcal{H}_\omega$ (take $z = \zeta(\lambda)$). Hence, the theorem states that ψ_λ is the global minimizer of E_ω in $\mathcal{H}_\omega$.

Proof :
Consider first functions in $\mathcal{H}_\omega$ which are given in the form

$$(\xi_1,\xi_2) \mapsto \psi(\xi_1,\xi_2) := f_\zeta(\xi_1;\lambda)e^{-i\zeta\xi_2}v\,, \tag{3.4}$$

with v periodic,

$$v(\xi_1,\xi_2) = v(\xi_1,\xi_2+2\pi/\omega)\,, \tag{3.5}$$

and

$$\zeta = \zeta(\lambda) \,.$$

Then,

$$E_\omega(\psi,\lambda) = \int_{-\pi/\omega}^{\pi/\omega}\int_0^\infty \left[|(i\nabla + (\xi_1+\zeta)\hat{i}_2)f_\zeta v|^2 + \frac{1}{2}\lambda|f_\zeta v|^4 - \lambda|f_\zeta v|^2\right] d\xi_1 d\xi_2 \,.$$

Clearly,

$$\int_{-\pi/\omega}^{\pi/\omega}\int_0^\infty |(i\nabla + (\xi_1+\zeta)\hat{i}_2)f_\zeta v|^2 \, d\xi_1 d\xi_2 =$$

$$= \int_{-\pi/\omega}^{\pi/\omega}\int_0^\infty \Bigg[|v|^2\left[|f_\zeta'|^2 + (\xi_1+\zeta)^2|f_\zeta|^2\right] + f_\zeta^2|\nabla v|^2 + \frac{1}{2}(f_\zeta^2)'\frac{\partial}{\partial\xi_1}(|v|^2) +$$

$$+ i(\xi_1+\zeta)f_\zeta^2\left(\bar{v}\frac{\partial v}{\partial\xi_2} - v\frac{\partial\bar{v}}{\partial\xi_2}\right)\Bigg] \, d\xi_1 d\xi_2 \;.$$

Furthermore, integration by parts and (2.4) yield

$$\int_{-\pi/\omega}^{\pi/\omega}\int_0^\infty \left[|v|^2\left[|f_\zeta'|^2 + (\xi_1+\zeta)^2|f_\zeta|^2\right] + \frac{1}{2}(f_\zeta^2)'\frac{\partial}{\partial\xi_1}(|v|^2)\right] \, d\xi_1 d\xi_2$$

$$= \lambda\int_{-\pi/\omega}^{\pi/\omega}\int_0^\infty |v|^2 f_\zeta^2(1-f_\zeta^2)\, d\xi_1 d\xi_2 \,. \quad (3.6)$$

Hence,

$$\Delta E_\omega = E_\omega(\psi,\lambda) - E_\omega(f_\zeta e^{-i\zeta\xi_2},\lambda)$$

$$= \int_{-\pi/\omega}^{\pi/\omega}\int_0^\infty f_\zeta^2\left[|\nabla v|^2 + i(\xi_1+\zeta)\left(\bar{v}\frac{\partial v}{\partial\xi_2} - v\frac{\partial\bar{v}}{\partial\xi_2}\right)\right] \, d\xi_1 d\xi_2$$

$$+ \frac{1}{2}\int_{-\pi/\omega}^{\pi/\omega}\int_0^\infty f_\zeta(\xi_1;\lambda)^4(1-|v(\xi_1,\xi_2)|^2)^2 \, d\xi_1 d\xi_2 \,.$$

Using (3.5), we can write

$$v(\xi_1,\xi_2) = \sum_{n=-\infty}^{\infty} v_n(\xi_1)e^{in\omega\xi_2} \,.$$

Then,

$$\Delta E_\omega = \sum_{n=-\infty}^{\infty}\int_0^\infty f_\zeta(\xi_1;\lambda)^2\left[|v_n'(\xi_1)|^2 + (n^2\omega^2 + 2n\omega\xi_1)|v_n(\xi_1)|^2\right] \, d\xi_1 +$$

$$+ \frac{1}{2}\int_{-\pi/\omega}^{\pi/\omega}\int_0^\infty f_\zeta(\xi_1;\lambda)^4(1-|v(\xi_1,\xi_2)|^2)^2 \, d\xi_1 d\xi_2 \,. \quad (3.7)$$

Consider now the functional

$$u \mapsto \mathcal{J}(u,\alpha) = \int_0^\infty |f_\zeta(\xi_1;\lambda)|^2 \left[|u'(\xi_1)|^2 + [\alpha^2 + 2\alpha(\xi_1+\zeta)]|u(\xi_1)|^2\right] d\xi_1 \,.$$

Substituting $w(\xi_1) = f_\zeta(\xi_1;\lambda)\, u(\xi_1)$ and utilizing (2.4), we obtain :

$$\begin{aligned} \textstyle\int_0^\infty |f_\zeta(\xi_1)|^2 |u'(\xi_1)|^2\, d\xi_1 &= \textstyle\int_0^\infty \left[-\left(w^2 \frac{f_\zeta'}{f_\zeta}\right)' + w^2 \frac{f_\zeta''}{f_\zeta} + |w'|^2\right] d\xi_1 \\ &= \textstyle\int_0^\infty \left[|w'|^2 + [(\xi_1+\zeta)^2 - \lambda(1-f_\zeta^2)]|w|^2\right] d\xi_1 \,. \end{aligned}$$

Consequently,

$$\mathcal{J}(w/f_\zeta,\alpha) = \int_0^\infty \left(|w'|^2 + [(\xi_1+\zeta+\alpha)^2 - \lambda(1-f_\zeta^2)]|w|^2\right) d\xi_1 \geq \gamma(\alpha,\lambda) \int_0^\infty |w|^2\, d\xi_1 \,.$$

Combining the above with (3.7), we obtain

$$\Delta E_\omega \geq \sum_{n=-\infty}^{\infty} \gamma(n\omega,\lambda) \int_0^\infty f_\zeta^2 |v_n|^2\, d\xi_1 + \frac{1}{2}\int_{-\pi/\omega}^{\pi/\omega}\int_0^\infty f_\zeta^4 (1-|v|^2)^2\, d\xi_1 d\xi_2 \geq 0 \,,$$

which proves, using (2.8), Inequality (3.3) for every function in $\mathcal{H}_\omega$ satisfying (3.4). Note for later use that this implies

$$|E_\omega(\psi,\lambda) - E_\omega(f_\zeta\, e^{-i\zeta\xi_2},\lambda)| \geq \frac{1}{2}\int_{-\pi/\omega}^{\pi/\omega}\int_0^\infty f_\zeta(\xi_1;\lambda)^4 (1-|v(\xi_1,\xi_2)|^2)^2\, d\xi_1 d\xi_2 \,. \qquad (3.8)$$

To prove (3.3) **for all** $\psi \in \mathcal{H}_\omega$, we consider now functions of the form

$$(\xi_1,\xi_2) \mapsto \psi_0(\xi_1,\xi_2) = f_\zeta(\xi_1;\lambda) e^{-iz\xi_2} v \ , \text{ with } v(\xi_1,\xi_2) = v(\xi_1,\xi_2+2\pi/\omega) \,. \qquad (3.9)$$

Consider first the case when $\omega \in \mathbb{R}_+$ satisfies

$$\frac{\zeta - z}{\omega} = \frac{p}{q} \text{ for some pair } (p,q) \in \mathbb{Z}\times\mathbb{N} \,. \qquad (3.10)$$

Clearly, if ψ_0 satisfies (3.9) for some $\omega \in \mathbb{R}_+$, then it also satisfies (3.9) for $\omega/\hat{q}$, for every $\hat{q} \in (\mathbb{N}\setminus\{0\})$. Moreover, it is easy to show that

$$E_{\omega/\hat{q}}(\psi_0) = \hat{q}E_\omega(\psi_0) \ , \ E_{\omega/\hat{q}}(\psi_\lambda) = \hat{q}E_\omega(\psi_\lambda) \,. \qquad (3.11)$$

We now choose $\hat{q} = q$, and observe that, according to (3.10), $\hat{\omega} = \omega/q$ satisfies :

$$\frac{\zeta - z}{\hat{\omega}} \in \mathbb{Z}. \qquad (3.12)$$

But in this case, ψ_0 admits the representation (3.4), and hence

$$E_{\hat{\omega}}(\psi_0) \geq E_{\hat{\omega}}(\psi_\lambda) \,.$$

Coming back to ω and using (3.11), we have the proof of (3.3) when ω satisfies (3.10) (with the additional condition that z is fixed).

The proof of (3.3) in the general case follows now immediately from the density of the rational numbers in $\mathbb{R}$.

□

4 Surface superconductivity

Let $\mathcal{J}$ be given by (1.1). Let (Ψ_κ, A_κ) denote a minimizer of $\mathcal{J}$ in $H^1(\Omega,\mathbb{C}) \times H^1(\Omega,\mathbb{R}^2)$. We prove in this section that $|\Psi_\kappa|^2$ is nearly a constant along the boundary, in $L^2(\partial\Omega)$ sense, as $\kappa \to \infty$, and for

$$\beta_0 < \lambda = \frac{\kappa}{h_{ex}} < \beta_0 + \epsilon \,,$$

where ϵ is defined in (2.8).

To this end we need to adapt the results in [19]. Let then

$$x = \mathcal{F}(t,s) \tag{4.1}$$

denote a diffeomorphism from

$$D(\delta) = \{(s,t) \,:\, |s| \leq |\partial\Omega|/2 \,,\ 0 \leq t \leq \delta\} \,,$$

to

$$\Omega_\delta = \left\{x \in \bar{\Omega} \,:\, d(x,\partial\Omega) \leq \delta\right\} \,.$$

In the above $t = d(x,\partial\Omega)$ and s denotes the arclength along $\partial\Omega$.

In order to formulate and prove the results of this section it is necessary to fix a specific gauge for (ψ_κ, A_κ). To this end we first define the magnetic potential P_κ to be the solution of

$$\begin{cases} \nabla \times P_\kappa = \nabla \times A_\kappa - h_{ex} \,, & \text{in } \Omega \\ \nabla \cdot P_\kappa = 0 \,, & \text{in } \Omega \\ P_\kappa \cdot \hat{\nu} = 0 \,, & \text{in } \partial\Omega \,. \end{cases} \tag{4.2}$$

(cf. for example [8] for the proof of existence of a unique solution for (4.2).) Moreover, the map associating the solution P_κ of (4.2) to the field $(h_\kappa - h_{ex})$ is linear continuous from $L^p(\Omega)$ into $W^{1,p}(\Omega)$ for any $p \in [1, +\infty[$, and, using the Sobolev injection Theorem, one can show that

$$||P_\kappa||_{L^\infty(\Omega)} \leq C_\Omega \, ||\nabla \times A_\kappa - h_{ex}||_{L^\infty(\Omega)} \,. \tag{4.3}$$

Let then $e_1 = -\hat{\nu}$ denote an inward unit normal vector on $\partial\Omega$ and let e_2 denote a unit tangential vector. Let further

$$g = \mathrm{Det}(D\mathcal{F}) = 1 - t\kappa_r(s) \,,$$

where κ_r denote the local curvature on $\partial\Omega$. Let (see [19]) F be any vector potential such that $\nabla \times F = h_{ex}$ and let a be defined by

$$a = a_1 e_1 + a_2 e_2 = [F \cdot e_1] e_1 + [gF \cdot e_2] e_2 \,. \tag{4.4}$$

By [14], or the appendix in [12], there exists $\hat{A}_\kappa$ such that if we substitute $F = \hat{A}_\kappa$ in (4.4), we obtain

$$a_1(s,t) = 0 \quad ; \quad a_2(s,t) = h_{ex}[c_2 + t - t^2\kappa_r(s)] \,,$$

where

$$c_2 = \frac{|\Omega|}{|\partial\Omega|}\,.$$

The gauge in (1.2) is now fixed by the condition that the $\hat{A}_\kappa$ has in the new coordinates the normal form given above and that A_κ satisfies

$$A_\kappa - \hat{A}_\kappa = P_\kappa\ .$$

We now introduce the change of variables :

$$(\xi_1, \xi_2) = (\frac{\kappa}{\sqrt{\lambda}}t\,,\ \frac{\kappa}{\sqrt{\lambda}}s)\,, \tag{4.5}$$

and prove the following lemma :

Lemma 4.1
Let

$$\tilde{\Psi}_\kappa(\xi_1,\xi_2) = \begin{cases} \Psi_\kappa(\frac{\sqrt{\lambda}}{\kappa}\xi_1, \frac{\sqrt{\lambda}}{\kappa}\xi_2)e^{-ic_2\xi_2} & \text{for } 0 \leq \xi_1 \leq \frac{\kappa}{\sqrt{\lambda}}\delta\ , \\ \Psi_\kappa(\kappa\delta/\sqrt{\lambda}, \xi_2)e^{-ic_2\xi_2}e^{-(\xi_1-\kappa\delta/\sqrt{\lambda})} & \text{for } \xi_1 \geq \frac{\kappa}{\sqrt{\lambda}}\delta\ , \end{cases}$$

where $c_2 = c_2(\Omega)$, and let

$$\omega_\kappa = \frac{2\pi\sqrt{\lambda}}{\kappa|\partial\Omega|}\,.$$

Then, as κ tends to $+\infty$,

$$\mathcal{J}(\Psi_\kappa, A_\kappa) \geq \frac{1}{\kappa^2}E_{\omega_\kappa}(\tilde{\Psi}_\kappa, \lambda) + \mathcal{O}(1/\kappa^2)\,. \tag{4.6}$$

We will later prove (see (4.18)), that $\left|E_{\omega_\kappa}(\tilde{\Psi}_\kappa,\lambda)\right| \geq C\kappa$, and hence the correction term on the right-hand-side of (4.6) is much smaller than the first term as $\kappa \to \infty$.

Proof :
In [2] (see also [16]), it was proved that for $\lambda < 1$, there exists $\mu > 0$ such that

$$|\nabla(\nabla \times A_\kappa)| \leq Ce^{-\mu\kappa d(x,\partial\Omega)}\,. \tag{4.7}$$

Consequently, for $x \in \Omega_\delta$, we have

$$|\nabla \times A_\kappa - h_{ex}|\,(x) \leq \int_0^{d(x,\partial\Omega)} |\nabla(\nabla \times A_\kappa)(t, s(x))|dt \leq C\int_0^\infty e^{-\mu\kappa t}dt\,.$$

Hence, there exists $C_1 > 0$ such that

$$\|\nabla \times A_\kappa - h_{ex}\|_{L^\infty(\Omega_\delta)} \leq \frac{C_1}{\kappa}\,.$$

In view of (4.7) we can state the above inequality for the $L^\infty(\Omega)$ norm of $\nabla\times A_\kappa - h_{ex}$ and thus (4.3) gives that, for some $C_2>0$,

$$\|A_\kappa - \hat{A}_\kappa\|_{L^\infty(\Omega)} \le \frac{C_2}{\kappa}\,.$$

Hence, for some $C_3>0$,

$$\int_\Omega |\nabla\times A_\kappa - h_{ex}|^2\, dx_1dx_2 \le \frac{C_3}{\kappa^2}\,, \tag{4.8}$$

and

$$\begin{aligned}\int_\Omega \left|\left(\frac{i}{\kappa}\nabla + A_\kappa\right)\Psi_\kappa\right|^2 dx &= \int_\Omega \left|\left(\frac{i}{\kappa}\nabla + \hat{A}_\kappa\right)\Psi_\kappa\right|^2 dx \\ &+ \int_\Omega |\Psi_\kappa|^2|A_\kappa - \hat{A}_\kappa|^2\, dx \;+\; \int_\Omega (A_\kappa - \hat{A}_\kappa)\cdot\left[\frac{i}{\kappa}(\overline{\Psi_\kappa}\nabla\Psi_\kappa - \Psi_\kappa\nabla\overline{\Psi_\kappa}) + 2\hat{A}_\kappa\right] dx \ge \\ &\ge \int_\Omega \left|\left(\frac{i}{\kappa}\nabla + \hat{A}_\kappa\right)\Psi_\kappa\right|^2 dx \;-2||A_\kappa - \hat{A}_\kappa||_{L^\infty(\Omega)}\,||\Psi_\kappa||_{L^\infty}\int_\Omega \left|\left(\frac{i}{\kappa}\nabla + \hat{A}_\kappa\right)\Psi_\kappa\right| dx\,.\end{aligned}$$

In [2] it is shown that

$$|\Psi_\kappa| + \left|\left(\frac{i}{\kappa}\nabla + \hat{A}_\kappa\right)\Psi_\kappa\right| \le Ce^{-\mu\kappa d(x,\partial\Omega)}\,, \tag{4.9}$$

for some $\mu>0$ when $\lambda<1$. Hence,

$$\int_\Omega \left|\left(\frac{i}{\kappa}\nabla + A_\kappa\right)\Psi_\kappa\right|^2 dx \ge \int_\Omega \left|\left(\frac{i}{\kappa}\nabla + \hat{A}_\kappa\right)\Psi_\kappa\right|^2 dx \;-\frac{C}{\kappa^2}\,. \tag{4.10}$$

Combining (4.8) and (4.10) we obtain

$$\mathcal{J}(\Psi_\kappa, A_\kappa) \ge \int_\Omega \left(\left|\left(\frac{i}{\kappa}\nabla + \hat{A}_\kappa\right)\Psi_\kappa\right|^2 + \frac{1}{2}|\Psi_\kappa|^4 - |\Psi_\kappa|^2\right) dx \;-\frac{C}{\kappa^2}\,. \tag{4.11}$$

Using the coordinates (4.1) we obtain

$$\begin{aligned}\int_{\Omega_\delta}\left(\left|\left(\frac{i}{\kappa}\nabla + \hat{A}_\kappa\right)\Psi_\kappa\right|^2 + \frac{1}{2}|\Psi_\kappa|^4 - |\Psi_\kappa|^2\right) dx_1dx_2 = \\ = \int_{D(\delta)}\left\{\frac{1}{\kappa^2}\left|\frac{\partial\Psi_\kappa}{\partial t}\right|^2 + \frac{1}{g^2}\left|\left(\frac{i}{\kappa}\frac{\partial}{\partial s} + a_2\right)\Psi_\kappa\right|^2 + \frac{1}{2}|\Psi_\kappa|^4 - |\Psi_\kappa|^2\right\} g\, dsdt\,.\end{aligned}$$

Applying the transformation (4.5), we obtain

$$\int_0^{\frac{\kappa\delta}{\sqrt{\lambda}}} d\xi_1 \int_{-\frac{\kappa|\partial\Omega|}{2\sqrt{\lambda}}}^{\frac{\kappa}{2\sqrt{\lambda}|\partial\Omega|}} d\xi_2 \frac{\tilde{g}}{\kappa^2}\left\{\left|\frac{\partial\tilde{\Psi}_\kappa}{\partial\xi_1}\right|^2 + \frac{1}{\tilde{g}^2}\left|\left(i\frac{\partial}{\partial\xi_2} + \xi_1 - \kappa_r(s)\frac{\sqrt{\lambda}}{\kappa}\xi_1^2\right)\tilde{\Psi}_\kappa\right|^2 + \frac{1}{2}\lambda|\tilde{\Psi}_\kappa|^4 - \lambda|\tilde{\Psi}_\kappa|^2\right\}\,,$$

where $\tilde{g}$ is defined by :

$$\tilde{g}(\xi_1,\xi_2) = 1 - \frac{\sqrt{\lambda}}{\kappa}\xi_1\kappa_r(\sqrt{\lambda}\xi_2/\kappa)\,.$$

Since by (4.9), there exist $\tilde{\mu}>0$ and $\tilde{C}$ such that on $[\,0,\frac{\kappa\delta}{\sqrt{\lambda}}\,[\,\times\,]-\frac{\kappa|\partial\Omega|}{2\sqrt{\lambda}}\,,+\frac{\kappa|\partial\Omega|}{2\sqrt{\lambda}}\,[$,

$$\left|\left(i\frac{\partial}{\partial\xi_2}+\xi_1-\kappa_r(\sqrt{\lambda}\xi_2/\kappa)\frac{\sqrt{\lambda}}{\kappa}\xi_1^2\right)\tilde{\Psi}_\kappa\right|^2 + |\xi_1|^2|\tilde{\Psi}_\kappa|^2 \leq \tilde{C}\,e^{-\tilde{\mu}\xi_1}\,,$$

there exist $\mu>0$ and C such that :

$$\left|\left(i\frac{\partial}{\partial\xi_2}+\xi_1\right)\tilde{\Psi}_\kappa\right|^2 \leq Ce^{-\mu\xi_1}\,.$$

We thus obtain

$$\int_0^{\frac{\kappa}{\sqrt{\lambda}}\delta} d\xi_1 \int_{-\frac{\kappa}{2\sqrt{\lambda}}|\partial\Omega|}^{\frac{\kappa}{2\sqrt{\lambda}}|\partial\Omega|} d\xi_2 \frac{1}{\tilde{g}}\left|\left(i\frac{\partial}{\partial\xi_2}+\xi_1-\kappa_r(\sqrt{\lambda}\xi_2/\kappa)\frac{\sqrt{\lambda}}{\kappa}\xi_1^2\right)\tilde{\Psi}_\kappa\right|^2 =$$

$$= \int_0^{\infty} d\xi_1 \int_{-\frac{\kappa}{2\sqrt{\lambda}}|\partial\Omega|}^{\frac{\kappa}{2\sqrt{\lambda}}|\partial\Omega|} d\xi_2 \left|\left(i\frac{\partial}{\partial\xi_2}+\xi_1\right)\tilde{\Psi}_\kappa\right|^2 + \mathcal{O}(1)\,.$$

Using the above arguments for the remaining terms yields

$$\mathcal{J}(\Psi_\kappa,A_\kappa) =$$

$$\frac{1}{\kappa^2}\int_0^{\frac{\kappa}{\sqrt{\lambda}}\delta} d\xi_1 \int_{-\frac{\kappa}{2\sqrt{\lambda}}|\partial\Omega|}^{\frac{\kappa}{2\sqrt{\lambda}}|\partial\Omega|} d\xi_2 \left(\left|\frac{\partial\tilde{\Psi}_\kappa}{\partial\xi_1}\right|^2 + \left|\left(i\frac{\partial}{\partial\xi_2}+\xi_1\right)\tilde{\Psi}_\kappa\right|^2 + \frac{1}{2}\lambda|\Psi_\kappa|^4 - \lambda|\Psi_\kappa|^2\right) + \mathcal{O}(\kappa^{-2})\,,$$

so

$$\mathcal{J}(\Psi_\kappa,A_\kappa) = \frac{1}{\kappa^2}E_{\omega_\kappa}(\tilde{\Psi}_\kappa,\lambda) + \mathcal{O}(\kappa^{-2})\,. \tag{4.12}$$

Combining (4.12) with (4.11) yields (4.6).

□

We can now prove the main result of this work.

Proof of Theorem 1.1:

Let $\hat{\psi}_{\lambda,\kappa}:\Omega_\delta\to\mathbb{C}$ be given by

$$\hat{\psi}_{\lambda,\kappa}(x) = \psi_\lambda\Big(\frac{\kappa}{\sqrt{\lambda}}t(x)\Big)\exp\left\{-ic_2\Big(\frac{\kappa}{\sqrt{\lambda}}s(x)\Big)\right\}\,.$$

Let further $\chi:\mathbb{R}_+\to[0,1]$ denote a smooth cutoff function satisfying

$$\chi(t) = \begin{cases} 1 & t\leq\frac{1}{2} \\ 0 & t\geq 1\end{cases}\,.$$

Then, $\chi\big(t(x)/\delta\big)\hat{\psi}_{\lambda,\kappa}(x)$ is in $H^1(\Omega,\mathbb{C})$, and it is not difficult to show that

$$\mathcal{J}(\Psi_\kappa, A_\kappa) \leq \mathcal{J}(\hat{\psi}_{\lambda,\kappa}, \hat{A}_\kappa) = -\frac{C_\lambda|\partial\Omega|}{\kappa\sqrt{\lambda}} + \mathcal{O}(\kappa^{-2}) \tag{4.13a}$$

where

$$C_\lambda = -\frac{\omega}{2\pi}E_\omega(\psi_\lambda, \lambda)\,. \tag{4.13b}$$

By Theorem 3.1 there exists $\epsilon > 0$ such that, for $\beta_0 < \lambda < \beta_0 + \epsilon$, we have, for all ω,

$$C_\lambda = -\frac{\omega}{2\pi}\inf_{\psi\in\mathcal{H}_\omega} E_\omega(\psi,\lambda) \tag{4.14}$$

Note that this implies in particular :

$$C_\lambda = -\lim_{\omega\to 0}\frac{\omega}{2\pi}\inf_{\psi\in\mathcal{H}_\omega} E_\omega(\psi,\lambda)\,. \tag{4.15}$$

Combining (4.6) and (4.13) we obtain

$$\mathcal{J}(\Psi_\kappa, A_\kappa) = -\frac{C_\lambda|\partial\Omega|}{\kappa\sqrt{\lambda}} + \mathcal{O}(\kappa^{-2})\,, \tag{4.16}$$

In [19] (Lemma 7.3), Pan proves (4.16), for any fixed $\beta_0 < \lambda < 1$, by using as a test functions the unknown minimizer of E_{ω_κ} in $\mathcal{H}_{\omega_\kappa}$ instead of ψ_λ, and (4.15) as the definition of C_λ. He also proves (4.16) when $\lambda(\kappa) \to \lambda$ (with $\lambda(\kappa) = \frac{\kappa}{h_{ex}(\kappa)}$) but with an additional $\mathcal{O}\Big([\lambda(\kappa) - \lambda]/\kappa\Big)$ error. Note that when $\lambda = \beta_0$ this result is no more useful since $C_{\beta_0} = 0$, and hence the leading order term of $\mathcal{J}$ is unknown in this case (see [10] for results in this case).

By (2.5) and (3.3), we have

$$C_\lambda = -b(\zeta(\lambda),\lambda)\,,$$

which shows that $C_\lambda > 0$ for every $\beta_0 < \lambda < 1$. Consequently, we have by (4.6) and (4.16),

$$E_{\omega_\kappa}(\tilde{\Psi}_\kappa,\lambda) \leq E_{\omega_\kappa}(\psi_\lambda,\lambda) + C\,. \tag{4.17}$$

Thus, by (4.13b)

$$E_{\omega_\kappa}(\tilde{\Psi}_\kappa,\lambda) \leq -C_\lambda\frac{|\partial\Omega|}{\sqrt{\lambda}}\kappa + C,$$

which proves that indeed

$$\big|E_{\omega_\kappa}(\tilde{\Psi}_\kappa,\lambda)\big| \geq C\kappa\,, \tag{4.18}$$

and that the correction term on the right hand side of (4.6) is much smaller than the leading order term.

Let w_κ be defined by

$$\tilde{\Psi}_\kappa(\xi_1,\xi_2) = f_\zeta(\xi_1;\lambda) w_\kappa(\xi_1,\xi_2)\, e^{-ic_2\xi_2}\,.$$

Clearly, w_κ is periodic in ξ_2. Thus, by (3.8), we get

$$\left| E_{\omega_\kappa}(\tilde{\Psi}_\kappa,\lambda) - E_{\omega_\kappa}(\psi_\lambda,\lambda)\right| \geq \frac{1}{2}\int_{-\pi/\omega_\kappa}^{\pi/\omega_\kappa} d\xi_2 \int_0^\infty d\xi_1\, |f_\zeta|^4(1-|w_\kappa|^2)^2\,.$$

Consequently, there exists $C_0>0$ such that

$$\int_{-\pi/\omega_\kappa}^{\pi/\omega_\kappa} d\xi_2 \int_0^\infty d\xi_1 |f_\zeta|^4(1-|w_\kappa|^2)^2 \leq C_0\,,$$

and hence, for suitable constants C_1 and C_2,

$$\int_{\Omega_\delta} [|\Psi_\kappa|^2 - |\hat{\psi}_{\lambda,\kappa}|^2]^2\,dx \leq \frac{C_1}{\kappa^2}\int_{-\pi/\omega_\kappa}^{\pi/\omega_\kappa} d\xi_2 \int_0^\infty d\xi_1\, |f_\zeta|^4(1-|w_\kappa|^2)^2 \leq \frac{C_2}{\kappa^2}\,,$$

which proves (1.8a).
To prove (1.8b), we first notice that it is proved in [2] that :

$$|\Psi_\kappa| + \frac{1}{\kappa}|\nabla\Psi_\kappa| \leq C\,,$$

and using the explicit form of $\hat{\psi}_{\lambda,\kappa}$, we obtain :

$$\left|\nabla(|\Psi_\kappa|^2 - |\hat{\psi}_{\lambda,\kappa}|^2)\right| \leq C\kappa\,. \tag{4.19}$$

Evidently, as a consequence of the mean value formula, there exist $C>0$ and $\delta_0>0$, such that, for every $0<\delta'\leq\delta_0$, there exists $0\leq\delta''\leq\delta'$ such that

$$\int_{t=\delta''}\left[|\Psi_\kappa|^2 - |\hat{\psi}_{\lambda,\kappa}|^2\right]^2 ds \leq \frac{C}{\delta'}\int_{\Omega_{\delta'}}\left[|\Psi_\kappa|^2 - |\hat{\psi}_{\lambda,\kappa}|^2\right]^2 dx_1dx_2\,.$$

Furthermore by (4.19), we have

$$\int_{\partial\Omega}\left[|\Psi_\kappa|^2 - |\hat{\psi}_{\lambda,\kappa}|^2\right]^2 ds \leq C\int_{t=\delta''}[|\Psi_\kappa|^2 - |\hat{\psi}_{\lambda,\kappa}|^2]^2\,ds \;+\; C\kappa\delta''\,.$$

Consequently, there exists $C>0$ such that :

$$\int_{\partial\Omega}\left[|\Psi_\kappa|^2 - |\hat{\psi}_{\lambda,\kappa}|^2\right]^2 ds \leq \frac{C}{\delta'}\frac{1}{\kappa^2} + C\kappa\delta'\,.$$

Choosing $\delta' = \kappa^{-3/2}$ proves (1.8b).

□

Finally, we compare Theorem 1.1 with the results in [10] (Remark 1.5). As was stated already in the introduction, when $\rho(\kappa) = o(\kappa^{1/2})$ and tends to ∞ as $\kappa \to +\infty$, (1.4) holds. The function u in (1.4) is given by

$$u(\tau) = \beta_0 \frac{|u_0(\tau)|^2}{\|u_0\|_4^4} ,$$

u_0 denoting the minimizer of (2.1).

We first note that, since as $\lambda \to \beta_0$, we have

$$f_{\zeta_\lambda}^2(\tau) \sim \frac{\lambda - \beta_0}{\beta_0^2} u(\tau) ,$$

and since

$$\frac{\lambda - \beta_0}{\beta_0^2} \sim \frac{\rho}{\kappa} \quad \text{as } \kappa \to \infty ,$$

(1.4) and (1.8a) match. The error in (1.4) is substantially smaller than in (1.8a). The difference is explained by the fact that ψ_κ itself is small on $\partial\Omega$ when $\lambda \to \beta_0$. Thus, if we extrapolate the error term in (1.4) to external fields for which $\rho/\kappa \approx 1$, it becomes $\mathcal{O}(\kappa^{-2})$ exactly as in (1.8a).

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